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Generators for maximal subgroups of Conway group Co_1

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Abstract: The Conway groups are the three sporadic simple groups Co_1, Co_2 and Co_3. There are total of 22 maximal subgroups of Co_1 and generators of 6 maximal subgroups are provided in web Atlas of finite simple groups. The aim of this paper is to give generators of remaining 16 maximal subgroups.

Keywords: Conway group, maximal subgroup, generators, finite group, normalizer

MSC: 20D05, 20B40

1 Introduction

The Conway group Co_1 is one of the 26 sporadic simple groups. The largest of the Conway groups, Co_0, is the group of automorphisms of the Leech lattice Γ with respect to addition and inner product. It has order 8,315,553,613,086,720,000 [1] but it is not a simple group. The simple group Co_1 of order $2^{21}.3^9.5^4.7^2.11.13.23$ is defined as the quotient of Co_0 by its center, which consists of the scalar matrices ±1 [1]. The local subgroups of Co_1 are found in [2] and the maximal subgroups of Co_1 in [3]. There is also a valuable discussion in [4].
The following theorem is crucial in determining the maximal subgroups of Co_1:

Theorem 1.1. *[3] If K is a non-Abelian characteristically simple subgroup of Co_1, then $N_{Co_1}(K)$ is contained either in a local subgroup of Co_1 or in a conjugate of one of six particular groups:*
1. $NA_5 \cong (A_5 \times J_2).2$,
2. $NA_6 \cong (A_6 \times U_3(3)).2$,
3. $NA_7 \cong (A_7 \times L_2(7)).2$,
4. $S(2) \cong Co_2$,
5. $S(3) \cong Co_3$,
6. $S(2^3) \cong U_6(2).S_3$.

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Note on notation. We use $A.B$ to denote an arbitrary extension of A by B, while $A : B$ and $A \cdot B$ denote split and non-split extensions, respectively. The symbol n denotes a cyclic group of that order, while $[n]$ denotes an arbitrary group of order n. We follow the ATLAS [5] notation for conjugacy classes. Moreover, we denote x^y by $y^{-1}xy$ and $[x, y]$ by $x^{-1}y^{-1}xy$.

To facilitate the computations in finite simple and almost simple groups, [6], [7] provides the representations and words for generators of most of the maximal subgroups. However, there are still some cases to deal with. A research problem "Words for maximal subgroups in sporadic groups" appears on the web page of R. A. Wilson. We pursue the work initiated by R.A. Wilson of finding words for maximal subgroups of Co_1. According to [3] there are 24 conjugacy classes of maximal subgroups, but later on R.A. Wilson pointed out a few errors (in his own paper) in the list of maximal subgroups of Co_1 [8] in which he mentions that the two subgroups $3^2.[2.3^6].2A_4$ and $3^2.[2^3.3^4].2A_4$ are not the maximal subgroups of Co_1, so the list contains total 22 conjugacy classes of maximal subgroups. There are 22 maximal subgroups of the group Co_1. The maximal local subgroups have been determined in [3]. The Atlas of Group Representations [6] contains the words for maximal subgroups of Co_1 except 16. The maximal subgroups of Co_1 are given below.

1. $\star\ (A_9 \times S_3)$
2. $\star\ (D_{10} \times (A_5 \times A_5).2).2$
3. $\star\ 5^{1+2}GL_2(5)$
4. $\star\ 3^{1+4} : 2.S_4(3).2$
5. $\star\ 3^6 : 2.M_{12}$.
6. $\star\ 3^2.U_4(3).D_8$
7. $\star\ 3^{3+4} : 2.(S_4 \times S_4)$
8. $\star\ 2^{4+12}.(S_3 \times 3S_6)$
9. $\star\ 5^3 : (4 \times A_5).2$
10. $\star\ 7^2 : (3 \times 2.S_4)$
11. $\star\ 5^2 : 2A_5$
12. $\star\ (A_7 \times L_2(7)) : 2$
13. $\star\ (A_6 \times U_3(3)) : 2$
14. $\star\ (A_4 \times G_2(4)) : 2$
15. $\star\ (A_5 \times J_2) : 2$
16. $\star\ 2^{2+12}(A_8 \times S_3)$
17. $U_6(2) : S_3$
18. $2^{1+8}.O_8(2)$
19. Co_3
20. $2^{11} : M_{24}$
21. $3.Suz : 2$
22. Co_2

Next we proceed to find words for 16 maximal subgroups which are marked by asterisk.

2 Methods

Most of the maximal subgroups on our list can be generated by two elements. If the group is small enough, a random search will produce the subgroup required. This method was successfully used in [9] but in Co_1 the subgroups are too large to use brute force. One more focused way of generating a subgroup is by choosing a pair of conjugacy classes A and B in G such that conjugates of random elements $a \in A$, $b \in B$ have a reasonable high probability of generating a conjugate of the desired subgroup.

In most of the cases we present here, even though the subgroup we wish to construct may be generated by two elements, it may be hard to tell which conjugacy classes they belong to. Even if we know a suitable pair of conjugacy classes it may be that the probability the random elements in these classes generate the desired subgroup is relatively small. In this case, we find some part of the desired subgroup, and work inside

another subgroup, usually an involution centralizer, to find the rest. Once we have found a copy of the desired subgroup, we can get information regarding the generating sets.
The maximal subgroups often occur as normalizers of elementary abelian groups. So normalizers, which are crux of the matter here, were mostly computed by methods given in [10] and [11].
The generators of subgroups, wherever possible, have been obtained from [6].
The subgroups can be identified by determining order, composition series and orbit sizes in several permutation representations. Moreover, comparing our result with the list of maximal subgroups in [5] we find that there is only one possibility of the subgroup.

3 Main Results

We have extensively used the information given in [5], [3] and Atlas of finite group representations [6]. We use GAP [12] for group theoratic calculations. Throughout this paper a and b are the standard generators of Co_1 in permutation representation on 98280 points available at [6].

3.1 Construction of $(A_9 \times S_3)$ inside Co_1

The required maximal subgroup is the normalizer of an element of class $3D$. Here we use power maps to find the representative of class $3D$ to say that a_1 is the element of said class given by $a_1 = ((ba)^2b)^4$. Now the normalizer of a_1 inside Co_1 gives us the required maximal subgroup. The normalizer can be computed by the technique given in [10] and the programs given in [11]. Before computing the normalizer we will give some random words of Co_1 which will be used later. These words are given below:

$$b_1 = (ab)^2ba, \quad b_2 = aba, \quad b_3 = ab$$
$$b_4 = (ab)^2, \quad b_5 = (ab)^2a, \quad b_6 = (ab)^2b.$$

Consider the group generated by a_1 and b say $H_2 =< a_1, b >$, then compute the normalizer of a_1 inside H_2. From here we get the partial normalizer of a_1 inside Co_1. Before computing the partial normalizer we will give some random elements of H_2 which will facilitates our computations. These elements are given by:

$$c_1 = a_1b, \quad c_2 = a_1ba_1, \quad c_3 = a_1ba_1bb,$$
$$c_4 = a_1ba_1bba_1, \quad c_5 = a_1ba_1bba_1b.$$

The words for partial normalizer are given below:

$$k_1 = b_2b_5b_2b_5^2b_2b_5^6b_2b_5^6b_2b_5, \quad k_2 = b_2b_5^2b_2b_5^2b_2b_5^3b_2b_5^7b_2b_5,$$
$$k_3 = a_1c_1a_1c_1a_1c_1^5a_1c_1a_1c_1, \quad k_4 = a_1c_1a_1c_1^3a_1c_1^2a_1c_1a_1c_1^5,$$
$$k_5 = a_1c_1a_1c_1^3a_1c_1^3a_1c_1^4a_1c_1^4, \quad k_6 = a_1c_1a_1c_1^5a_1c_1^3a_1c_1^5a_1c_1.$$

Next we will find an involution inside the above partial normalizer. This involution is given by $d_1 = k_1^3$, then find the centralizer of d_1 inside Co_1 by using the method given by J.Bray [13]. The generators of the centralizer of d_1 Inside Co_1 are given by:

$$d_2 = a[(d_1, a)]^2, \quad d_3 = [(d_1, b)]^2, \quad d_4 = ab[(d_1, ab)]^{10},$$
$$d_5 = aba[(d_1, aba)]^7, \quad d_6 = [(d_1, abab)]^7, \quad d_7 = [(d_1, ababa)]^7,$$
$$d_8 = d_2d_4, \quad d_9 = d_2d_5, \quad d_{10} = d_2d_6, \quad d_{11} = d_2d_7.$$

$H_3 =< d_2, d_3, d_4, d_5, d_6, d_7 >$.
The words for the normalizer of a_1 inside the above centralizer (H_3) are given below:

$$k_7 = d_2d_8^2d_2d_8^5d_2d_8^3d_2d_8^5d_2d_8^6, \quad k_8 = d_3d_9^4d_3d_9^7d_3d_9d_3d_9^4d_3d_9$$

$$k_9 = d_3 d_9^7 d_3 d_9^3 d_3 d_9^4 d_3 d_9^4 d_3 d_9, \qquad k_{10} = d_5 d_{11}^2 d_5 d_{11}^2 d_5 d_{11}^2 d_5 d_{11}^2 d_5 d_{11}^2.$$

By looking at the [5] we see that these words generate only partial normalizer so we repeat the above process untill we find the complete normalizer. Now we give some other elements of Co_1 which will be used in further computations.

$$m_1 = (b_1 b_2 b_3 b_4^2 b_5^2 b_6^3)^4, \qquad m_2 = (b_1 b_2 b_3 b_4^2 b_5^3 b_6^2)^6, \qquad m_3 = (b_1 b_2 b_3 b_4^3 b_5 b_6^2)^3$$
$$m_4 = (b_1 b_2 b_3 b_4^3 b_5 b_6^2)^6, \qquad m_5 = (b_1 b_2 b_3 b_4^3 b_5 b_6^3)^3.$$

Consider the group generated by a_1 and m_5 say $H_4 =< a_1, m_5 >$, then compute the normalizer of a_1 inside H_4. From here we get the partial normalizer of a_1 inside Co_1. Before computing the partial normalizer we will give some random elements of H_4. These elements are given by:

$$n_1 = a_1 m_5, \qquad n_2 = a_1 m_5 m_5,$$
$$n_3 = a_1 m_5 m_5 a_1, \qquad n_4 = a_1 m_5 m_5 a_1 m_5,$$
$$n_5 = a_1 m_5 m_5 a_1 m_5 a_1, \qquad n_6 = a_1 m_5 m_5 a_1 m_5 a_1 m_5,$$
$$n_7 = a_1 m_5 m_5 a_1 m_5 a_1 m_5 m_5.$$

The word for the normalizer of a_1 inside H_4 is given by:

$$k_{11} = a_1 n_7^4 a_1 n_7^4 a_1 n_7^4 a_1 n_7^4 a_1 n_7^4.$$

Now by combining the words for the normalizer of a_1 inside H_2, H_3 and H_4 we get the required complete normalizer given by k_2, k_7 and k_{11}. The generators for $(A_9 \times S_3)$ are k_2 and $k_7 k_{11}$.

3.2 Construction of $(D_{10} \times (A_5 \times A_5).2).2$ inside Co_1

The required maximal subgroup is the normalizer of an element of class 5B. Here we first find an element of class 5B and then find the normalizer of that element inside Co_1, which gives us the required maximal subgroup. The element of class 5B can be calculated by using the power maps. The normalizer can be computed by the technique given in [10] and the programs given in [11] will facilitate us in computing the normalizer. Before computing the normalizer we will give some elements of Co_1 which will be used later. These elements are given below:

$$b_1 = ababba, \qquad b_2 = aba, \qquad b_3 = ab,$$
$$b_4 = abab, \qquad b_5 = ababa, \qquad b_6 = ababb,$$
$$b_7 = ababbab, \qquad b_{12} = babbababab.$$

The element of class $5C$ is given by $c = (b_1 b_{12})^2$. Next we will give the strategy of finding the normalizer of c inside Co_1. Consider the group generated by c and a say $H_2 =< c, a >$, then compute the normalizer of c inside H_2. From here we get the partial normalizer of c inside Co_1. Before computing the partial normalizer we will give some random elements of H_2 which will facilitates us in computations.

$$c_1 = ca, \qquad c_2 = cac, \qquad c_6 = caccac,$$
$$c_7 = accaca, \qquad c_8 = accacaca, \qquad c_9 = accacacac.$$

The words for normalizer of c inside H_2 are given by:

$$k_1 = cc_1 cc_1^3 cc_1^3 cc_1^3 cc_1^2, \qquad k_2 = cc_1^3 cc_1^6 cc_1^2 cc_1^6 cc_1^3,$$
$$k_3 = cc_1^4 cc_1^7 cc_1^6 cc_1^7 cc_1^4, \qquad k_4 = ac_6^6 ac_6^3 ac_6^3 ac_6^5 ac_6^4,$$

$$k_5 = ac_9ac_9ac_9^7ac_9^4ac_9^6.$$

Here we will give some more elements of Co_1.

$$d_1 = (b_1b_2b_3b_7^2b_4b_5^3)^6, \quad d_2 = (b_1b_2b_3b_7^3b_4b_5^2)^6,$$
$$d_3 = (b_1b_2b_3b_7^3b_4^2b_5^3)^4, \quad d_4 = (b_1b_2b_3^2b_7^2b_4^2b_5^3)^6.$$

Consider the group generated by fixing c and a is replaced by d_2 say $H_3 =< c, d_2 >$, then compute the normalizer of c inside H_3. Some elements of H_3, which are used in further computations, are given below:

$$e_1 = cd_2, \quad e_2 = cd_2c, \quad e_3 = cd_2cd_2$$

The words for normalizer of c inside H_3 are given by:

$$k_6 = d_2e_1d_2e_1^5d_2e_1^{15}d_2e_1^5d_2e_1^{13}, \quad k_7 = d_2e_1^5d_2e_1^6d_2e_1^6d_2e_1^6d_2e_1^5,$$
$$k_8 = d_2e_1^{14}d_2e_1^{14}d_2e_1^{14}d_2e_1^{14}d_2e_1^{14}.$$

Now combining the normalizer of c inside H_2 and H_3 we get the complete normalizer of Co_1 given by k_5, k_6 and k_7. The words for $(D_{10} \times (A_5 \times A_5).2).2$ are k_5k_6 and k_7.

3.3 Construction of $5^{1+2}GL_2(5)$ inside Co_1

From the information given in [5], the required maximal subgroup is the normalizer of an element of class 5c. The construction of this group consist of two steps given below.
Step 1
Here we first find an element of class 5c. For that we give some words of Co_1. These words are given by:

$$b_1 = ababba, \quad b_2 = aba, \quad b_3 = ab$$
$$b_4 = abab, \quad b_5 = ababa, \quad b_6 = ababb$$
$$b_7 = ababbab, \quad b_8 = ababbaba, \quad b_9 = ababbabab$$
$$b_{10} = ababbababa, \quad b_{11} = ababbababab, \quad b_{12} = babbababab$$
$$b_{13} = babbabababa, \quad b_{14} = babbabababb, \quad b_{15} = babbabababba.$$

Then we use the power maps to find an element of order 5 and next check its centralizer order which confirms that the element belongs to class 5c. The element of class 5c is given by $5c = (b_{13}b_{15})^3$.
Step 2
In this step we will find the normalizer of 5c inside Co_1. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of 5c inside different subgroups of of Co_1. Then we combine these partial normalizer to get the required normalizer. The computations of these partial normalizers are given below.
Consider the group generated by $5c$ and c_1 say $H_1 =< 5c, c_1 >$, then compute the normalizer of $5c$ inside H_1. Before computing the partial normalizer we will give some words of H_1 which will facilitates us in computations. These words are given by:

$$d_1 = b_{10}e_1(b_{10}^{-1}), \quad d_2 = 5cd_1, \quad d_3 = 5cd_15c,$$
$$d_4 = 5cd_15c5c, \quad d_5 = 5cd_15c5cd_1.$$

Next we use the "TKnormalizertest" given in [11] to compute the words for the partial normalizer of $5c$ inside H_1. These words are given by:

$$k_1 = d_1d_2^1d_1d_2^4d_1d_2^4d_1d_2^4d_1d_2^2, \quad k_2 = d_1d_2^4d_1d_2^4d_1d_2^7d_1d_2^3d_1d_2^7,$$

$$k_3 = d_1 d_2^7 d_1 d_2^3 d_1 d_2^7 d_1 d_2^4 d_1 d_2^4, \qquad k_4 = d_1 d_2^5 d_1 d_2^7 d_1 d_2^5 d_1 d_2^{10} d_1 d_2,$$
$$k_5 = d_1 d_2^{10} d_1 d2^2 d_1 d_2^7 d_1 d_2^5 d_1 d_2^6, \qquad k_{80} = (e_1 e_2 e_3^2 a^2 b^2 e_1)^6,$$
$$k_{81} = (e_1 e_2 e_3^2 a^2 b^2 e_1^2)^4.$$

Again consider the group generated by $5c$ inside k_{81} say $H_2 =< 5c, k_{81} >$. Here we compute the partial normalizer of $5c$ inside H_2 by giving the similar arguments as mentioned above. We will give some words for H_2 which are used in computations. These words are given below:

$$d_6 = 5ck_{81}, \qquad d_7 = 5ck_{81}5c, \qquad d_8 = 5ck_{81}5ck_{81},$$
$$d_9 = 5ck_{81}5ck_{81}k_{81}, \qquad d_{10} = 5ck_{81}5ck_{81}k_{81}5c, \qquad d_{11} = 5ck_{81}5ck_{81}k_{81}5c5c,$$
$$d_{12} = 5ck_{81}5ck_{81}k_{81}5c5ck_{81}, \qquad d_{13} = 5ck_{81}5ck_{81}k_{81}5c5ck_{81}k_{81},$$
$$d_{14} = 5ck_{81}5ck_{81}k_{81}5c5ck_{81}k_{81}5c, \qquad d_{15} = d_6 d_7,$$
$$d_{16} = d_6 d_8, \qquad d_{17} = d_6 d_9, \qquad d_{18} = d_6 d_{10},$$
$$d_{19} = d_6 d_{11}, \qquad d_{20} = d_6 d_{12}, \qquad d_{21} = d_6 d_{13},$$
$$d_{22} = d_6 d_{14}, \qquad c_2 = (k_2)^5.$$

The words for the partial normalizer are given below:

$$k_7 = c_2 d_8^3 c_2 d_8^3 c_2 d_8^9 c_2 d_8^3 c_2 d_8^3, \qquad k_8 = c_2 d_8^3 c_2 d_8^3 c_2 d_8^9 c_2 d_8^6 c_2 d_8^9,$$
$$k_9 = c_2 d_8^3 c_2 d_8^6 c_2 d_8^3 c_2 d_8^6 c_2 d_8^3.$$

Consider the group generated by k_6, k_7 and k_8 say $H_3 =< k_6, k_7, k_8 >$. The words for H_3 are given below:

$$d_{23} = k_6 k_7, \qquad d_{24} = k_6 k_8, \qquad d_{25} = k_7 k_8, \qquad d_{26} = k_6 k_7 k_8,$$
$$d_{27} = k_6 k_7 k_8 k_6, \qquad d_{28} = k_6 k_7 k_8 k_7, \qquad d_{29} = k_6 k_7 k_8 k_7 k_8,$$
$$d_{30} = k_6 k_7 k_8 k_7 k_8 k_6, \qquad d_{31} = k_6 k_7 k_8 k_7 k_8 k_7, \qquad d_{32} = k_6 k_7 k_8 k_7 k_8 k_8.$$

The words for the normalizer of $5c$ inside H_3 are given below:

$$k_9 = 5ck_6^3 5ck_6^3 5ck_6^3 5ck_6^3 5ck_6^3, \qquad k_{10} = 5ck_6^6 5ck_6^6 5ck_6^6 5ck_6^6 5ck_6^6,$$
$$k_{11} = 5ck_7^2 5ck_7^2 5ck_7^2 5ck_7^2 5ck_7^2, \qquad k_{12} = 5ck_8 5ck_8^2 5ck_8^2 5ck_8^2 5ck_8,$$
$$k_{13} = 5cd_{25} 5cd_{25}^5 5cd_{25}^5 5cd_{25}^2 5cd_{25}^6.$$

Consider the involution $c_3 = k_4^2$. Since c_3 is an involution so its normalizer can easily be calculated by using the method given by J.Bray [13]. The generators of the normalizer of c_3 inside Co_1 are given below:

$$f_1 = 5c[(c_3, 5c)]^2, \qquad f_2 = k_{81}5ck_{81}[c_3, k_{81}5ck_{81}]^{16},$$
$$f_3 = [c_3, k_{81}5ck_{81}5c]^{12}, \qquad f_4 = [c_3, k_{81}5ck_{81}5ck_{81}]^6.$$

Consider the group generated by f_2 and f_3 say $H_4 =< f_2, f_3 >$. Now compute the normalizer of 5c inside H_4. Before computing the normalizer we give some words of H_4. These words are given below:

$$d_{33} = f_2 f_3, \qquad d_{34} = f_2 f_3 f_2, \qquad d_{40} = f_3 f_2 f_3 f_3 f_2 f_2 f_3.$$

The word for the normalizer of 5c inside H_4 is given below:
$k_{14} = d_{33} d_{40}^6 d_{33} d_{40}^7 d_{33} d_{40}^6 d_{33} d_{40}^2 d_{33} d_{40}^7$.
Now combining the above partial normalizers will give us the words for the required maximal subgroup. These words are given by k_4, k_{13} and k_{14}.

3.4 Construction of $3^{1+4} : 2.S_4(3).2$ inside Co_1

From the information given in Atlas [5] the required maximal subgroup is the normalizer of an element of class 3B. So here we first find an element of class 3B and then find the normalizer of it. We will give some words of Co_1. These words are given by:

$$b_1 = ababba, \quad b_2 = aba, \quad b_3 = ab, \quad b_4 = abab,$$
$$b_5 = ababa, \quad b_6 = ababb, \quad b_7 = ababbab,$$
$$c_1 = b_1b_2b_3b_7b_4b_5^2, \quad c_2 = (b_1b_2b_3b_7b_4b_5^2)^3, \quad c_3 = (b_1b_2b_3b_7b_4b_5^2)^5,$$
$$c_4 = (b_1b_2b_3b_7b_4^2b_5^3)^6, \quad c_5 = (b_1b_2b_3b_7b_4^3b_5^3)^1, \quad c_6 = (b_1b_2b_3b_7b_4^3b_5^3)^4.$$

The construction of this group consist of two steps given below.

Step 1

In this step we will find an element of class 3B. This can be done by using the power maps of the above generated elements, then checking whether the centralizer order confirms that the element under consideration belongs to class 3B or not. The element of class 3B is given by "b".

Step 2

In this step we will find the normalizer of b inside Co_1. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of b inside different subgroups of of Co_1. Then we combine these partial normalizer to get the required normalizer. The computations of these partial normalizers are given below.

Consider the group generated by b and c_1 say $H_1 =< b, c_1 >$, then compute the normalizer of b inside H_1. Before computing the partial normalizer we will give some words of H_1 which will facilitates our computations. These words are given below:

$$e_1 = bc_1, \quad e_2 = bc_1b, \quad e_3 = bc_1bc_1.$$

Next we use the "TKnormalizertest" [11] to compute the words for the partial normalizer of b inside H_1.

$$k_1 = be_1be_1be_1^{13}be_1^{11}be_1^{12}, \quad k_2 = be_1be_1^6be_1^{10}be_1^5be_1^{12},$$
$$k_3 = be_1be_1^{13}be_1^2be_1^8be_1^4, \quad k_4 = be_1^2be_1^{11}be_1^6be_1^{11}be_1^2,$$
$$k_5 = be_1^4be_1^9be_1^2be_1^{11}be_1^5, \quad k_6 = be_1^4be_1^9be_1^3be_1be_1^6.$$

Again consider the group generated by b and c_4 say $H_2 =< b, c_4 >$. Here we compute the partial normalizer of b inside H_2 by giving similar arguments as those mentioned above. We will give some words for H_2 which are used in further computations. These words are given by:

$$e_9 = bc_4, \quad e_{10} = bc_4b, \quad e_{11} = bc_4bc_4, \quad e_{12} = bc_4bc_4b.$$

The words for the partial normalizer are given below:

$$k_7 = be_9be_9be_93be_9^{11}be_9^{12}, \quad k_8 = be_9be_9^2be_9^2be_9^2be_9^2,$$
$$k_9 = be_9be_9^{13}be_9^2be_9^9be_9^4, \quad k_{10} = be_9be_9^{13}be_9^4be_9^5be_9^6.$$

Now combining the above partial normalizers will give the words for the required maximal subgroup. These words are given by k_5 and k_{10}.

3.5 Construction of $(A_4 \times G_2(4)) : 2$ inside Co_1

From the information given in Atlas [5] the required maximal subgroup is the normalizer of $2B^2$ (a four group whose involutions are in class 2B). The construction of this subgroup consist of two steps.

Step1
In this step we find $2B^2$. First we find an involution of class 2B. This involution is given by a, then we find the centralizer inside Co_1. This can be done by using the technique given by J. Bray given in [13]. The generators of the centralizer inside Co_1 are given by:

$$a_1 = [a, b]^3, \quad a_2 = [a, ba]^3, \quad a_3 = bab[a, bab]^7,$$
$$a_4 = baba[a, baba]^7, \quad a_5 = babab[a, babab]^5, \quad a_6 = abab[a, abab]^7,$$
$$a_7 = ababa[a, ababa]^7.$$

Then we search inside this centralizer for an involution of class 2B to find an elementary abelian group of order 4. This involution is given by $c = a_5^{15}$. Now we have the required $2B^2$ generated by a and c where $c = a_5^{15}$.
Step2
In this step we will calculate the normalizer of $2B^2$ inside Co_1. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of $2B^2$ inside different subgroups of Co_1. Then we combine these partial normalizer to get the required normalizer. We also give some elements of Co_1 which are used in computations:

$$b_1 = ab, \quad b_2 = aba, \quad b_3 = abab, \quad b_4 = ababa$$
$$b_5 = ababb, \quad b_6 = ababba, \quad b_7 = babba, \quad b_8 = babbab,$$
$$d_{12} = (b_1 b_2 b_3 b_7 b_4^2 b_5^3)^6, \quad d_{13} = (b_1 b_2 b_3 b_7 b_4^3 b_5^2)^4.$$

Consider the group generated by a, c and d_{12} say $H_1 =< a, c, d_{12} >$, then compute the normalizer of $2B^2$ inside H_1. Before computing the partial normalizer we will some words of H_1 which will facilitates our computations. These words are given below:

$$e_1 = ad_{12}, \quad e_2 = cd_{12}, \quad e_3 = cd_{12}a, \quad e_4 = cd_{12}ac.$$

Next we use the "TKnormalizertest" [11] to compute the words for the partial normalizer of $2B^2$ inside Co_1. These words are given below:

$$k_1 = ae_1ae_1ae_1ae_1ae_1^{12}, \quad k_2 = ae_1ae_1ae_1ae_1^2ae_1, \quad k_3 = ae_2ae_2ae_2^{11}ae_2^9ae_2^{13}$$
$$k_4 = ae_2ae_2^3ae_2^6ae_2^9ae_2^2, \quad k_5 = ae_2ae_2^3ae_2^8ae_2^3ae_2$$

Again consider the group generated by a, c and d_{13} say $H_2 =< a, c, d_{13} >$. Here we compute the partial normalizer of $2B^2$ inside H_2 by giving similar arguments to those mentioned above. We give some words for H_2 which are used in further computations:

$$e_{13} = ad_{13}, \quad e_{14} = cd_{13}, \quad e_{15} = ad_{13}c, \quad e_{16} = ad_{13}ca,$$

The words for the partial normalizer are given by:

$$k_9 = ae_{13}ae_{13}ae_{13}ae_{13}ae_{13}^9, \quad k_{10} = ae_{13}ae_{13}ae_{13}ae_{13}^6ae_{13}^7.$$

Now combining the above partial normalizers gives us the words for the required maximal subgroup. These words are given by k_4, k_5 and k_{10}.

3.6 Construction of $(2^{2+12})A_8 \times S_3$ inside Co_1

From the information given in Atlas [5] the required maximal subgroup is the normalizer of $2A^2$ (a four group whose involutions are in class 2A). The construction of this subgroup consist of two steps.

Step1

In this step we find $2A^2$. First we find an involution of class 2A. This involution is given by $c = (ab)^{20}$, then we find the centralizer of c inside Co_1. This can be done by using the technique in[13]. The generators of the centralizer of c inside Co_1 are given below:

$$a_2 = a[a_1, a]^2, \quad a_3 = b[a_1, b]^2, \quad a_4 = aba[a_1, aba]^2,$$
$$a_5 = bab[a_1, bab]^2, \quad a_6 = baba[a_1, baba], \quad a_7 = babab[a_1, babab]^2.$$

Then searching inside this centralizer for an involution of class 2A, which combines with c, gives an elementary abelian group of order 4. This involution is given by $d = (a_2a_6)^4$. Now we have the required $2A^2$ generated by c and d.

Step2

In this step we will calculate the normalizer of $2A^2$ inside Co_1. The normalizer can be found by using the technique given in [10] i.e. we construct the partial normalizer of $2A^2$ inside different subgroups of of Co_1. Then we combine these partial normalizer to get the required normalizer. We also give some elements of Co_1 which are used in computations:

$$b_1 = ab, \quad b_2 = aba, \quad b_3 = abab, \quad b_4 = ababa,$$
$$b_5 = ababb, \quad b_6 = ababba, \quad b_7 = babba, \quad b_8 = babbab,$$
$$d_1 = (b_1b_2b_3b_7b_4b_5^2)^1, \quad d_2 = (b_1b_2b_3b_7b_4b_5^2)^3, \quad d_3 = (b_1b_2b_3b_7b_4b_5^2)^5,$$
$$d_4 = (b_1b_2b_3b_7b_4^2b_5^3)^6, \quad d_{12} = (b_1b_2b_3b_7^3b_4^3b_5)^5.$$

Consider the group generated by c, d and d_1 say $H_1 =< c, d, d_1 >$, then compute the normalizer of $2A^2$ inside H_1. Before computing the partial normalizer we give some words of H_1 which will facilitate our computations:

$$e_1 = cd_1, \quad e_2 = cd_1d, \quad e_3 = cd_1dc, \quad e_4 = cd_1dcdd_1.$$

Next we use the "TKnormalizertest" [11] to compute the words for the partial normalizer of $2B^2$ inside Co_1. These words are given by:

$$k_1 = ce_1ce_1ce_1ce_1^2ce_1, \quad k_2 = de_1de_1^2de_1^2de_1^5de_1^4, \quad k_3 = de_1de_1^2de_1^4de_1^2de_1^3,$$
$$k_4 = de_1^2de_1de_1^4de_1de_1^2, \quad k_5 = de_1^2de_1de_1^4de_1^3de_1^6, \quad k_6 = de_1^2de_1de_1^4de_1^5de_1^2,$$
$$k_7 = de_2de_2^2de_2^2de_2^5de_2^4, \quad k_8 = de_2^2de_2de_2^4de_2de_2^2.$$

Again consider the group generated by c, d and d_4 say $H_2 =< c, d, d_4 >$. Here we compute the partial normalizer of $2A^2$ inside H_2 by giving similar arguments to those mentioned above. We give some words for H_2 which are used in further computations:

$$e_{11} = cd_4, \quad e_{12} = dd_4, \quad e_{13} = dd_4c, \quad e_{14} = dd_4cd.$$

The words for the partial normalizer are given by:

$$k_9 = de_{11}de_{11}de_{11}^2de_{11}de_{11}, \quad k_{10} = de_{11}de_{11}de_{11}^2de_{11}de_{11}^6,$$
$$k_{11} = de_{11}de_{11}de_{11}^5de_{11}^4de_{11}^4, \quad k_{12} = de_{11}de_{11}^2de_{11}^2de_{11}de_{11}^5.$$

Now consider the group generated by c, d and d_{12} say $H_3 =< c, d, d_{12} >$. Here we compute the partial normalizer of $2A^2$ inside H_3 by by giving similar arguments to those mentioned above. We give some words for H_3 which are used in further computations:

$$e_{22} = cd_{12}, \quad e_{23} = dd_{12}, \quad e_{24} = dd_{12}c, \quad e_{25} = dd_{12}cd.$$

The words for the partial normalizer are given below:

$$k_{15} = ce_{22}^14ce_{22}^{20}ce_{22}^{20}ce_{22}^{20}ce_{22}^6, \quad k_{16} = ce_{23}^7ce_{23}^5ce_{23}^6ce_{23}^7ce_{23}^5,$$
$$k_{17} = de_{23}de_{23}^4de_{23}^4de_{23}^8de_{23}^2.$$

Now combining the above partial normalizers gives the words for the required maximal subgroup. These words are given by k_6, k_{16} and k_{17}.

3.7 Construction of $3^6 : 2.M_{12}$ inside Co_1

From the information given in Atlas [5] the required maximal subgroup is the normalizer of 3^6 (elementary abelian group of order 729). The construction of this subgroup consist of two steps given below.
Step1
Here we will construct 3^6. To construct 3^6 we adopt the following strategy.
i) Find an arbitrary element of order 3. This element is given by b.
ii) Find centralizer of b inside Co_1. Before calculating the centralizer we give some elements of Co_1 as follows:

$$b_1 = ab, \quad b_2 = aba, \quad b_3 = abab, \quad b_4 = ababa,$$
$$b_5 = ababb, \quad b_6 = ababba, \quad b_7 = babba, \quad b_8 = babbab,$$
$$c_1 = (b_1b_2b_3b_7b_4b_5^2), \quad c_2 = (b_1b_2b_3b_7b_4b_5^2)^3,$$
$$c_3 = (b_1b_2b_3b_7b_4b_5^2)^5, \quad c_4 = (b_1b_2b_3b_7b_4^2b_5^3)^6.$$

The centralizer of b can be found by using the technique given in [10] i.e. we start by constructing the partial centralizer of 3^6 inside different subgroups of of Co_1. Next we combine these partial centralizer to get the required centralizer of b inside Co_1. We also give some elements of Co_1 which are used in computations. Consider the group generated by b, c_1 say $H_1 =< c, c_1 >$. Compute the centralizer of b inside H_1. Before computing the partial centralizer we give some random elements of H_1 which are used in computations. These elements are given by:

$$e_1 = bc_1, \quad e_2 = bc_1b, \quad e_3 = bc_1bc_1, \quad e_4 = bc_1bc_1b.$$

The generators of centralizer of b inside Co_1 are given below:

$$k_1 = be_1be_1be_1^{13}be_1^{11}be_1^{12}, \quad k_2 = be_1be_1^6be_1^{10}be_1^5be_1^{12}, \quad k_3 = be_1be_1^{13}be_1^2be_1^8be_1^4,$$
$$k_4 = be_1^2be_1^{11}be_1^6be_1^{11}be_1^2, \quad k_5 = be_1^4be_1^9be_1^2be_1^{11}be_1^5, \quad k_6 = be_1^4be_1^9be_1^3be_1be_1^6.$$

Again consider the group generated by b and c_4 say $H_2 =< b, c_4 >$. Here we compute the partial normalizer of b inside H_2. We give some random elements of H_2 which are used in further computations:

$$e_9 = bc_4, \quad e_{10} = bc_4b, \quad e_{11} = bc_4bc_4.$$

The generators of centralizer of b inside H_2 are given below:

$$k_7 = be_9be_9be_9^{13}be_9^{11}be_9^{12}, \quad k_8 = be_9be_9^2be_9^2be_9^2be_9^2,$$
$$k_9 = be_9be_9^{13}be_9^2be_9^9be_9^4, \quad k_{10} = be_9be_9^{13}be_9^4be_9^5be_9^6.$$

Now combining the above partial normalizers gives us the generators for 3^6. These generators are given by k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8,k_9 and k_{10}.
Then we can easily find 3^6 inside the above centralizer. The generators for 3^6 are given by:

$$f_1 = k_1, \quad f_2 = k_2, \quad f_3 = k_3^2,$$
$$f_4 = k_4^2, \quad f_5 = (k_2k_7)^2, \quad f_6 = (k_2k_{10})^4.$$

Step2
In this step we will find the normalizer of 3^6 inside Co_1 which is the required maximal subgroup. The words for the normalizer of 3^6 are given below:

$$z_1 = b_1^2b_2^2b_3b_4^2b_5^3, \quad z_2 = b_1b_2^2b_3^4b_4b_5, \quad x_1 = f_2k_7,$$
$$z_3 = f_3x_1^2f_3x_1^6f_3x_1^{15}f_3x_1^6f_3x_1^2, \quad z_4 = f_3x_1^3f_3x_1^{12}f_3x_1^6f_3x_1^{12}f_3x_1^3,$$
$$z_5 = (f_3x_1^4)^3x_1^3(f_3x_1^4)^2, \quad x_2 = f_6z_2f_5f_6z_2, \quad z_6 = (f_5x_2^5)^5.$$

The words for $3^6 : 2.M_{12}$ are $w_1 = z_3z_4$ and $w_2 = z_5z_6$.

3.8 Construction of $3^2.U_4(2).D_8$ inside Co_1

From the information given in Atlas [5] the required maximal subgroup is the normalizer of 3^2 (elementary abelian group of order 9). Similarly as in the previous cases we will construct this group into two steps given below.

Step1
In this step we will find 3^2. This can be done by taking 3^6 which we constructed in section 3.7, then searching inside these 3^6 we can easily find the required 3^2 given by f_4 and f_5.
Step2
In this step we will find the normalizer of $H_1 =< f_4, f_5 >$ inside Co_1. The normalizer can be found by using the technique given in [10] i.e., we construct the partial normalizer of H_1 inside different subgroups of of Co_1. Then we combine these partial normalizer to get the required normalizer. The computations of these partial normalizers are given below.
Before computing the partial normalizer we give some words of Co_1 which will facilitate our computations. These words are given below:

$$l_1 = b_6b_7^2b_3^4b_4b_5^3, \quad l_2 = b_6b_7^3b_3^2b_4^3b_5^3, \quad l_3 = b_6b_7^3b_3^3b_4^6b_5^3,$$
$$l_4 = b_6b_7^4b_3b_4^2b_5^2, \quad l_5 = b_6b_7^4b_3^4b_4b_5, \quad l_6 = b_6b_7^5b_3^3b_4^5b_5,$$
$$l_7 = b_6b_7^6b_3^4b_4b_5, \quad l_8 = b_6^2b_7b_3b_4^6b_5^2, \quad l_9 = b_6^2b_7^7b_3b_4^6b_5^2,$$
$$l_{10} = b_6^3b_7^2b_3^4b_4b_5^3, \quad l_{11} = b_6^3b_7^3b_3^2b_4^3b_5^3, \quad l_{12} = b_6^3b_7^3b_3^3b_4^6b_5^3,$$
$$l_{13} = b_6^3b_7^4b_3b_4^2b_5^2, \quad l_{14} = b_6^3b_7^4b_3^4b_4b_5, \quad l_{15} = b_6^3b_7^5b_3^3b_4^5b_5,$$
$$l_{16} = b_6^3b_7^6b_3^4b_4^1b_5, \quad l_{17} = b_6^4b_7b_3b_4^6b_5^2, \quad l_{18} = b_6^4b_7^7b_3b_4^6b_5^2,$$
$$g_1 = f_5l_1, \quad g_2 = f_6l_1, \quad g_3 = f_6l_1f_5,$$
$$g_4 = f_6l_1f_5l_1, \quad g_5 = f_6l_1f_5l_1f_6, \quad g_6 = f_5l_1f_5l_1f_6.$$

the words for the words for the partial normalizer of H_1 are:

$$k_{11} = f_5g_4^2f_5g_4^5f_5g_4^4f_5g_4^5f_5g_4^2, \quad k_{12} = f_5g_4^7f_5g_4^6f_5g_4^5f_5g_4^6f_5g_4^7,$$
$$k_{13} = f_5g_1f_5g_1f_5g_1f_5g_1f_5g_1^4.$$

We find an involution inside the above calculated normalizer and then calculate its centralizer. This involution is given by $g_{15} = k_{15}^3$, generators of the centralizer of g_{15} inside Co_1 are:

$$h_1 = [g_{15}, a]^2, \quad h_2 = b[g_{15}, b], \quad h_3 = [g_{15}, ab]^3$$
$$h_4 = [g_{15}, aba]^3, \quad h_5 = [g_{15}, abab]^2.$$

Now we will find the partial normalizer of H_1 inside Co_1. The words for the centralizer are given below:

$$g_{16} = h_2h_3, \quad g_{17} = h_2h_4, \quad g_{18} = h_2h_5,$$
$$g_{19} = h_1h_4h_5, \quad g_{20} = h_1h_4h_5h_2, \quad g_{21} = h_1h_4h_5h_2h_3,$$
$$g_{22} = h_1h_4h_5h_2h_3h_4, \quad g_{23} = h_1h_4h_5h_2h_3h_4h_5, \quad g_{24} = h_1h_4h_5h_2h_3h_4h_5h_2,$$
$$g_{25} = h_1h_4h_5h_2h_3h_4h_5h_3, \quad g_{26} = h_1h_4h_5h_2h_3h_4h_5h_4, \quad g_{27} = h_1h_4h_5h_2h_3h_4h_5h_4h_2,$$
$$g_{28} = h_1h_2h_3h_4h_5.$$

The words for the partial normalizer of H_1 inside Co_1 are given by:

$$k_{17} = h_1g_{16}h_1g_{16}^2h_1g_{16}^2h_1g_{16}^{11}h_1g_{16}^8, \quad k_{18} = h_1g_{16}h_1g_{16}^2h_1g_{16}^8h_1g_{16}^{11}h_1g_{16}^8,$$
$$k_{19} = h_1g_{16}h_1g_{16}^4h_1g_{16}^1 0h_1g_{16}h_1g_{16}^8, \quad k_{20} = h_2g_{27}h_2g_{27}^4h_2g_{27}^3h_2g_{27}^5h_2g_{27}^3,$$
$$k_{31} = f_5g_{28}^2f_5g_{28}^7f_5g_{28}^4f_5g_{28}^6f_5g_{28}^2.$$

The words for the required maximal subgroup $3^2.U_4(2).D_8$ are k_{11} and $k_{19}k_{31}$.

3.9 Construction of $3^{3+4} : 2.(S_4 \times S_4)$ inside Co_1

Following [5], we see that the required subgroup is the normalizer of 3^3. We can easily find 3^3 from 3^6 calculated above in 3.7, then the normalizer of it gives us the required subgroup. The generators of 3^3 are f_4, f_5 and f_6. Before computing the normalizer we give some elements:

$$l_1 = b_6b_7^2b_3^4b_4b_5^3, \quad g_1 = f_4l_1, \quad g_7 = f_4l_1f_5l_1f_6,$$
$$g_8 = f_4f_5f_6l_1f_4l_1, \quad z_{13} = f_4g_8f_4g_8^5f_4g_8f_4g_8^6f_4g_8, \quad g_9 = z_{13}^3,$$
$$h_1 = [g_9, a]^3, \quad h_2 = b[g_9, b], \quad h_3 = ab[g_9, ab]^2,$$
$$h_4 = [g_9, aba]^3, \quad h_5 = abab[g_9, abab]^2, \quad g_{10} = h_2h_5,$$
$$g_{11} = h_1h_2h_3, \quad g_{12} = h_1h_3h_4, \quad g_{13} = g_{11}g_{12}.$$

The generators for the normalizer of 3^3 are given below:

$$z_{11} = f_6g_1^3(f_6g_1^7)^3f_6g_1^4, \quad z_{12} = f_6g_7(f_6g_7^3)^2f_6g_7^2f_6g_7^8,$$
$$z_{14} = h_1g_{10}^4(h_1g_{10}^3)^2h_1g_{10}^4h_1g_{10}^6, \quad z_{15} = h_1g_{13}h_1g_{13}^2h_1g_{13}^{13}h_1g_{13}h_1g_{13}^8,$$
$$z_{16} = (z_{14}z_{15})^4z_{15}^2z_{14}z_{15}^7, \quad z_{17} = (z_{14}z_{15})^2z_{14}z_{15}^7z_{14}z_{15}^5z_{14}z_{15}^2.$$

The words for $3^{3+4} : 2.(S_4 \times S_4)$ are given by $w_5 = z_{16}z_{17}$ and $w_6 = z_{12}$.

3.10 Construction of $2^{4+12}.(S_3 \times 3S_6)$ inside Co_1

From [5], the required subgroup is the normalizer of $2A^4$ (inside Co_1) and can be constructed by taking an involution of class $2A$, then searching inside its centralizer. We have already constructed $N_{Co_1}(2A^2)$ in section 3.6, and now we will search $2A^4$ inside $N_{Co_1}(2A^2)$. Now $2A^4 = \langle k_1, k_3, g_1, g_2\rangle$, where k_1, k_3 are same as in section 3.6 and $g_1 = (k_2k_8)^2$, $g_2 = (k_2k_9)^2$. The words for k_8 and k_9 are given in section 3.6.

$$h_1 = k_1, \quad h_2 = k_3, \quad h_5 = a[h_1, a]^2,$$
$$h_7 = ba[h_1, ba]^2, \quad h_8 = [h_2, ab]^2, \quad h_9 = [h_2, ba]^3, \quad h_{10} = aba[h_2, aba]^2,$$
$$l_1 = h_8h_9, \quad l_2 = h_8h_{10}.$$

The generator for the normalizer of $2A^4$ are given below.

$$k_{10} = h_5h_7h_5h_7^2h_5h_7^5h_5h_7h_5h_7^4, \quad k_{11} = h_5h_7h_5h_7^2h_5h_7^5h_5h_7^3h_5h_7^6,$$
$$k_{12} = l_1l_2l_1l_2^6l_1l_2^4l_1l_2l_1l_2^6.$$

The words for $2^{4+12}.(S_3 \times 3S_6)$ are found to be k_{11} and $k_{13} = k_{12}k_{10}$.

3.11 Construction of $5^3 : (4 \times A_5).2$ inside Co_1

From [5], $5^3 : (4 \times A_5).2$ is the normalizer of 5^3 inside Co_1. We give some random elements of Co_1 below.

$$b_1 = (ab)^2ba, \quad b_3 = ab, \quad b_6 = (ab)^2b,$$
$$b_{10} = (ab)^2(ba)^3b, \quad b_{13} = bab(ba)^4, \quad b_{15} = babb(ab)^3ba,$$
$$5c = (b_{13}b_{15})^3, \quad e_1 = (ab)^{20}, \quad e_2 = (ab)^8,$$
$$e_3 = b_6^3, \quad c_1 = (ababa^2b^3)^4, \quad d_1 = b_{10}e_1(b_{10}^{-1}),$$
$$d_2 = 5cd_1, \quad k_2 = (d_1d_2^4)^3(d_1d_2^3)^2d_1d_2^7,$$

$k_4 = d_1d_2^5d_1d_2^7d_1d_2^5d_1d_2^{10}d_1d_2$, $k_{10} = (e_1e_2e_3^2a^2b^2e_1^2)^4$,
$d_6 = 5ck_{10}$, $d_8 = 5ck_{10}5ck_{10}$, $c_2 = (k_2)^5$,
$k_6 = c_2d_6^7c_2d_6^5c_2d_6^6c_2d_6^5c_2d_6^7$, $k_7 = c_2d_8^3c_2d_8^3c_2d_8^9c_2d_8^3c_2d_8^3$,
$k_8 = c_2d_8^3c_2d_8^3c_2d_8^9c_2d_8^6c_2d_8^9$, $k_9 = 5ck_6^35ck_6^35ck_6^35ck_6^35ck_6^3$,
$k_{12} = 5ck_85ck_8^25ck_8^25ck_8^25ck_8$, $c_3 = k_4^2$,
$f_2 = (k_{10}5ck_{10})[c_3, k_{10}5ck_{10}]^{16}$, $f_3 = [c_3, k_{10}5ck_{10}5c]^{12}$,
$d_{33} = f_2f_3$, $d_{40} = f_3f_2f_3f_3f_2f_2f_3$.

$k_{14} = d_{33}d_{40}^6d_{33}d_{40}^7d_{33}d_{40}^6d_{33}d_{40}^2d_{33}d_{40}^7$,
$g_0 = k_2^2$, $g_1 = k_2k_{12}$, $g_4 = k_9k_{12}$,
$g_5 = k_{12}k_{14}$, $k_{16} = (g_0g_1)^3g_0g_1^2$,
$k_{22} = (g_0g_4)^4g_0g_4^2$, $k_{23} = (g_0g_5)^4g_0g_5^2$.

The generators of 5^3 are $h_1 = k_{16}$,$h_2 = k_{22}$ and $h_3 = k_{23}$.

$h_4 = k_4^2$, $h_5 = [h_4, a]^7$, $h_6 = [h_4, b]^{15}$,
$h_9 = [h_4, ba]^{15}$, $e_7 = h_5h_9$,
$z_1 = h_6e_7^3h_6e_7^2h_6e_7h_6e_7^6h_6e_7^3$, $z_2 = h_6e_7^3h_6e_7^6h_6e_7^9h_6e_7^4h_6e_7^2$, $e_{13} = (z_1)^6$,
$e_{14} = [e_{13}, a]^7$, $e_{15} = [e_{13}, b]^{15}$, $e_{16} = [e_{13}, abb]^{15}$,
$e_{22} = e_{14}e_{15}e_{16}e_{15}$, $e_{24} = e_{14}e_{15}e_{16}e_{15}e_{16}e_{14}$,
$z_3 = e_{15}e_{22}^9e_{15}e_{22}^2e_{15}e_{22}^4e_{15}e_{22}^7e_{15}e_{22}^5$, $z_4 = e_{24}e_{22}^6e_{24}e_{22}^6e_{24}e_{22}^8e_{24}e_{22}e_{24}e_{22}^4$.

The generators of the normalizer of 5^3 are h_1, z_1, z_3 and z_4. The words for $5^3 : (4 \times A_5).2$ are $z_5 = h_1z_1$ and $z_6 = z_3z_4z_3$.

3.12 Construction of $7^2 : (3 \times 2.S_4)$ inside Co_1

Following [5], the required subgroup is the normalizer of 7^2. It is constructed by taking an element of class $7B$ and by searching inside its centralizer we find $7B^2$. Before computations we give some random elements of Co_1:

$b_1 = (ab)^2ba$, $b_2 = aba$, $b_3 = ab$, $b_4 = (ab)^2$,
$b_5 = (ab)^2a$, $b_6 = (ab)^2b$, $b_7 = (ab)^2bab$, $b_8 = (ab)^2(ba)^2$,
$b_9 = (ab)^2(ba)^2b$, $b_{10} = bab(ba)^3b^2a$.

The elements of $7B$ are given by $a_2 = b_{10}^6$. The generators of $7B^2$ are given below:

$f_1 = a_2e_1a_2e_1a_2e_1^3a_2e_1^7a_2e_1^7$, $f_2 = a_2e_1a_2e_1^2a_2e_1^8a_2e_1a_2e_1^6$.

Now we find the normalizer of $7B^2$.

$k_1 = a_2e_1a_2e_1^2a_2e_1^7a_2e_1^7a_2e_1^{10}$, $k_2 = a_2e_1a_2e_1^2a_2e_1^8a_2e_1a_2e_1^6$, $g_2 = (b_7b_8b_9^6b_4^4b_6^2)^8$,
$e_3 = a_2g_2$, $e_2 = a_2g_2a_2g_2^2a_2^2$,
$k_3 = (e_2e_3e_2e_3^2)^2e_3^9e_2e_3^8$.

The words for $7^2 : (3 \times 2.S_4)$ are k_3 and $k_4 = k_1k_2$.

3.13 Construction of $5^2 : 2A_5$ inside Co_1

The required subgroup is the normalizer of $5C^2[5]$ and it can be constructed by taking an element of order 5 and then search inside its centralizer. We can easily find $5C^2$.

$$
\begin{array}{lll}
b_1 = ababb, & b_2 = ababbababa, & b_3 = babbabababa, \\
b_4 = babbababababba, & 5c = (b_3 b_4)^3, & e_1 = (ab)^{20}, \\
e_2 = (ab)^8, & e_3 = b_6^3, & d_1 = b_2 e_1 (b_2^{-1}), \\
d_2 = 5cd_1, & k_1 = d_1 d_2 d_1 d_2^4 d_1 d_2^4 d_1 d_2^4 d_1 d_2^2, & \\
k_2 = d_1 d_2^4 d_1 d_2^4 d_1 d_2^7 d_1 d_2^3 d_1 d_2^7, & k_4 = d_1 d_2^5 d_1 d_2^7 d_1 d_2^5 d_1 d_2^{10} d_1 d_2, & \\
k_4 = (e_1 e_2 e_3^2 a^2 b^2 e_1^2)^4, & d_3 = 5ck_4, & d_4 = 5ck_4 5ck_4, \\
c_2 = k_2^5, & k_5 = c_2 d_6^7 c_2 d_6^5 c_2 d_6^6 c_2 d_6^5 c_2 d_6^7, & \\
k_6 = c_2 d_4^3 c_2 d_4^3 c_2 d_4^9 c_2 d_4^3 c_2 d_4^3, & k_7 = c_2 d_4^3 c_2 d_4^3 c_2 d_4^9 c_2 d_4^6 c_2 d_4^9, & \\
k_8 = 5ck_5^3 5ck_5^3 5ck_5^3 5ck_5^3 5ck_5^3, & d_5 = k_6 k_7, &
\end{array}
$$

$$
\begin{array}{lll}
k_9 = 5cd_5 5cd_5^5 5cd_5^5 5cd_5^2 5cd_5^6, & c_3 = k_3^2, & f_2 = k_4 5ck_4 [c_3, k_4 5ck_4]^{16}, \\
f_3 = [c_3, k_4 5ck_4 5c]^{12}, & d_6 = f_2 f_3, & d_7 = f_3 f_2 f_3^2 f_2^2 f_3, \\
k_{10} = d_6 d_7^6 d_6 d_7^7 d_6 d_7^6 d_6 d_7^2 d_6 d_7^7, & z_1 = k_2 k_{10}. &
\end{array}
$$

The generators of $5C^2$ are given by $5c$ and z_1.

$$
\begin{array}{lll}
g_1 = k_2 k_8 k_{10}, & k_{11} = (z_1 g_1)^5 g_1, & h_1 = k_2 k_9, \\
k_{12} = (k_9 h_1)^4 h_1^5 k_9 h_1^{19}, & j_1 = k_{12}^2, & \\
j_2 = a[j_1, a]^6, & j_3 = b[j_1, b]^7, & j_4 = [j_1, ab]^{15}, \\
l_1 = j_2 j_3 j_4, & k_{13} = j_2 l_1^5 j_2 l_1^2 j_2 l_1^{12} j_2 l_1^7 j_2 l_1^{16}. &
\end{array}
$$

The words for $5^2 : 2A_5$ are k_{11} and k_{13}.

3.14 Construction of $(A_7 \times L_2(7)) : 2$ inside Co_1

The whole strategy for locating $(A_7 \times L_2(7)) : 2$ is given in [3]. First we take an A_5 of the type $(2B, 3A, 5A)$ which is in the unique class of Co_1 with normalizer $N(A_5) = (A_5 \times J_2).2$ [5], then find A_4 inside A_5. Then we find an element of class $3A$ which commutes with A_4 but not with A_5. This $3A$ element with A_5 extends A_5 to our required A_7. Finally we found that $N(A_5) = (A_7 \times L_2(7)) : 2$. We give some random elements of Co_1 to facilitate computations.

$$
\begin{array}{lll}
b_1 = (ab)^2 ba, & b_2 = aba, & b_3 = ab, \\
b_4 = (ab)^2, & b_5 = (ab)^2 a, & b_6 = (ab)^2 b, \\
b_7 = (ab)^2 bab, & b_8 = (ab)^2 (ba)^2, & b_9 = (ab)^2 (ba)^2 b, \\
b_{10} = (ab)^2 (ba)^3, & b_{13} = bab(ba)^4, & \\
b_{15} = bab^2 (ab)^3 ba, & 2b = b_1, & 3a = b_{15}^{14},
\end{array}
$$

The generators for A_5 are:

$$c_1 = (2b)^{b_3^{-12}},\ c_2 = (3a)^{b_{13}^{-4}},\ c_3 = (2b)^{b_3^{-12}} (3a)^{b_{13}^{-4}}.$$

The generators of A_4 are given by $d_1 = c_1$, and $d_2 = (c_2 c_3)^2 c_3 c_2 c_3^2 c_1 c_2 c_3 c_2$.

$$e_1 = [d_1, ab]^3, \quad e_2 = [d_1, ba]^6, \quad e_3 = [d_1, baba]^6,$$
$$e_6 = e_2e_3, \quad e_7 = e_1e_2e_3, \quad f_1 = e_1e_6e_1e_6e_1e_6e_1e_6e_1e_6^2.$$

Here $f_5 = f_1^7$ commutes with A_4 but not with A_5, so the generators of A_7 are c_1, c_2, c_3 and f_5.

$$e_8 = e_1e_6, \quad e_9 = e_1e_7, \quad k_1 = e_1e_8e_1e_8^5e_1e_8^4e_1e_8^{11}e_1e_8^6,$$
$$k_4 = e_2e_8^5e_2e_8^2e_2e_8^9e_2e_8^2e_2e_8^7, \quad g_1 = (c_3f_5c_2c_1c_3)^2, \quad g_2 = [g_1, b]^{10},$$
$$g_3 = ab[g_1, ab]^5, \quad g_5 = g_2g_3, \quad k_6 = g_2g_5^5g_2g_5^{10}g_2g_5^5g_2g_5^3g_2g_5^{11}.$$

The words for $(A_7 \times L_2(7)) : 2$ are k_1 and $k_7 = k_4k_6$.

3.15 Construction of $(A_6 \times U_3(3)) : 2$ inside Co_1

The required group is the normalizer of A_6 which lies in the suzuki chain[3]. We easily find A_6 inside 3.14. The generators of A_6 are given by $g_1 = c_1$ and $g_2 = (f_5c_3)^2c_1c_2(c_3c_2)^2f_5c_3f_5c_1c_2c_3c_2$. Next we find the normalizer of A_6.

$$h_1 = (b_7b_8b_9^{11}b_4^6b_6^2)^7, \quad e_8 = g_2h_1, \quad k_2 = g_1e_8^2g_1e_8g_1e_8^3g_1e_8g_1e_8,$$
$$k_4 = g_1e_8^2g_1e_8^2g_1e_8^3g_1e_8^2g_1e_8, \quad l_6 = ababa[g_1, ababa]^5, \quad l_8 = l_4l_6,$$
$$k_7 = l_4l_8^3l_4l_8^6l_4l_8l_4l_8^2l_4l_8^6.$$

The words for $(A_6 \times U_3(3)) : 2$ are $k_8 = k_7k_4$ and $k_9 = k_2k_4k_7$.

3.16 Construction of $(A_5 \times J_2) : 2$ inside Co_1

The required group is the normalizer of A_5 [5] which we already constructed in 3.14. It can also be constructed by taking an involution of class $2B$, an element of class $3A$ and product of these two elements belongs to class $5A$ [5], then $N(2B, 3A, 5A) = (A_5 \times J_2).2$.

$$b_1 = (ab)^2a, \quad b_2 = bab^2(ab)^3, \quad b_3 = bab^2(ab)^3ba,$$
$$3a = b_3^{14}, \quad 2b = a.$$

The generators of the required group whose normalizer is to be computed are given below:

$$c_1 = (3a)^{b_1^{-18}}, \ c_2 = (2b)^{b_2^{-1}}, \ 3 = (3a)^{b_1^{-18}}(2b)^{b_2^{-1}}.$$

Before computing the normalizer we give some elements:

$$d_1 = a[c_2, a], \quad d_2 = [c_2, b]^{12}, \quad d_3 = aba[c_2, aba]^7,$$
$$d_4 = d_1d_3, \quad k_1 = d_2d_4d_2d_4d_2d_4^4d_2d_4^6d_2d_4^3, \quad e_1 = k_1^{12},$$
$$k_2 = (d_2d_4)^3d_4^6(d_2d_4^6)^2, \quad e_3 = [e_1, b]^2, \quad e_4 = [e_1, ab]^3,$$
$$e_5 = aba[e_1, aba]^2, \quad e_6 = e_4e_5, \quad k_3 = e_3e_6^2e_3e_6^3e_3e_6^4e_3e_6^7e_3e_6^6.$$

The words for $(A_5 \times J_2) : 2$ are k_2 and k_3.

The orders and orbit-shapes of above computed maximal subgroups are mentioned in table given below.

Group	*Order*	*Orbit – shape*
$(A_9 \times S_3)$	1088640	$360^1 3240^2 20160^1 25920^1 45360^1$
$(D_{10} \times (A_5 \times A_5).2).2$	144000	$60^1 600^1 720^1 1500^1 3000^3$
		$12000^1 14400^1 18000^2 24000^1$
$5^{1+2} GL_2(5)$	60000	$5^1 150^1 1250^1 1500^1 1875^1 2500^2$
		$3000^2 7500^1 15000^3 30000^1$
$3^{1+4} : 2.S_4(3).2$	25194240	$27^1 3240^1 9720^1 32805^1 52488^1$
$(2^{2+12})A_8 \times S_3$	1981808640	$360^1 5760^1 92160^1$
$3^6 : 2.M_{12}$	138568320	$594^1 17496^1 32076^1 48114^1$
$3^2.U_4(3).D_8$	235146240	$756^1 36288^1 61236^1$
$3^{3+4} : 2.(S_4 \times S_4)$	2519424	$108^1 1944^1 8748^1 34992^1 52488^1$
$2^{4+12}.(S_3 \times 3S_6)$	849346560	$72^1 1440^1 23040^1 73728^1$
$5^3 : (4 \times A_5).2$	60000	$30^1 150^1 600^1 750^6 1500^4 3000^1 36000^8$
$7^2 : (3 \times 2.S_4)$	3528	$84^1 588^3 882^2 1176^7 1764^5 3528^{22}$
$5^2 : 4A_5$	3000	$30^1 150^1 300^1 600^3 750^4 1500^2 3000^3 0$
$(A_7 \times L_2(7)) : 2$	846720	$2520^1 35280^1 60480^1$
$(A_6 \times U_3(3)) : 2$	4353560	$7560^1 90720^1$
$(A_4 \times G_2(4)) : 2$	6038323200	98280^1
$(A_5 \times J_2) : 2$	72576000	$37800^1 60480^1$

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Competing Interest

The authors do not have any competing interests.

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