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On split Lie color triple systems

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Abstract: In order to begin an approach to the structure of arbitrary Lie color triple systems, (with no restrictions neither on the dimension nor on the base field), we introduce the class of split Lie color triple systems as the natural generalization of split Lie triple systems. By developing techniques of connections of roots for this kind of triple systems, we show that any of such Lie color triple systems T with a symmetric root system is of the form $T = U + \sum_{[\alpha]\in \Lambda^1/\sim} I_{[\alpha]}$ with U a subspace of T_0 and any $I_{[\alpha]}$ a well described (graded) ideal of T, satisfying $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. Under certain conditions, in the case of T being of maximal length, the simplicity of the triple system is characterized.

Keywords: split, Lie color triple system, root system, root space

MSC: 17A32, 17A60, 17B22, 17B65

1 Introduction

The concept of Lie triple systems was introduced by Nathan Jacobson in 1949 to study subspaces of associative algebras closed under triple commutators $[[u, v], w]$. The role played by Lie triple systems in the theory of symmetric spaces is parallel to that of Lie algebras in the theory of Lie groups: the tangent space at every point of a symmetric space has the structure of a Lie triple system. Because of close relation to Lie algebras and theoretical physics, Lie triple systems have recently been widely studied [1–4]. The notion of Lie color algebras was introduced as generalized Lie algebras in 1960 by Ree [5]. So far, many results of this kind of algebras have been considered in the frameworks of enveloping algebras, representations and related problems [6–8]. Lie color triple systems were introduced as generalized Lie triple systems by Zhang in his doctoral thesis in 2007. Furthermore, Lie color triple systems are related to Lie color algebras in the same way that Lie triple systems related to Lie algebras. So it is natural to prove analogs of results from the theory of Lie triple systems to Lie color triple systems.

In the framework of infinite dimensional Lie algebras, Neeb, Stumme and other authors have successfully developed over the recent years a theory of split and locally finite Lie algebras [9, 10]. Calderón introduced the concept of split Lie triple systems of arbitrary dimension [11–13]. In [14], Calderón introduced techniques of connections of roots in the field of split Lie color algebras. Recently, the structure of different classes of split algebras have been studied by using techniques of connections of roots (see for instance [15–18]). Our work is based on [13, 14] and our aim is to consider the structure of split Lie color triple systems by the techniques of connections of roots.

Throughout this paper, Lie color triple systems T are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper proceeds as follows. In section 2, we establish the preliminaries on split Lie color triple systems theory. In section 3, we show that such an arbitrary split Lie color triple system with

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a symmetric root system is of the form $T = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ with U a subspace of T_0 and any $I_{[\alpha]}$ a well described (graded) ideal of T, satisfying $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. In section 4, we show that under certain conditions, in the case of T being of maximal length, the simplicity of the triple system is characterized.

2 Preliminaries

First we recall the definitions of Lie color algebras and Lie triple systems.

Definition 2.1. [14] *Let Γ be an abelian group. A bi-character on Γ is a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K} \setminus \{0\}$ satisfying*
(1) $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$,
(2) $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$,
(3) $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$
for all $\alpha, \beta, \gamma \in \Gamma$.

It is clear that $\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1$ for any $\alpha \in \Gamma$, where 0 denotes the identity element of Γ.

Definition 2.2. [14] *Let $L = \oplus_{g\in\Gamma} L_g$ be a Γ-graded $\mathbb{K}$-vector space. For a nonzero homogeneous element $v \in L$, denote by $\bar{v}$ the unique group element in Γ such that $v \in L_{\bar{v}}$, which will be called the homogeneous degree of v. We shall say that L is a **Lie color algebra** if it is endowed with a $\mathbb{K}$-bilinear map $[\cdot, \cdot] : L \times L \to L$ satisfying*
(1) $[v, w] = -\varepsilon(\bar{v}, \bar{w})[w, v]$, (Skew- symmetry)
(2) $[v, [w, t]] = [[v, w], t] + \varepsilon(\bar{v}, \bar{w})[w, [v, t]]$ (Jacobi identity)
for all homogeneous elements $v, w, t \in L$.

Lie superalgebras are examples of Lie color algebras with $\Gamma = \mathbb{Z}_2$ and $\varepsilon(i, j) = (-1)^{ij}$, for any $i, j \in \mathbb{Z}_2$. We also note that L_0 is a Lie algebra.

Definition 2.3. [13] *A **Lie triple system** is a vector space T endowed with a trilinear map $\{\cdot, \cdot, \cdot\} : T \times T \times T \to T$ satisfying*
(1) $\{x, y, z\} = -\{y, x, z\}$,
(2) $\{x, y, z\} + \{y, z, x\} + \{z, x, y\} = 0$,
(3) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} + \{a, \{x, y, b\}, c\} + \{a, b, \{x, y, c\}\}$
for $x, y, z, a, b, c \in T$.

Definition 2.4. *Let $T = \oplus_{g\in\Gamma} T_g$ be a Γ-graded $\mathbb{K}$-vector space. For a nonzero homogeneous element $v \in T$, denote by $\bar{v}$ the unique group element in Γ such that $v \in T_{\bar{v}}$, which will be called the homogeneous degree of v. We shall say that T is a **Lie color triple system** if it is endowed with a $\mathbb{K}$-trilinear map*
(1) $\{x, y, z\} = -\varepsilon(\bar{x}, \bar{y})\{y, x, z\}$,
(2) $\varepsilon(\bar{z}, \bar{x})\{x, y, z\} + \varepsilon(\bar{x}, \bar{y})\{y, z, x\} + \varepsilon(\bar{y}, \bar{z})\{z, x, y\} = 0$,,
(3) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} + \varepsilon(\bar{x} + \bar{y}, \bar{a})\{a, \{x, y, b\}, c\}$
$+\varepsilon(\bar{x} + \bar{y}, \bar{a} + \bar{b})\{a, b, \{x, y, c\}\}$
for all homogeneous elements $x, y, z, a, b, c \in T$.

Example 2.5. *Lie triple systems are examples of Lie color triple systems with $\Gamma = \{0\}$ and $\varepsilon(0, 0) = 1$.*

Example 2.6. *Lie triple supersystems are examples of Lie color triple systems with $\Gamma = \{\mathbb{Z}_2\} = \{\bar{0}, \bar{1}\}$ and $\varepsilon(\bar{\alpha}, \bar{\beta}) = (-1)^{\bar{\alpha}\bar{\beta}}$, for any $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$.*

Example 2.7. *If L is a Lie color algebra with product $[\cdot, \cdot]$, then L becomes a Lie color triple system by putting $\{x, y, z\} = [[x, y], z]$.*

Definition 2.8. *Let $I = \oplus_{g\in\Gamma} I_g$ be a graded subspace of a Lie color triple system T. Then I is called a* ***subsystem*** *of T, if $\{I, I, I\} \subseteq I$; I is called an* ***ideal*** *of T, if $\{I, T, T\} \subseteq I$.*

Definition 2.9. *The* ***standard embedding*** *of a Lie color triple system T is the $\mathbb{Z}_2$-graded Lie color algebra $L = L^0 \oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{L(x, y) : x, y \in T\}$, where $L(x, y)$ denotes the left multiplication operator in T, $L(x, y)(z) := \{x, y, z\}$; $L^1 := T$ and where the product is given by*

$$[(L(x, y), z), (L(u, v), w)] :$$
$$= (L(\{x, y, u\}, v) + \varepsilon(\bar{x} + \bar{y}, \bar{u})L(u, \{x, y, v\}) + L(z, w), \{x, y, w\} - \varepsilon(\bar{z}, \bar{u} + \bar{v})\{u, v, z\}).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a Lie color algebra.

Let us recall the concept of split Lie color algebra. Denote by $H = \oplus_{g\in\Gamma} H_g$ a maximal abelian (graded) subalgebra, (MAGSA), of a Lie color algebra L. For a linear functional

$$\alpha : H_0 \to \mathbb{K},$$

we define the root space of L (with respect to H) associated to α as the subspace

$$L_\alpha = \{v_\alpha \in L : [h_0, v_\alpha] = \alpha(h_0)v_\alpha \text{ for any } h_0 \in H_0\}.$$

The elements $\alpha : H_0 \to \mathbb{K}$ satisfying $L_\alpha \neq 0$ are called roots of L with respect to H. We denote $\Lambda := \{\alpha \in H_0^* \setminus \{0\} : L_\alpha \neq 0\}$. We say that L is a split Lie color algebra, with respect to H, if $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$. We also say that Λ is the root system of L.

Definition 2.10. *Let T be a Lie color triple system, $L = L^0 \oplus L^1$ be its standard embedding, and $H^0 = \oplus_{g\in\Gamma} H_g^0$ be a maximal abelian (graded) subalgebra* (MAGSA) of L^0. For a linear functional $\alpha \in (H_0^0)^*$, we define the root space of T (with respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [h, t_\alpha] = \alpha(h)t_\alpha$ for any $h \in H_0^0\}$. The elements $\alpha \in (H_0^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T with respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H_0^0)^* \setminus \{0\} : T_\alpha \neq 0\}$.

Let us observe that $T_0 := \{t_0 \in T : [h, t_0] = 0$ for any $h \in H_0^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H_0^0)^*$ such that $L_\alpha^0 := \{v_\alpha^0 \in L^0 : [h, v_\alpha^0] = \alpha(h)v_\alpha^0$ for any $h \in H_0^0\} \neq 0$.

Lemma 2.11. *Let T be a Lie color triple system, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a* MAGSA *of L^0. For $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\xi, q \in \Lambda^0 \cup \{0\}$, the following assertions hold.*

(1) *If $[T_\alpha, T_\beta] \neq 0$, then $\alpha + \beta \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L_{\alpha+\beta}^0$.*
(2) *If $[L_\xi^0, T_\alpha] \neq 0$, then $\xi + \alpha \in \Lambda^1 \cup \{0\}$ and $[L_\xi^0, T_\alpha] \subseteq T_{\xi+\alpha}$.*
(3) *If $[L_\xi^0, L_q^0] \neq 0$, then $\xi + q \in \Lambda^0 \cup \{0\}$ and $[L_\xi^0, L_q^0] \subseteq L_{\xi+q}^0$.*
(4) *If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, then $\alpha + \beta + \gamma \in \Lambda^1 \cup \{0\}$ and $\{T_\alpha, T_\beta, T_\gamma\} \subseteq T_{\alpha+\beta+\gamma}$.*

Proof. (1) For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H_0^0$, by Definition 2.2 (2), one has $[h, [x, y]] = \varepsilon(0, \bar{x})[x, [h, y]] + [[h, x], y] = [x, \beta(h)y] + [\alpha(h)x, y] = (\alpha + \beta)(h)[x, y]$.

(2) For any $x \in L_\xi^0$, $y \in T_\alpha$ and $h \in H_0^0$, by Definition 2.2 (2), one has $[h, [x, y]] = [x, [h, y]] + [[h, x], y] = \varepsilon(0, \bar{x})[x, \alpha(h)y] + [\xi(h)x, y] = (\xi + \alpha)(h)[x, y]$.

(3) For any $x \in L_\xi^0$, $y \in L_q^0$ and $h \in H_0^0$, by Definition 2.2 (2), one has $[h, [x, y]] = \varepsilon(0, \bar{x})[x, [h, y]] + [[h, x], y] = [x, q(h)y] + [\xi(h)x, y] = (\xi + q)(h)[x, y]$.

(4) It is a consequence of Lemma 2.11 (1) and (2). □

Definition 2.12. *Let T be a Lie color triple system, $L = L^0 \oplus L^1$ be its standard embedding, and $H^0 = \oplus_{g\in\Gamma} H_g^0$ be a* MAGSA *of L^0. We shall call that T is a* ***split Lie color triple system*** *(with respect to H^0) if $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$. We say that Λ^1 is the root system of T.*

We observe that it is straightforward to prove that if T is a split Lie color triple system with respect to H^0 and we let $L = L^0 \oplus L^1$ be its standard embedding algebra, then L^0 is a split Lie color algebra with respect to the splitting Cartan subalgebra H^0, with set of nonzero roots Λ^0. We also note that the facts $H^0 \subset L^0 = [T, T]$ and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ imply

$$H^0 = [T_0, T_0] + \sum_{\alpha\in\Lambda^1} [T_\alpha, T_{-\alpha}]. \tag{2.1}$$

Lemma 2.13. *Let $T = \oplus_{g\in\Gamma} T_g$ be a split Lie color triple system with corresponding root space decomposition $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$. If we denote by $T_{\alpha,g} = T_\alpha \cap T_g$, the following assertions hold.*

(1) $T_\alpha = \oplus_{g\in\Gamma} T_{\alpha,g}$ *for any* $\alpha \in \Lambda^1 \cup \{0\}$.

(2) $H^0_g = \sum_{g_1+g_2=g, g_1,g_2\in\Gamma} [T_{0,g_1}, T_{0,g_2}] + \sum_{\substack{\alpha\in\Lambda^1 \\ g_1+g_2=g, g_1,g_2\in\Gamma}} [T_{\alpha,g_1}, T_{-\alpha,g_2}]$.

(3) T_0 *is a split Lie triple system, with respect to* H^0_0*, with root space decomposition* $T_0 = T_{0,0} \oplus (\oplus_{\alpha\in\Lambda^1} T_{\alpha,0})$.

Proof. (1) By the Γ-grading of T we may express any $v_\alpha \in T_\alpha$, $\alpha \in \Lambda^1 \cup \{0\}$, in the form $v_\alpha = v_{\alpha,g_1} + \cdots + v_{\alpha,g_n}$ with $v_{\alpha,g_i} \in T_{g_i}$ for distinct $g_1, \cdots, g_n \in \Gamma$. If $h_0 \in H^0_0$ then $[h_0, v_{\alpha,g_i}] = \alpha(h_0) v_{\alpha,g_i}$ for $i = 1, \cdots, n$. Hence $T_\alpha = \oplus_{g\in\Gamma}(T_\alpha \cap T_g)$ and we can write $T_\alpha = \oplus_{g\in\Gamma} T_{\alpha,g}$ for any $\alpha \in \Lambda^1 \cup \{0\}$.

(2) Consequence of Eq. (2.1) and Lemma 2.13 (1).

(3) By considering $g = 0$ we get $T_0 = T_{0,0} \oplus (\oplus_{\alpha\in\Lambda^1} T_{\alpha,0})$. Hence, the direct character of the sum and the fact that $\alpha \neq 0$ for any $\alpha \in \Lambda^1$ give us that H^0_0 is a maximal abelian subalgebra of the Lie algebra L^0_0. Hence T_0 is a split Lie triple system respect to H^0_0. □

Definition 2.14. *A root system Λ^1 of a split Lie color triple system T is called* ***symmetric*** *if it satisfies that $\alpha \in \Lambda^1$ implies $-\alpha \in \Lambda^1$.*

A similar concept applies to the set Λ^0 of nonzero roots of L^0.

In the following, T denotes a split Lie color triple system with a symmetric root system Λ^1, and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ the corresponding root decomposition. We begin the study of split Lie color triple system by developing the concept of connections of roots.

Definition 2.15. *Let α and β be two nonzero roots, we shall say that α and β are* ***connected*** *if there exists a family $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\} \subset \Lambda^1 \cup \{0\}$ of roots of T such that*

(1) $\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \cdots, \alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1}\} \subset \Lambda^1$;

(2) $\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \cdots, \alpha_1 + \cdots + \alpha_{2n}\} \subset \Lambda^0$;

(3) $\alpha_1 = \alpha$ *and* $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} \in \pm\beta$.

We shall also say that $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β.

We denote by

$$\Lambda^1_\alpha := \{\beta \in \Lambda^1 : \alpha \text{ and } \beta \text{ are connected}\},$$

we can easily get that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore $\pm\alpha \in \Lambda^1_\alpha$.

Definition 2.16. *A subset Ω^1 of a root system Λ^1, associated to a split Lie color triple system T, is called a* ***root subsystem*** *if it is symmetric, and for $\alpha, \beta, \gamma \in \Omega^1 \cup \{0\}$ such that $\alpha + \beta \in \Lambda^0$ and $\alpha + \beta + \gamma \in \Lambda^1$ then $\alpha + \beta + \gamma \in \Omega^1$.*

Let Ω^1 be a root subsystem of Λ^1. We define

$$T_{0,\Omega^1} := \text{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \gamma = 0;\ \alpha, \beta, \gamma \in \Omega^1 \cup \{0\}\} \subset T_0$$

and $V_{\Omega^1} := \oplus_{\alpha\in\Omega^1} T_\alpha$. Taking into account the fact that $\{T_0, T_0, T_0\} = 0$, it is straightforward to verify that $T_{\Omega^1} := T_{0,\Omega^1} \oplus V_{\Omega^1}$ is a subsystem of T. We will say that T_{Ω^1} is a subsystem associated to the root subsystem Ω^1.

Proposition 2.17. *If Λ^0 is symmetric, then the relation $\sim$ in Λ^1, defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda^1_\alpha$, is of equivalence.*

Proof. $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$.

If $\alpha \sim \beta$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β, then

$$\{\alpha_1 + \cdots + \alpha_{2n+1}, -\alpha_{2n+1}, -\alpha_{2n}, \cdots, -\alpha_2\}$$

is a connection from β to α in the case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = \beta$, and

$$\{-\alpha_1 - \cdots - \alpha_{2n+1}, \alpha_{2n+1}, \alpha_{2n}, \cdots, \alpha_2\}$$

in the case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = -\beta$. Therefore $\beta \sim \alpha$.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β and $\{\beta_1, \cdots, \beta_{2m+1}\}$ is a connection from β to γ. If $m \neq 0$, then

$$\{\alpha_1, \cdots, \alpha_{2n+1}, \beta_2, \cdots, \beta_{2m+1}\}$$

is a connection from α to γ in the case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = \beta$, and

$$\{\alpha_1, \cdots, \alpha_{2n+1}, -\beta_2, \cdots, -\beta_{2m+1}\}$$

in the case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = -\beta$. If $m = 0$, then $\gamma \in \pm\beta$ and so

$$\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$$

is a connection from α to γ. Therefore $\alpha \sim \gamma$ and $\sim$ is of equivalence. □

Proposition 2.18. *Let α be a nonzero root and suppose Λ^0 is symmetric. Then Λ^1_α is a root subsystem.*

Proof. If $\beta \in \Lambda^1_\alpha$ then there exists a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β. It is clear that $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ also connects α to $-\beta$ and therefore $-\beta \in \Lambda^1_\alpha$. Let $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\alpha \cup \{0\}$ be such that $\beta_1 + \beta_2 \in \Lambda^0$ and $\beta_1 + \beta_2 + \beta_3 \in \Lambda^1$. If $\beta_1 = 0$, as $\beta_1 + \beta_2 \in \Lambda^0$ then $\beta_2 \neq 0$ and there exists a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β_2. We have $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, 0, \beta_3\}$ is a connection from α to $\beta_2 + \beta_3$ in case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = \beta_2$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, 0, -\beta_3\}$ in case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = -\beta_2$. So $\beta_1 + \beta_2 + \beta_3 = \beta_2 + \beta_3 \in \Lambda^1_\alpha$. Suppose $\beta_1 \neq 0$, then there exists a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β_1. Hence, $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, \beta_2, \beta_3\}$ is a connection from α to $\beta_1 + \beta_2 + \beta_3$ in case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = \beta_1$ and $\{\alpha_1, \alpha_2, \cdots, \alpha_{2n+1}, -\beta_2, -\beta_3\}$ in case $\alpha_1 + \cdots + \alpha_{2n} + \alpha_{2n+1} = -\beta_1$. Therefore $\beta_1 + \beta_2 + \beta_3 \in \Lambda^1_\alpha$. □

3 Decompositions

In this section, we will state a series of previous results in order to show that for a fixed $\alpha_0 \in \Lambda^1$, the subsystem $T_{\Lambda^1_{\alpha_0}}$ associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

Lemma 3.1. *The following assertions hold.*

(1) *If $\alpha, \beta \in \Lambda^1$ with $[T_\alpha, T_\beta] \neq 0$, then α is connected with β.*

(2) *If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[L^0_\alpha, T_\beta] \neq 0$, then α is connected with β.*

(3) *If $\alpha, \beta \in \Lambda^1$, $\alpha, \beta \in \Lambda^0$ and $[L^0_\alpha, L^0_\beta] \neq 0$, then α is connected with β.*

(4) *If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[T_\alpha, T_{\overline{\beta}}] = 0$, $[L^0_\alpha, T_{\overline{\beta}}] = 0$ and $[T_{\overline{\beta}}, L^0_\alpha] = 0$ if furthermore $\alpha \in \Lambda^0$. If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[L^0_\alpha, L^0_{\overline{\beta}}] = 0$ if furthermore $\alpha, \overline{\beta} \in \Lambda^0$.*

Proof. (1) Suppose $[T_\alpha, T_\beta] \neq 0$, by Lemma 2.11 (1), one gets $\alpha + \beta \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, one gets $\{\alpha, \beta, -\alpha\}$ is a connection from α to β.

(2) Suppose $[L^0_\alpha, T_\beta] \neq 0$, by Lemma 2.11 (2), one gets $\alpha + \beta \in \Lambda^1 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^1$, we obtain $\{\alpha, 0, -\alpha - \beta\}$ is a connection from α to β.

(3) Suppose $[L^0_\alpha, L^0_\beta] \neq 0$, by Lemma 2.11 (3), one has $\alpha + \beta \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, one gets $\{\alpha, \beta, -\alpha\}$ is a connection from α to β.

(4) It is a consequence of Lemma 3.1 (1), (2) and (3). □

Lemma 3.2. *If $\alpha, \overline{\beta} \in \Lambda^1$ are not connected, then $\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} = 0$.*

Proof. If $[T_\alpha, T_{-\alpha}] = 0$, it is clear. One may suppose that $[T_\alpha, T_{-\alpha}] \neq 0$ and $\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} \neq 0$. By the definition of Lie color triple systems, one has $\{T_{-\alpha}, T_{\overline{\beta}}, T_\alpha\} \neq 0$ or $\{T_{\overline{\beta}}, T_\alpha, T_{-\alpha}\} \neq 0$, contradicting Lemma 3.1 (4). Hence, $\{T_\alpha, T_{-\alpha}, T_{\overline{\beta}}\} = 0$. □

Lemma 3.3. *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. For $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta, \gamma \in \Lambda^1 \cup \{0\}$, then the following assertions hold.*

(1) *If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, then $\beta, \gamma, \alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

(2) *If $\{T_\gamma, T_\alpha, T_\beta\} \neq 0$, then $\gamma, \beta, \gamma + \alpha + \beta \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

(3) *If $\{T_\beta, T_\gamma, T_\alpha\} \neq 0$, then $\beta, \gamma, \beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

Proof. (1) It is easy to see that $[T_\alpha, T_\beta] \neq 0$, for $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \in \Lambda^1 \cup \{0\}$. By Lemma 3.1 (1), one gets $\alpha \sim \beta$ in the case $\beta \neq 0$. From here, $\beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. In order to complete the proof, we will show $\gamma, \alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. We distinguish two cases.

Case 1. Suppose $\alpha + \beta + \gamma = 0$. It is clear that $\alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. The fact that $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$ gives us $\gamma \neq 0$. By Lemma 2.11 (1), one gets $\alpha + \beta \in \Lambda^0$. As $\alpha + \beta = -\gamma$, $\{\alpha, \beta, 0\}$ would be a connection from α to γ and we conclude $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\alpha + \beta + \gamma \neq 0$. We treat separately two cases.

Suppose $\alpha + \beta \neq 0$. By Lemma 2.11 (1), one gets $\alpha + \beta \in \Lambda^0$ and so $\{\alpha, \beta, \gamma\}$ is a connection from α to $\alpha + \beta + \gamma$. Hence $\alpha + \beta + \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. In the case $\gamma \neq 0$, $\{\alpha, \beta, -\alpha - \beta - \gamma\}$ is a connection from α to γ. So $\gamma \in \Lambda^1_{\alpha_0}$. Hence $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Suppose $\alpha + \beta = 0$. Then necessarily $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. Indeed, if $\gamma \neq 0$ and α is not connected with γ, by Lemma 3.2, $\{T_\alpha, T_\beta, T_\gamma\} = \{T_\alpha, T_{-\alpha}, T_\gamma\} = 0$, a contradiction. We also have $\alpha + \beta + \gamma = \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(2) The fact that $[T_\gamma, T_\alpha] \neq 0$ implies by Lemma 3.1 (1) that $\alpha \sim \gamma$ in the case $\gamma \neq 0$. From here, $\gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. In order to complete the proof, we will show $\beta, \gamma + \alpha + \beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. We distinguish two cases.

Case 1. Suppose $\gamma + \alpha + \beta = 0$. It is clear that $\gamma + \alpha + \beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. The fact that $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$ gives us $\beta \neq 0$. By Lemma 2.11 (1), one has $\gamma + \alpha \in \Lambda^0$. As $\gamma + \alpha = -\beta$, $\{\alpha, \gamma, 0\}$ would be a connection from α to β and we conclude $\beta \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\gamma + \alpha + \beta \neq 0$. We treat separately two cases.

Suppose $\gamma + \alpha \neq 0$. By Lemma 2.11 (1), one gets $\gamma + \alpha \in \Lambda^0$ and so $\{\alpha, \gamma, \beta\}$ is a connection from α to $\gamma + \alpha + \beta$. Hence $\gamma + \alpha + \beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. In the case $\beta \neq 0$, we have $\{\alpha, \gamma, -\alpha - \gamma - \beta\}$ is a connection from α to β. So $\beta \in \Lambda^1_{\alpha_0}$. Hence $\beta \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Suppose $\gamma + \alpha = 0$. Then necessarily $\beta \in \Lambda^1_{\alpha_0} \cup \{0\}$. Indeed, if $\beta \neq 0$ and α is not connected with β, by Lemma 3.2, $\{T_\gamma, T_\alpha, T_\beta\} = \{T_{-\alpha}, T_\alpha, T_\beta\} = 0$, a contradiction. We also have $\gamma + \alpha + \beta = \beta \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(3) By the definition of Lie color triple systems, one has

$$\varepsilon(\bar{\alpha}, \bar{\beta})\{T_\beta, T_\gamma, T_\alpha\} \subset \varepsilon(\bar{\gamma}, \bar{\alpha})\{T_\alpha, T_\beta, T_\gamma\} + \varepsilon(\bar{\beta}, \bar{\gamma})\{T_\gamma, T_\alpha, T_\beta\}.$$

So either $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$ or $\{T_\gamma, T_\alpha, T_\beta\} \neq 0$. By Lemma 3.3 (1) and (2), one gets $\beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$. Next we will show that $\beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$. We treat separately three cases.

Case 1. Suppose $\beta \neq 0$, then $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3 (1), one has $\beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\beta = 0$ and $\gamma \neq 0$, then $\gamma \in \Lambda^1_{\alpha_0}$. By Lemma 3.3 (2), one has $\beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 3. Suppose $\beta = 0$ and $\gamma = 0$, then $\beta + \gamma + \alpha = \alpha \in \Lambda^1_{\alpha_0}$, we also have $\beta + \gamma + \alpha \in \Lambda^1_{\alpha_0} \cup \{0\}$. □

Lemma 3.4. *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. For $\alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}$ with $\alpha + \beta + \gamma = 0$ and $\tau, \epsilon \in \Lambda^1 \cup \{0\}$, the following assertions hold.*

(1) *If $\{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} \neq 0$, then $\tau, \epsilon, \tau + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

(2) *If $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \neq 0$, then $\tau, \epsilon, \epsilon + \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

(3) *If $\{T_\tau, T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}\} \neq 0$, then $\tau, \epsilon, \tau + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.*

Proof. (1) From the fact that $\alpha+\beta+\gamma = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ whenever $\alpha \in \Lambda^1$, one may suppose that at least two distinct elements in $\{\alpha, \beta, \gamma\}$ are nonzero and one may consider the case $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, $\alpha+\beta \neq 0$ and $\gamma \neq 0$. Since

$$\begin{aligned}0 &\neq \{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\}\\ &\subset \{T_\alpha, T_\beta, \{T_\gamma, T_\tau, T_\epsilon\}\} - \varepsilon(\bar\alpha+\bar\beta, \bar\gamma)\{T_\gamma, \{T_\alpha, T_\beta, T_\tau\}, T_\epsilon\}\\ &\quad - \varepsilon(\bar\alpha+\bar\beta, \bar\gamma+\bar\tau)\{T_\gamma, T_\tau, \{T_\alpha, T_\beta, T_\epsilon\}\},\end{aligned}$$

any of the above three summands is nonzero. In order to complete the proof, we first will show $\tau, \epsilon \in \Lambda^1_{\alpha_0}\cup\{0\}$. We distinguish three cases.

Case 1. Suppose $\{T_\alpha, T_\beta, \{T_\gamma, T_\tau, T_\epsilon\}\} \neq 0$. As $\gamma \neq 0$ and $\{T_\gamma, T_\tau, T_\epsilon\} \neq 0$, Lemma 3.3 (1) shows that τ, ϵ are connected with γ in the case of being nonzero roots and so $\tau, \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\{T_\gamma, \{T_\alpha, T_\beta, T_\tau\}, T_\epsilon\} \neq 0$. As $\alpha+\beta \neq 0$ and $\gamma \neq 0$. So either $\alpha \neq 0$ or $\beta \neq 0$. By Lemma 3.3 (1) and (2), one has $\tau, \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 3. Suppose $\{T_\gamma, T_\tau, \{T_\alpha, T_\beta, T_\epsilon\}\} \neq 0$. As $\alpha+\beta \neq 0$ and $\gamma \neq 0$. So either $\alpha \neq 0$ or $\beta \neq 0$. By Lemma 3.3 (1) and (2), one has $\tau, \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Finally, we will show $\tau+\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. From the fact that $\alpha+\beta+\gamma = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} \neq 0$, let us suppose that at least one element in $\{\tau, \epsilon\}$ is nonzero. So either $\tau \in \Lambda^1_{\alpha_0}$ or $\epsilon \in \Lambda^1_{\alpha_0}$. Then $\{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} \subset \{T_0, T_\tau, T_\epsilon\}$. By Lemma 3.3 (2) and (3), one has $\tau+\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(2) From the fact that $\alpha+\beta+\gamma = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ whenever $\alpha \in \Lambda^1$, one may suppose that at least two distinct elements in $\{\alpha, \beta, \gamma\}$ are nonzero and one may consider the case $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, $\alpha+\beta \neq 0$ and $\gamma \neq 0$. Since

$$\begin{aligned}0 &\neq \varepsilon(\bar\alpha+\bar\beta, \bar\epsilon)\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\}\\ &\subset \{T_\alpha, T_\beta, \{T_\epsilon, T_\gamma, T_\tau\}\} - \varepsilon(\bar\alpha+\bar\beta, \bar\epsilon+\bar\gamma)\{T_\epsilon, T_\gamma, \{T_\alpha, T_\beta, T_\tau\}\}\\ &\quad - \{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\tau\},\end{aligned}$$

any of the above three summands is nonzero. In order to complete the proof, we firstly will show $\tau, \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. We distinguish three cases.

Case 1. Suppose $\{T_\alpha, T_\beta, \{T_\epsilon, T_\gamma, T_\tau\}\} \neq 0$. As $\gamma \neq 0$ and $\{T_\epsilon, T_\gamma, T_\tau\} \neq 0$, Lemma 3.3 (2) shows that ϵ, τ are connected with γ in the case of being nonzero roots and so $\epsilon, \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\{T_\epsilon, T_\gamma, \{T_\alpha, T_\beta, T_\tau\}\} \neq 0$. As $\alpha+\beta \neq 0$ and $\gamma \neq 0$. So either $\alpha \neq 0$ or $\beta \neq 0$. By Lemma 3.3 (1) and (2), one has $\epsilon, \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 3. Suppose $\{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\tau\} \neq 0$. As $\alpha+\beta \neq 0$ and $\gamma \neq 0$. So either $\alpha \neq 0$ or $\beta \neq 0$. By Lemma 3.3 (1) and (2), one has $\epsilon, \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Finally, we will show $\epsilon+\tau \in \Lambda^1_{\alpha_0} \cup \{0\}$. From the fact that $\alpha+\beta+\gamma = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \neq 0$, let us suppose that at least one element in $\{\epsilon, \tau\}$ is nonzero. So either $\epsilon \in \Lambda^1_{\alpha_0}$ or $\tau \in \Lambda^1_{\alpha_0}$. Then $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \subset \{T_\epsilon, T_0, T_\tau\}$. By Lemma 3.3 (1) and (3), one has $\epsilon+\tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(3) By the definition of Lie color triple systems, one has

$$0 \neq \{T_\tau, T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}\} \subset \{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} - \varepsilon(\bar\tau, \bar\epsilon)\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\}.$$

Suppose $\{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} \neq 0$, by Lemma 3.4 (1), one has $\tau, \epsilon, \tau+\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. Suppose $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \neq 0$, by Lemma 3.4 (2), one has $\tau, \epsilon, \epsilon+\tau \in \Lambda^1_{\alpha_0} \cup \{0\}$. Therefore, in these two cases, we get $\tau, \epsilon, \tau+\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. □

Lemma 3.5. *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_{\alpha_0} \cup \{0\}$ with $\alpha_1+\alpha_2+\alpha_3 = 0$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, then the following assertions hold.*

(1) $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] = 0$.

(2) *In case $\overline{\epsilon} \in \Lambda^0$, then* $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\overline{\epsilon}}] = 0$.

(3) $[[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_0], T_{\overline{\epsilon}}] = 0$.

Proof. (1) From the fact $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$, it is clear that if $\alpha_3 = 0$ one gets $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\bar{\epsilon}}] = 0$. Let us consider the case $\alpha_3 \neq 0$. By the definition of Lie color triple systems, we have

$$[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\bar{\epsilon}}] \subset [[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, T_{\bar{\epsilon}}]] - \varepsilon(\bar{\alpha_1} + \bar{\alpha_2}, \bar{\alpha_3})[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}]]. \tag{3.2}$$

Let us consider the first summand in Eq. (3.2). As $\alpha_3 \neq 0$, one has $\alpha_3 \in \Lambda^1_{\alpha_0}$. For $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$ and Lemma 3.1 (4), one easily gets $[T_{\alpha_3}, T_{\bar{\epsilon}}] = 0$. Therefore $[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, T_{\bar{\epsilon}}]] = 0$.

Let us now consider the second summand in Eq. (3.2), it is sufficient to verify that

$$[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}]] = 0.$$

To do so, we first assert that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$. Indeed, by the definition of Lie color algebras, we have

$$[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] \subset [T_{\alpha_1}, [T_{\alpha_2}, T_{\bar{\epsilon}}]] - \varepsilon(\bar{\alpha_1}, \bar{\alpha_2})[T_{\alpha_2}, [T_{\alpha_1}, T_{\bar{\epsilon}}]], \tag{3.3}$$

where $\alpha_1, \alpha_2 \in \Lambda^1_{\alpha_0} \cup \{0\}$, $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$. In the following, we distinguish three cases.

Case 1. $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. As $\alpha_1 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_1}, T_{\bar{\epsilon}}] = 0$. As $\alpha_2 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_2}, T_{\bar{\epsilon}}] = 0$. Therefore by Eq. (3.3), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$.

Case 2. $\alpha_1 \neq 0$ and $\alpha_2 = 0$. As $\alpha_1 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_1}, T_{\bar{\epsilon}}] = 0$. That is $[T_{\alpha_2}, [T_{\alpha_1}, T_{\bar{\epsilon}}]] = 0$. As $\alpha_2 = 0$, $[T_{\alpha_2}, T_{\bar{\epsilon}}] = [T_0, T_{\bar{\epsilon}}] \subset L^0_{\bar{\epsilon}}$. By Lemma 3.1 (4), one gets $[T_{\alpha_1}, [T_{\alpha_2}, T_{\bar{\epsilon}}]] = 0$. Therefore by Eq. (3.3), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$.

Case 3. $\alpha_1 = 0$ and $\alpha_2 \neq 0$. As $\alpha_2 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_2}, T_{\bar{\epsilon}}] = 0$. That is $[T_{\alpha_1}, [T_{\alpha_2}, T_{\bar{\epsilon}}]] = 0$. As $\alpha_1 = 0$, $[T_{\alpha_1}, T_{\bar{\epsilon}}] = [T_0, T_{\bar{\epsilon}}] \subset L^0_{\bar{\epsilon}}$. By Lemma 3.1 (5), we get $[T_{\alpha_2}, [T_{\alpha_1}, T_{\bar{\epsilon}}]] = 0$. Therefore by (3.3), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$.

So $[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}]]=0$ is a consequence of $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$. By (3.2), one gets $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\bar{\epsilon}}] = 0$. The proof is complete.

(2) From the fact $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$, one gets if $\alpha_3 = 0$ then it is clear that $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\bar{\epsilon}}] = 0$. Let us consider the case $\alpha_3 \neq 0$. Note that

$$[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\bar{\epsilon}}] \subset [[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, L^0_{\bar{\epsilon}}]] - \varepsilon(\bar{\alpha_1} + \bar{\alpha_2}, \bar{\alpha_3})[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], L^0_{\bar{\epsilon}}]]. \tag{3.4}$$

Let us consider the first summand in Eq. (3.4). As $\alpha_3 \neq 0$, one gets $[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, L^0_{\bar{\epsilon}}]] = 0$ by Lemma 3.1 (4). Let us now consider the second summand in Eq. (3.4). As either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$, the definition of Lie color algebras, the fact $[T_0, L^0_{\bar{\epsilon}}] \subset T_{\bar{\epsilon}}$ and Lemma 3.1 (4), we obtain that $[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], L^0_{\bar{\epsilon}}]] = 0$. So, the second summand in Eq. (3.4) is also zero and then $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\bar{\epsilon}}] = 0$.

(3) It is a consequence of Lemma 3.5 (1), (2) and

$$[[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_0], T_{\bar{\epsilon}}] \subset [[T_0, T_{\bar{\epsilon}}], \{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}] + [[T_{\bar{\epsilon}}, \{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}], T_0].$$

□

Definition 3.6. *A Lie color triple system T is said to be* ***simple****, if $\{T, T, T\} \neq 0$ and its only ideals are $\{0\}$ and T.*

Theorem 3.7. *Suppose Λ^0 is symmetric, the following assertions hold.*

(1) *For any $\alpha_0 \in \Lambda^1$, the subsystem*

$$T_{\Lambda^1_{\alpha_0}} = T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}$$

of T associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

(2) *If T is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda^1$.*

Proof. (1) Recall that

$$T_{0,\Lambda^1_{\alpha_0}} := \mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \gamma = 0;\ \alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}\} \subset T_0$$

and $V_{\Lambda^1_{\alpha_0}} := \oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma$. In order to complete the proof, it is sufficient to show that

$$\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}.$$

We first check that $\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. It is easy to see that

$$\{T_{\Lambda^1_{\alpha_0}}, T, T\} = \{T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}, T, T\} = \{T_{0,\Lambda^1_{\alpha_0}}, T, T\} + \{V_{\Lambda^1_{\alpha_0}}, T, T\}.$$

Next, we will show that $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Note that

$$\begin{aligned}\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} =& \{T_{0,\Lambda^1_{\alpha_0}}, T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha), T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)\} \\ =& \{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0\} + \{T_{0,\Lambda^1_{\alpha_0}}, T_0, \oplus_{\alpha \in \Lambda^1} T_\alpha\} \\ &+ \{T_{0,\Lambda^1_{\alpha_0}}, \oplus_{\alpha \in \Lambda^1} T_\alpha, T_0\} + \{T_{0,\Lambda^1_{\alpha_0}}, \oplus_{\alpha \in \Lambda^1} T_\alpha, \oplus_{\beta \in \Lambda^1} T_\beta\}.\end{aligned}$$

Here, it is clear that $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0\} \subset \{T_0, T_0, T_0\} = 0$. Taking into account $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\}$, for $\alpha \in \Lambda^1$, Lemma 3.4 (1) and the fact that either $\alpha \in \Lambda^1_{\alpha_0}$ or $\alpha \notin \Lambda^1_{\alpha_0}$, give us that $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\} \subset V_{\Lambda^1_{\alpha_0}}$ or $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\} = 0$. Similarly, one gets that $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0\} \subset V_{\Lambda^1_{\alpha_0}}$ or $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0\} = 0$. Next, we will consider $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\}$, where $\alpha, \beta \in \Lambda^1$. We treat five cases.

Case 1. If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha + \beta = 0$. One has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} \subset T_{0,\Lambda^1_{\alpha_0}}.$$

Case 2. If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha + \beta \neq 0$. By $\Lambda^1_{\alpha_0}$ is a root subsystem, one gets

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$. By Lemma 3.4 (1), one has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0.$$

Case 4. If $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha \notin \Lambda^1_{\alpha_0}$. By Lemma 3.4 (1), one has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0.$$

Case 5. If $\beta \notin \Lambda^1_{\alpha_0}$ and $\alpha \notin \Lambda^1_{\alpha_0}$. By Lemma 3.4 (1), one has

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0.$$

Therefore, $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$.

Next, we will show that $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. It is obvious that

$$\begin{aligned}\{V_{\Lambda^1_{\alpha_0}}, T, T\} &= \{\oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma, T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha), T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha)\} \\ &= \{\oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma, T_0, T_0\} + \{\oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma, T_0, \oplus_{\alpha \in \Lambda^1} T_\alpha\} \\ &+ \{\oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha \in \Lambda^1} T_\alpha, T_0\} + \{\oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha \in \Lambda^1} T_\alpha, \oplus_{\beta \in \Lambda^1} T_\beta\}.\end{aligned}$$

Here, it is clear that $\{T_\gamma, T_0, T_0\} \subset V_{\Lambda^1_{\alpha_0}}$, for $\gamma \in \Lambda^1_{\alpha_0}$. Next, we will consider $\{T_\gamma, T_0, T_\alpha\}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. We treat three cases.

Case 1. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$. By Lemma 3.3 (1), one has

$$\{T_\gamma, T_0, T_\alpha\} = 0.$$

Case 2. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha \neq 0$. By $\Lambda^1_{\alpha_0}$ is a root subsystem, one has

$$\{T_\gamma, T_0, T_\alpha\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha = 0$. It is clear that

$$\{T_\gamma, T_0, T_\alpha\} \subset T_{0,\Lambda^1_{\alpha_0}}.$$

Hence, $\{T_\gamma, T_0, T_\alpha\} \subset T_{\Lambda^1_{\alpha_0}}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. Similarly, it is easy to get $\{T_\gamma, T_\alpha, T_0\} \subset T_{\Lambda^1_{\alpha_0}}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. At last, we will consider $\{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^1$. We treat five cases.

Case 1. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha + \beta = 0$, one gets

$$\{T_\gamma, T_\alpha, T_\beta\} \subset T_{0,\Lambda^1_{\alpha_0}}.$$

Case 2. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha + \beta \neq 0$, one gets

$$\{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$. By Lemma 3.3 (1) and (2), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

Case 4. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$ and $\beta \in \Lambda^1_{\alpha_0}$. By Lemma 3.3 (1) and (3), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

Case 5. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$. By Lemma 3.3 (1), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

So, $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Therefore $\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$ is a consequence of $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$ and $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Consequently, this proves that $T_{\Lambda^1_{\alpha_0}}$ is an ideal of T.

(2) The simplicity of T implies $T_{\Lambda^1_{\alpha_0}} = T$. Hence $\Lambda^1_{\alpha_0} = \Lambda^1$. □

Theorem 3.8. *Suppose Λ^0 is symmetric. Then for a vector space complement U of* $\mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \gamma = 0$, *where* $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ *in* T_0, *we have*

$$T = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals $T_{\Lambda^1_{\alpha_0}}$ of T described in Theorem 3.7. Moreover $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ *if* $[\alpha] \neq [\beta]$.

Proof. Let us denote $\xi_0 :=\mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha+\beta+\gamma = 0$, where $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ in T_0. By Proposition 2.17, we can consider the quotient set $\Lambda^1/\sim := \{[\alpha] : \alpha \in \Lambda^1\}$. By denoting $I_{[\alpha]} := T_{\Lambda^1_\alpha}$, $T_{0,[\alpha]} := T_{0,\Lambda^1_\alpha}$ and $V_{[\alpha]} := V_{\Lambda^1_\alpha}$, one gets $I_{[\alpha]} := T_{0,[\alpha]} \oplus V_{[\alpha]}$. From

$$T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha) = (U + \xi_0) \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha),$$

it follows

$$\oplus_{\alpha\in\Lambda^1} T_\alpha = \oplus_{[\alpha]\in\Lambda^1/\sim} V_{[\alpha]}, \quad \xi_0 = \sum_{[\alpha]\in\Lambda^1/\sim} T_{0,[\alpha]},$$

which implies

$$T = U + \xi_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha) = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]},$$

where each $I_{[\alpha]}$ is an ideal of T by Theorem 3.7.

Next, it is sufficient to show that $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. Note that,

$$\{I_{[\alpha]}, T, I_{[\beta]}\} = \{T_{0,[\alpha]} \oplus V_{[\alpha]}, T_0 \oplus (\oplus_{\gamma\in\Lambda^1} T_\gamma), T_{0,[\beta]} \oplus V_{[\beta]}\}$$

$$
\begin{aligned}
&=\{T_{0,[\alpha]}, T_0, T_{0,[\beta]}\} + \{T_{0,[\alpha]}, T_0, V_{[\beta]}\} + \{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\}\\
&+\{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\} + \{V_{[\alpha]}, T_0, T_{0,[\beta]}\} + \{V_{[\alpha]}, T_0, V_{[\beta]}\}\\
&+\{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\} + \{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\}.
\end{aligned}
$$

Here, it is clear that $\{T_{0,[\alpha]}, T_0, T_{0,[\beta]}\} \subset \{T_0, T_0, T_0\} = 0$. If $[\alpha] \neq [\beta]$, by Lemmas 3.3 and 3.4, it is easy to see $\{T_{0,[\alpha]}, T_0, V_{[\beta]}\} = 0$, $\{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\} = 0$, $\{V_{[\alpha]}, T_0, T_{0,[\beta]}\} = 0$, $\{V_{[\alpha]}, T_0, V_{[\beta]}\} = 0$, $\{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\} = 0$, $\{V_{[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, V_{[\beta]}\} = 0$.

Next, we will show $\{T_{0,[\alpha]}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{0,[\beta]}\} = 0$. Indeed, for $\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\} \in T_{0,[\alpha]}$ with $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_\alpha \cup \{0\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, and for $\{T_{\beta_1}, T_{\beta_2}, T_{\beta_3}\} \in T_{0,[\beta]}$ with $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\beta \cup \{0\}$, $\beta_1 + \beta_2 + \beta_3 = 0$, by the definition of Lie color triple systems, one gets

$$
\begin{aligned}
&\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, \{T_{\beta_1}, T_{\beta_2}, T_{\beta_3}\}\}\\
\subset&\varepsilon(\bar{\gamma_1}, \bar{\beta_1} + \bar{\beta_2})\{T_{\beta_1}, T_{\beta_2}, \{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}\}\\
&+\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_1}\}, T_{\beta_2}, T_{\beta_3}\}\\
&+\varepsilon(\bar{\gamma_1}, \bar{\beta_1})\{T_{\beta_1}, \{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_2}\}, T_{\beta_3}\}.
\end{aligned}
$$

By Lemma 3.4, it is easy to see that

$$
\{T_{\beta_1}, T_{\beta_2}, \{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_3}\}\} = 0,
$$

$$
\{\{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_1}\}, T_{\beta_2}, T_{\beta_3}\} = 0,
$$

$$
\{T_{\beta_1}, \{\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, \oplus_{\gamma\in\Lambda^1} T_\gamma, T_{\beta_2}\}, T_{\beta_3}\} = 0.
$$

for $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_\alpha \cup \{0\}$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$, $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\beta \cup \{0\}$, $\beta_1 + \beta_2 + \beta_3 = 0$, $[\alpha] \neq [\beta]$. So $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. □

Definition 3.9. *The **annihilator** of a Lie color triple system T is the set* $\mathrm{Ann}(T) = \{x \in T : \{x, T, T\} = 0\}$.

Corollary 3.10. *Suppose Λ^0 is symmetric. If* $\mathrm{Ann}(T) = 0$, *and* $\{T, T, T\} = T$, *then T is the direct sum of the ideals given in Theorem 3.8,*

$$
T = \oplus_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.
$$

Proof. From $\{T, T, T\} = T$ and Theorem 3.8, we have

$$
\{U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}, U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}\} = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.
$$

Taking into account $U \subset T_0$, Lemma 3.3 and the fact that $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$ (see Theorem 3.8) give us that $U = 0$. That is,

$$
T = \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}.
$$

To finish, it is sufficient to show the direct character of the sum. For $x \in I_{[\alpha]} \cap \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\nsim\alpha}} I_{[\beta]}$, using again the equation $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ for $[\alpha] \neq [\beta]$, we obtain

$$
\{x, T, I_{[\alpha]}\} = \{x, T, \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\nsim\alpha}} I_{[\beta]}\} = 0.
$$

So $\{x, T, T\} = \{x, T, I_{[\alpha]} + \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\nsim\alpha}} I_{[\beta]}\} = \{x, T, I_{[\alpha]}\} + \{x, T, \sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\nsim\alpha}} I_{[\beta]}\} = 0 + 0 = 0$. That is, $x \in \mathrm{Ann}(T) = 0$. Thus $x = 0$, as desired. □

4 The simple components

In this section we study if any of the components in the decomposition given in Corollary 3.10 is simple. Under certain conditions we give an affirmative answer. From now on $\mathrm{char}(\mathbb{K}) = 0$.

Lemma 4.1. *Let $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ be a split Lie color triple system. If I is an ideal of T then $I = (I \cap T_0) \oplus (\oplus_{\alpha\in\Lambda^1}(I \cap T_\alpha))$.*

Proof. We can see that $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ as a weight module with respect to the split Lie color algebra L^0 with MAGSA H^0. The character of the ideal of I and the fact $L^0 = [T, T]$ give us that I is a submodule of T. It is well-known that a submodule of a weight module is again a weight module. From here, I is a weight module with respect to L^0 (and H^0) and so $I = (I \cap T_0) \oplus (\oplus_{\alpha\in\Lambda^1}(I \cap T_\alpha))$. □

Taking into account the above lemma, observe that the grading of I and Lemma 2.13 (1) let us write

$$I = \oplus_{g\in\Gamma} I_g = \oplus_{g\in\Gamma}((I_g \cap T_{0,g}) \oplus (\oplus_{\alpha\in\Lambda^1}(I_g \cap T_{\alpha,g}))). \tag{4.5}$$

Let us introduce the concepts of root-multiplicativity and maximal length in the frame work of split Lie color triple systems. For each $g \in \Gamma$, we denote by $\Lambda_g^1 := \{\alpha \in \Lambda^1, T_{\alpha,g} \neq 0\}$ and $\Lambda_g^0 := \{\alpha \in \Lambda^0, L^0_{\alpha,g} \neq 0\}$.

Definition 4.2. *We say that a split Lie color triple system T is* ***root-multiplicative*** *if given $\alpha \in \Lambda^1_{g_i}$, $\beta \in \Lambda^1_{g_j}$ and $\gamma \in \Lambda^1_{g_k}$, with $g_i, g_j, g_k \in \Gamma$, such that $\alpha + \beta \in \Lambda^0$, and $\alpha + \beta + \gamma \in \Lambda^1$, then $\{T_{\alpha,g_i}, T_{\beta,g_j}, T_{\gamma,g_k}\} \neq 0$.*

Definition 4.3. *We say that a split Lie color triple system T is of* ***maximal length*** *if for any $\alpha \in \Lambda^1_g$, $g \in \Gamma$, we have $\dim T_{\kappa\alpha,\kappa g} = 1$ for $\kappa \in \{\pm 1\}$.*

Observe that if T is of maximal length, then Eq. (4.5) enables us assert that given any nonzero ideal I of T then

$$I = \oplus_{g\in\Gamma} I_g = \oplus_{g\in\Gamma}((I_g \cap T_{0,g}) \oplus (\oplus_{\alpha\in\Lambda_g^{1,I}} T_{\alpha,g})). \tag{4.6}$$

where $\Lambda_g^{1,I} := \{\alpha \in \Lambda^1 : I_g \cap T_{\alpha,g} \neq 0\}$ for each $g \in \Gamma$.

Lemma 4.4. *Let T be a root-multiplicative split Lie color triple system with $\mathrm{Ann}(T) = 0$. If for any $\alpha \in \Lambda^1$, we have $\dim L^0_\alpha = 1$. Then there is not any nonzero ideal of T contained in T_0.*

Proof. Suppose there exists a nonzero ideal I of T such that $I \subset T_0$. Given $\alpha \in \Lambda^1$, as $\{I, T_0, T_\alpha\} \subset T_\alpha \cap T_0$ and $\{I, T_\alpha, T_0\} \subset T_\alpha \cap T_0$, $\{I, T_0, T_\alpha\} = \{I, T_\alpha, T_0\} = 0$. Given also $\beta \in \Lambda^1$ with $\alpha + \beta \neq 0$, $\{I, T_\alpha, T_\beta\} \subset T_{\alpha+\beta} \cap T_0 = 0$. As $\mathrm{Ann}(T) = 0$, $\{I, T_\alpha, T_{-\alpha}\} \neq 0$ for some $\alpha \in \Lambda^1$. Thus, there exist $t_{\pm\alpha} \in T_{\pm\alpha}$ and $t_0 \in I$ such that $\{t_0, t_\alpha, t_{-\alpha}\} \neq 0$. Hence $0 \neq [t_0, t_\alpha] \in L^0_\alpha$. As $\dim L^0_\alpha = 1$, the root-multiplicativity of T (consider the roots $0, \alpha, 0 \in \Lambda^1 \cup \{0\}$), and the fact that $\dim L^0_\alpha = 1$ give us the existence of $0 \neq t_0' \in T_0$ such that $0 \neq \{t_0, t_\alpha, t_0'\} \in T_\alpha$. As $t_0 \in I$, we conclude $0 \neq t_\alpha' := \{t_0, t_\alpha, t_0'\} \in I \subset T_0$, a contradiction. Hence, I is not contained in T_0. □

Theorem 4.5. *Let T be a split Lie color triple system of maximal length, root-multiplicative, with $\mathrm{Ann}(T) = 0$ and satisfying $T = \{T, T, T\}$. If Λ^0 is symmetric and for any $\alpha \in \Lambda^1$, we have $\dim L^0_\alpha \leq 1$. Then T is simple if and only if it has all its nonzero roots connected.*

Proof. The first implication is Theorem 3.7 (2). To prove converse, consider I a nonzero ideal of T, by Lemma 4.4 and Eq. (4.6) we can write

$$I = \oplus_{g\in\Gamma} I_g = \oplus_{g\in\Gamma}((I_g \cap T_{0,g}) \oplus (\oplus_{\alpha\in\Lambda_g^{1,I}} T_{\alpha,g})).$$

with $\Lambda_g^{1,I} \subset \Lambda_g^1$ for any $g \in \Gamma$ and some $\Lambda_g^{1,I} \neq \emptyset$. Hence by the maximal length of T, we may choose $\alpha_0 \in \Lambda_g^{1,I}$ such that

$$0 \neq T_{\alpha_0,g} \subset I. \tag{4.7}$$

Given any $\beta_0 \in \Lambda^1$ with $\beta_0 \notin \{\alpha_0, -\alpha_0\}$, the fact that α_0 and β_0 are connected gives us a connection $\{\alpha_1, \cdots, \alpha_{2n+1}\}$ from α_0 to β_0 such that $\alpha_1 = \alpha_0, \alpha_1+\alpha_2+\alpha_3, \cdots, \alpha_1+\cdots+\alpha_{2n}+\alpha_{2n+1} \in \Lambda^1$, $\alpha_1+\alpha_2, \cdots, \alpha_1+\cdots+\alpha_{2n} \in \Lambda^0$ and $\alpha_1+\cdots+\alpha_{2n}+\alpha_{2n+1} \in \{\beta_0, -\beta_0\}$. Consider $\alpha_1 = \alpha_0$, α_2, α_3 and $\alpha_1+\alpha_2+\alpha_3$. Since $\alpha_2 \in \Lambda^1$ there exists $g_1 \in \Gamma$ such that $T_{\alpha_2,g_1} \neq 0$. Since $\alpha_3 \in \Lambda^1$ there exists $g_2 \in \Gamma$ such that $T_{\alpha_3,g_2} \neq 0$. From here, the root-multiplicativity and maximal length of T shows $0 \neq \{T_{\alpha_0,g}, T_{\alpha_2,g_1}, T_{\alpha_3,g_2}\} = T_{\alpha_1+\alpha_2+\alpha_3,g+g_1+g_2}$, and by Eq. (4.7)

$$0 \neq T_{\alpha_1+\alpha_2+\alpha_3,g+g_1+g_2} \subset I.$$

We can argue in a similar way from $\alpha_1+\alpha_2+\alpha_3$, α_4, α_5 and $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$ to get

$$0 \neq T_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,g_3} \subset I.$$

for some $g_3 \in \Gamma$. Following this process with the connection $\{\alpha_1, \cdots, \alpha_{2n+1}\}$ we obtain that

$$0 \neq T_{\alpha_1+\alpha_2+\cdots+\alpha_{2n+1},g_4} \subset I$$

and so either $T_{\beta_0,g_4} \subset I$ or $T_{-\beta_0,g_4} \subset I$ for some $g_4 \in \Gamma$. That is,

$$0 \neq T_{\epsilon\beta_0,g_4} \subset I \tag{4.8}$$

for some $\epsilon \in \{\pm 1\}$, some $g_4 \in \Gamma$ and for any $\beta_0 \in \Lambda^1$ with $\beta_0 \notin \{\alpha_0, -\alpha_0\}$.

Taking into account $H^0 = [T_0, T_0] + \sum_{\alpha\in\Lambda^1}[T_\alpha, T_{-\alpha}]$, the grading of T gives us $H^0_g = \sum_{g_1+g_2=g, g_1,g_2\in\Gamma}[T_{0,g_1}, T_{0,g_2}] + \sum_{\substack{\alpha\in\Lambda^1 \\ g_1+g_2=g, g_1,g_2\in\Gamma}}[T_{\alpha,g_1}, T_{-\alpha,g_2}]$. We can suppose that either $H^0_0 = \sum_{g\in\Gamma}[T_{0,g}, T_{0,-g}]$ or $H^0_0 = \sum_{\alpha\in\Lambda^1, g\in\Gamma}[T_{\alpha,g}, T_{-\alpha,-g}]$. We treat two cases.

In the first case, by Eq. (4.8), there exists $g_5 \in \Gamma$ such that

$$[[T_{0,g_5}, T_{0,-g_5}], T_{\epsilon\beta_0,g_4}] \neq 0. \tag{4.9}$$

By the Jacobi identity of the Lie color algebras, either $[T_{0,g_5}, T_{\epsilon\beta_0,g_4}] \neq 0$ or $[T_{0,-g_5}, T_{\epsilon\beta_0,g_4}] \neq 0$ and so $L^0_{\epsilon\beta_0,g_4+g_5} \neq 0$ or $L^0_{\epsilon\beta_0,g_4-g_5} \neq 0$. That is

$$0 \neq L^0_{\epsilon\beta_0,\kappa g_5+g_4} \subset I \tag{4.10}$$

for some $\kappa \in \{\pm 1\}$. Since $\epsilon\beta_0 \in \Lambda^1_{g_4}$, we have by the maximal length of T such that $-\epsilon\beta_0 \in \Lambda^1_{-g_4}$. By Eq. (4.10), and the root-multiplicativity and maximal length of T, we obtain

$$0 \neq [L^0_{\epsilon\beta_0,\kappa g_5+g_4}, T_{-\epsilon\beta_0,-g_4}] = T_{0,\kappa g_5} \subset I. \tag{4.11}$$

Taking into account Eq. (4.11) and that Eq. (4.9) gives us

$$\beta_0([T_{0,g_5}, T_{0,-g_5}]) \neq 0,$$

we have that for any $g_6 \in \Gamma$ such that $T_{\epsilon\beta_0,g_6} \neq 0$ necessarily

$$0 \neq [[T_{0,g_5}, T_{0,-g_5}], T_{\epsilon\beta_0,g_6}] = T_{\epsilon\beta_0,g_6} \subset I,$$

and so

$$T_{\epsilon\beta_0} \subset I, \tag{4.12}$$

for any $\beta_0 \in \Lambda^1$ with $\beta_0 \notin \{\alpha_0, -\alpha_0\}$, and some $\epsilon \in \{\pm 1\}$.

In the second case, by Eq. (4.8), there exists $g_7 \in \Gamma$ such that

$$[[T_{\alpha,g_7}, T_{-\alpha,-g_7}], T_{\epsilon\beta_0,g_4}] \neq 0. \tag{4.13}$$

By the Jacobi identity of the Lie color algebras, either

$$[T_{\alpha,g_7}, T_{\epsilon\beta_0,g_4}] \neq 0,$$

or

$$[T_{-\alpha,-g_7}, T_{\epsilon\beta_0,g_4}] \neq 0,$$

and so $L^0_{\alpha+\epsilon\beta_0,g_4+g_7} \neq 0$ or $L^0_{-\alpha+\epsilon\beta_0,g_4-g_7} \neq 0$. That is

$$0 \neq L^0_{\kappa\alpha+\epsilon\beta_0,\kappa g_7+g_4} \subset I, \tag{4.14}$$

for some $\kappa \in \{\pm 1\}$. Since $\epsilon\beta_0 \in \Lambda^1_{g_4}$, we have by the maximal length of T such that $-\epsilon\beta_0 \in \Lambda^1_{-g_4}$. By Eq. (4.14), and the root-multiplicativity and maximal length of T we obtain

$$0 \neq [L^0_{\kappa\alpha+\epsilon\beta_0,\kappa g_7+g_4}, T_{-\epsilon\beta_0,-g_4}] = T_{\kappa\alpha,\kappa g_7} \subset I. \tag{4.15}$$

Taking into account Eq. (4.15) and that Eq. (4.13) gives us

$$\beta_0([T_{\alpha,g_7}, T_{-\alpha,-g_7}]) \neq 0,$$

we have that for any $g_8 \in \Gamma$ such that $T_{\epsilon\beta_0,g_8} \neq 0$ necessarily

$$0 \neq [[T_{\alpha,g_7}, T_{-\alpha,-g_7}], T_{\epsilon\beta_0,g_8}] = T_{\epsilon\beta_0,g_8} \subset I,$$

and so

$$T_{\epsilon\beta_0} \subset I, \tag{4.16}$$

for any $\beta_0 \in \Lambda^1$ with $\beta_0 \notin \{\alpha_0, -\alpha_0\}$, and some $\epsilon \in \{\pm 1\}$.

Observe that as a consequence of $T = \{T, T, T\}$, we have

$$T_0 = \sum_{\substack{\alpha+\beta+\gamma=0 \\ \alpha,\beta,\gamma\in\Lambda^1\cup\{0\}}} \{T_\alpha, T_\beta, T_\gamma\}. \tag{4.17}$$

Let us study the products $\{T_\alpha, T_\beta, T_\gamma\}$ of Eq. (4.17) in order to show $T_0 \subset I$. Taking into account $\{T_0, T_0, T_0\} = 0$, and the fact $\alpha+\beta+\gamma = 0$ with $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$, we can suppose $\gamma \neq 0$ and either $\alpha \neq 0$ or $\beta \neq 0$. Suppose $\alpha \neq 0$ and $\beta = 0$ (resp. $\alpha = 0$ and $\beta \neq 0$), then $\alpha = -\gamma$ (resp. $\beta = -\gamma$) and by Eqs. (4.12) and (4.16), $\{T_\alpha, T_\beta, T_\gamma\} = \{T_{-\gamma}, T_0, T_\gamma\} \subset I$, (resp. $\{T_\alpha, T_\beta, T_\gamma\} = \{T_0, T_{-\gamma}, T_\gamma\} \subset I$). If the three elements in $\{\alpha, \beta, \gamma\}$ are nonzero, in case some $T_\epsilon \subset I$, $\epsilon \in \{\alpha, \beta, \gamma\}$, then clearly $\{T_\alpha, T_\beta, T_\gamma\} \subset I$. Finally, consider the case in which any of the T_ϵ does not belong to I. If $\{T_\alpha, T_\beta, T_\gamma\} = 0$ then $\{T_\alpha, T_\beta, T_\gamma\} \subset I$. If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, necessarily $\alpha+\beta \neq 0$ and so $\alpha+\beta \in \Lambda^0$. From here, we have by root-multiplicativity $\{T_\alpha, T_\beta, T_{-\beta}\} = T_\alpha$. Eqs. (4.12) and (4.16) give us $T_{-\beta} \subset I$, then $T_\alpha \subset I$ and so $\{T_\alpha, T_\beta, T_\gamma\} \subset I$. Therefore Eq. (4.17) implies

$$T_0 \subset I. \tag{4.18}$$

Fix now any $\alpha_0 \in \Lambda^1$. By Eqs. (4.12) and (4.16), either $T_{\alpha_0} \subset I$ or $T_{-\alpha_0} \subset I$. Write $T_{\rho\alpha_0} \subset I$ with $\rho \in \pm 1$, then we can show $T_{-\rho\alpha_0} \subset I$. Indeed, since $\alpha_0 \neq 0$, there exists $h_0 \in H^0_0$ such that $\alpha_0(h_0) \neq 0$ and so we have

$$t_{-\rho\alpha_0} = -\rho\alpha_0(h_0)^{-1}[h_0, t_{-\rho\alpha_0}], \tag{4.19}$$

for any $t_{-\rho\alpha_0} \in T_{-\rho\alpha_0}$.

As

$$H^0_0 = \sum_{g\in\Gamma}[T_{0,g}, T_{0,-g}] + \sum_{\alpha\in\Lambda^1, g\in\Gamma}[T_{\alpha,g}, T_{-\alpha,-g}],$$

we can suppose that either $h_0 = [t_{0,g}, t'_{0,-g}]$ with $t_{0,g} \in T_{0,g}$, $t'_{0,g} \in T_{0,-g}$ or $h_0 = [t_{\alpha,g'}, t_{-\alpha,-g'}]$ with $t_{\alpha,g'} \in T_{\alpha,g'}$, $t_{-\alpha,-g'} \in T_{-\alpha,-g'}$. From here, in order to prove that $T_{\pm\alpha_0} \subset I$ for any $\alpha_0 \in \Lambda^1$. we treat two cases.

In the first case, we have by Eqs (4.18) and (4.19), that

$$t_{-\rho\alpha_0} = -\rho\alpha_0(h_0)^{-1}[[t_{0,g}, t'_{0,g}], t_{-\rho\alpha_0}] \in I, \tag{4.20}$$

for any $t_{-\rho\alpha_0} \in T_{-\rho\alpha_0}$.

In the second case, we have by Eqs (4.18) and (4.19), that

$$t_{-\rho\alpha_0} = -\rho\alpha_0(h_0)^{-1}[[t_{\alpha,g}, t_{-\alpha,-g}], t_{-\rho\alpha_0}] \in I, \tag{4.21}$$

for any $t_{-\rho\alpha_0} \in T_{-\rho\alpha_0}$.

Since $\dim T_{-\rho\alpha_0} = 1$, we conclude $T_{-\rho\alpha_0} \subset I$ and so $T_{\pm\alpha_0} \subset I$ for any $\alpha_0 \in \Lambda^1$. From here, and taking into account Eqs (4.16) and (4.18), we conclude $I = T$ and so T is simple. □

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