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Characterizations of Convex spaces and Anti-matroids via Derived Operators

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Abstract: In this paper we use the notion of derived sets to study convex spaces. By axiomatizing the derived sets on convex spaces, we define c -derived operators and restricted c -derived operators. Results show that convex structures can be characterized in terms of c -derived operators. Furthermore, the link between c -derived operators and Shi's m -derived operators is studied. Specifically, it is proved that a c -derived operator is an m -derived operator if and only if it satisfies the Exchange Law. At last, we show an application of c -derived operators to anti-matroids.

Keywords: convex space, derived operator, m -derived operator, matroid, anti-matroid

MSC: 52A01, 14P25.

1 Introduction

Convexity theory has been accepted to be increasingly important in recent years in the study of extremum problems of applied mathematics. In fact, convex structure exists in many mathematical research areas, such as lattices [1, 2], algebras [3, 4], metric spaces [5], graphs [6–8] and topological spaces [9, 10]. In 1993, M. van de Vel collected the convexity theory systematically in the famous book [11].

A convex structure [11] on a set is a family of subsets which contains the empty set and is closed under arbitrary intersections and directed unions. Often it is more convenient not to describe the family of convex sets directly, and thus some other characterizations of convex structures become especially important, which can be found in [11] consisting of hull operators, restricted hull operators, betweenness relations and independence structures.

The notion of derived sets is first introduced by Georg Cantor in 1872 and he developed set theory in large part to study derived sets on the real line. In mathematics, more specially in point-set topology, the derived sets of a subset S of a topological space is the set of all limit points of S . Various topological notions can be characterized in terms of derived sets (refer to [12, 13]):

- (T1) A subset is closed precisely when it contains its derived set.
- (T2) Two subsets are separated if and only if they are disjoint and each is disjoint from the other's derived set.
- (T3) A bijection between two topological spaces is a homeomorphism if and only if the derived set of the image of any subset is the image of the derived set of that subset.

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- (T4) Any Polish space can be written as the union of a countable set and a perfect set, where a set is called perfect if its derived set coincides with itself.

As a set-structure that is similar to topologies, a natural idea is whether convex structures can be characterized by derived operators. Another motivation comes from m -derived operators [14] introduced by Xin and Shi in 2010 which provide a new description for a matroid. As we know, a convex structure is a matroid if and only if its hull operator satisfies the Exchange Law. This encourages us to bring the notion of derived operators from matroids to convex structures directly, which can be regarded as an extension of m -derived operators. In this paper, we introduce the notion of derived operators defined in convex spaces, called c -derived operators, which turns out to be important in the study of both convex spaces and anti-matroids.

The layout of the paper is organized as follows. In Section 2, some preliminaries on convex structures and (anti-) matroids are introduced. In Sections 3 and 4, we introduce and study the notion of c -derived operators, and construct the isomorphism between c -derived operators and convex structures. Also, we proved that CP mappings can be completely described in terms of c -derived operators. In Section 5, the notion of restricted c -derived operators is introduced, which is isomorphic to convex structures. At last, the link between c -derived operators and m -derived operators are studied, and also some equivalent descriptions of anti-matroids is provided.

2 Preliminary

In this section, we shall review some basic concepts and results on the convex structures and (anti-) matroids. For undefined notions in this paper, the reader can refer to [11, 15].

Let X be a nonempty set, let $\mathcal{P}(X)$ denote the power set of X , and $\mathcal{P}_{fin}(X)$ the family of all finite subsets of X . A family $\{D_i \mid i \in I\} \subseteq \mathcal{P}(X)$ is called *directed* if for each pairs $i_1, i_2 \in I$, there exists an $i_3 \in I$ such that $D_{i_1} \subseteq D_{i_3}$ and $D_{i_2} \subseteq D_{i_3}$. It is trivial that $\mathcal{P}_{fin}(X)$ is a directed family.

Definition 2.1 ([11]). A subset \mathcal{C} of $\mathcal{P}(X)$ is called a *convex structure*, if it satisfies the following conditions:

- (CS1) $\emptyset, X \in \mathcal{C}$;
- (CS2) if $\{A_i \mid i \in I\} \subseteq \mathcal{C}$, then $\bigcap_{i \in I} A_i \in \mathcal{C}$;
- (CS3) if $\{D_i \mid i \in I\} \subseteq \mathcal{C}$ is directed, then $\bigcup_{i \in I} D_i \in \mathcal{C}$.

The pair (X, \mathcal{C}) is called a *convex space* if \mathcal{C} is a convex structure on X , and each $A \in \mathcal{C}$ a *convex set*.

Definition 2.2 ([11]). Let (X, \mathcal{C}) be a convex space. For each $A \in \mathcal{P}(X)$, define

$$co(A) = \bigcap \{B \in \mathcal{C} \mid A \subseteq B\}.$$

Then $co(A)$ is the least element of \mathcal{C} that contains A , called the (*convex*) *hull* of A . The operator co is called the *hull operator* on (X, \mathcal{C}) .

Convex structures can be characterized by hull operators.

Proposition 2.3 ([11]). Let co be the hull operator on (X, \mathcal{C}) . Then it satisfies the following properties:

- (AC1) *Normalization Law*: $co(\emptyset) = \emptyset$;
- (AC2) *Extensive Law*: $A \subseteq co(A)$;
- (AC3) *Idempotent Law*: $co(co(A)) = co(A)$;
- (AC4) *Algebraic Law*: $co(A) = \bigcup \{co(F) \mid F \in \mathcal{P}_{fin}(A)\}$.

Conversely, a mapping $co : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ with the properties (AC1)–(AC4), determines a convex structure \mathcal{C} on X defined as follows:

$$\mathcal{C} = \{A \in \mathcal{P}(X) \mid co(A) = A\}.$$

Further, the relationship between hull operators and convex structures on a given set is bijective.

Definition 2.4 ([15]). A convex structure \mathcal{C} on a finite set X is called

- (1) a *matroid* provided its hull operator co satisfies the Exchange Law: if $A \subseteq X$ and $p, q \in X - co(A)$ with $p \neq q$, then $p \in co(\{q\} \cup A)$ implies $q \in co(\{p\} \cup A)$;
- (2) a *anti-matroid* (or a *convex geometry*) provided its hull operator co satisfies the Anti-Exchange Law: if $A \subseteq X$ and $p, q \in X - co(A)$ with $p \neq q$, then $p \in co(\{q\} \cup A)$ implies $q \notin co(\{p\} \cup A)$.

Proposition 2.5 ([16]). Let \mathcal{C} be an anti-matroid on a finite set X and let $A, B \subseteq X$. If $co(A) = co(B) = C$, then $co(A \cap B) = C$.

Definition 2.6 ([16]). Let C be a convex set in a convex space (X, \mathcal{C}) . A subset $A \subseteq C$ is said to *generate* C if $co(A) = C$. More specially, if A is the minimal generating set, it is called the *generator* of C . When there is only a single generating set for any convex set in \mathcal{C} , we say that the convex structure \mathcal{C} is *uniquely generated*.

Proposition 2.7 ([16]). Let X be a finite nonempty set. Then a convex structure \mathcal{C} on X is anti-matroid if and only if it is uniquely generated.

Definition 2.8 ([11]). Let $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ be a mapping between convex spaces. Then f is called

- (1) *convex structure-preserving* (CP, in short) provided $B \in \mathcal{C}_Y$ implies $f^{-1}(B) \in \mathcal{C}_X$;
- (2) *convex-to-convex* (CC, in short) provided $A \in \mathcal{C}_X$ implies $f(A) \in \mathcal{C}_Y$.

Theorem 2.9 ([11]). Let $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ be a mapping between two convex spaces. Then the following conditions are equivalent.

- (1) f is CP (resp., CC).
- (2) For any $F \in \mathcal{P}_{fin}(X)$, $f(co_X(F)) \subseteq co_Y(f(F))$ (resp., $co_Y(f(F)) \subseteq f(co_X(F))$).
- (3) For any $A \in \mathcal{P}(X)$, $f(co_X(A)) \subseteq co_Y(f(A))$ (resp., $co_Y(f(A)) \subseteq f(co_X(A))$).

The category of convex spaces and CP mappings is denoted **Conx**.

3 Derived operators

In this section, the notion of c -derived operators is presented. Further, it is proved that every c -derived operator can induce a convex structure.

Definition 3.1. Let X be a set. A mapping $d : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ is called a *convexly derived operator* (c -derived operator for short) on X provided d satisfies the following conditions:

- (CD1) Normalization Law: $d(\emptyset) = \emptyset$;
- (CD2) Representative Law: $x \in d(A)$ implies $x \in d(A - \{x\})$;
- (CD3) Idempotent Law: $d(d(A) \cup A) \subseteq d(A) \cup A$;
- (CD4) Algebraic Law: $d(A) = \bigcup \{d(F) \mid F \in \mathcal{P}_{fin}(A)\}$.

We call the pair (X, d) a *convexly derived space* (*c-derived space* for short) if d is a c-derived operator on X .

Proposition 3.2. *Conditions (CD2) and (CD4) can be replaced by*

(CD2^{*}) $A \subseteq B$ implies $d(A) \subseteq d(B)$;

(CD4^{*}) $d(A) = \bigcup \{d(F) \cap (X - F) \mid F \in \mathcal{P}_{fin}(A)\}$.

Proof. Necessity. By (CD4), the verification of (CD2^{*}) is straightforward. It suffices to verify (CD4^{*}). Take any $x \in d(A)$. By (CD2) and (CD4), we obtain

$$x \in d(A - \{x\}) = \bigcup \{d(F) \mid F \in \mathcal{P}_{fin}(A - \{x\})\}.$$

Then there exists $G \in \mathcal{P}_{fin}(A - \{x\})$ such that $x \in d(G)$. This implies

$$x \in d(G) \cap (X - G) \subseteq \bigcup \{d(F) \cap (X - F) \mid F \in \mathcal{P}_{fin}(A)\}.$$

Therefore, $d(A) \subseteq \bigcup \{d(F) \cap (X - F) \mid F \in \mathcal{P}_{fin}(A)\}$. The opposite inclusion holds obviously. Thus (CD4^{*}) holds.

Sufficiency. (CD2) Take any $x \in d(A)$. By (CD4^{*}), there exists $F \in \mathcal{P}_{fin}(A)$ such that $x \in d(F) \cap (X - F)$. This means $x \in d(F - \{x\})$. By (CD2^{*}), we have $x \in d(A - \{x\})$. Thus (CD2) holds.

(CD4) By using (CD4^{*}), we obtain

$$d(A) = \bigcup \{d(F) \cap (X - F) \mid F \in \mathcal{P}_{fin}(A)\} \subseteq \bigcup \{d(F) \mid F \in \mathcal{P}_{fin}(A)\}.$$

It follows from (CD2^{*}) that $\bigcup_{F \in \mathcal{P}_{fin}(A)} d(F) \subseteq d(A)$. Thus $d(A) = \bigcup \{d(F) \mid F \in \mathcal{P}_{fin}(A)\}$. Therefore (CD4) holds. \square

Definition 3.3. A mapping $f : (X, d_X) \rightarrow (Y, d_Y)$ between c-derived spaces is called *c-derived-preserving* (DP for short) provided that

$$\forall A \subseteq X, f(d_X(A)) \subseteq f(A) \cup d_Y(f(A)).$$

Proposition 3.4. *Let $f : (X, d_X) \rightarrow (Y, d_Y)$ be a mapping between c-derived spaces. Then f is a DP mapping if and only if $f(d_X(F)) \subseteq f(F) \cup d_Y(f(F))$ for all $F \in \mathcal{P}_{fin}(X)$.*

Proof. The necessity is obvious. For sufficiency, take any $A \subseteq X$. By (CD4) and (CD2^{*}), we obtain

$$\begin{aligned} f(d_X(A)) &= f\left(\bigcup \{d_X(F) \mid F \in \mathcal{P}_{fin}(A)\}\right) \\ &= \bigcup \{f(d_X(F)) \mid F \in \mathcal{P}_{fin}(A)\} \\ &\subseteq \bigcup \{f(F) \cup d_Y(f(F)) \mid F \in \mathcal{P}_{fin}(A)\} \\ &\subseteq f(A) \cup d_Y(f(A)). \end{aligned}$$

The proof is completed. \square

The following conclusion is straightforward.

Proposition 3.5. *Let $f : (X, d_X) \rightarrow (Y, d_Y)$ and $g : (Y, d_Y) \rightarrow (Z, d_Z)$ be two DP mappings between c-derived spaces. Then the composite mapping $g \circ f$ is also a DP mapping.*

The category of c-derived spaces and DP mappings is denoted by **Deri**.

Proposition 3.6. *Let d be a c-derived operator on X . Then the family $\mathcal{C}_d \subseteq \mathcal{P}(X)$ defined by*

$$\mathcal{C}_d = \{C \subseteq X \mid d(C) \subseteq C\}$$

forms a convex structure on X . Moreover, $\mathcal{C}_d = \{d(A) \cup A \mid A \subseteq X\}$.

Proof. The verifications of (CS1) and (CS2) are straightforward. For (CS3), take any directed family $\{D_i \mid i \in I\} \subseteq \mathcal{C}_d$. Then $d(D_i) \subseteq D_i$ for all $i \in I$. Furthermore, by (CD4), we have

$$\begin{aligned} d\left(\bigcup_{i \in I} D_i\right) &= \bigcup \{d(F) \mid F \subseteq \bigcup_{i \in I} D_i\} = \bigcup \{d(F) \mid \exists i \in I, F \subseteq D_i\} \\ &= \bigcup_{i \in I} \bigcup \{d(F) \mid F \subseteq D_i\} = \bigcup_{i \in I} d(D_i) \subseteq \bigcup_{i \in I} D_i. \end{aligned}$$

This shows that $\bigcup_{i \in I} D_i \in \mathcal{C}_d$.

For convenience, denote $\mathcal{C}_d^* := \{A \cup d(A) \mid A \subseteq X\}$. It remains to show $\mathcal{C}_d = \mathcal{C}_d^*$. First by (CD3), we have $d(A \cup d(A)) \subseteq A \cup d(A)$ for all $A \in \mathcal{P}(X)$, which implies that $\mathcal{C}_d^* \subseteq \mathcal{C}_d$. In addition, if $A \in \mathcal{C}_d$, then $d(A) \subseteq A$, implying that $A = A \cup d(A) \in \mathcal{C}_d^*$. It follows that $\mathcal{C}_d \subseteq \mathcal{C}_d^*$. \square

Proposition 3.7. *Let d be a c -derived operator on X , let $\mathcal{C}_d \subseteq \mathcal{P}(X)$ be the convex structure induced by d and let co_d be the hull operator on \mathcal{C}_d . Then $co_d(A) = d(A) \cup A$ for all $A \in \mathcal{P}(X)$.*

Proof. On one hand, since $A \subseteq d(A) \cup A \in \mathcal{C}_d$, it follows that $co_d(A) = \bigcap \{C \in \mathcal{C}_d \mid A \subseteq C\} \subseteq d(A) \cup A$. On the other hand, if $C \in \mathcal{C}_d$ satisfying $A \subseteq C$, then $d(C) \subseteq C$, implying that $d(A) \cup A \subseteq d(C) \cup C = C$. Hence $co_d(A) = d(A) \cup A$. \square

Proposition 3.8. *Let $f : (X, d_X) \rightarrow (Y, d_Y)$ be a DP mapping between c -derived spaces. Then $f : (X, \mathcal{C}_{d_X}) \rightarrow (Y, \mathcal{C}_{d_Y})$ is a CP mapping.*

Proof. It's trivial by the Proposition 3.7. \square

By Proposition 3.6 and Proposition 3.8, we obtain a functor $\mathbb{F} : \mathbf{Conx} \rightarrow \mathbf{Deri}$ defined by

$$\mathbb{F}(X, d) = (X, \mathcal{C}_d) \text{ and } \mathbb{F}(f) = f.$$

4 Derived Operators Induced by Convexities

In this part, we show that every c -derived operator can be induced by a convex structure. Moreover, it is proved that the category of c -derived spaces is isomorphic to that of convex spaces.

Definition 4.1. Let (X, \mathcal{C}) be a convex space. Then for each $A \subseteq X$, the set $d_{\mathcal{C}}(A)$ defined as follows:

$$d_{\mathcal{C}}(A) = \{x \in X \mid x \in co(A - \{x\})\}$$

is called the c -derived set of A .

Proposition 4.2. *Let (X, \mathcal{C}) be a convex space with the corresponding hull operator co . Then the following statements hold for any $A \in \mathcal{P}(X)$.*

(1)

$$\begin{aligned} d_{\mathcal{C}}(A) &= \{x \in X \mid \forall C \in \mathcal{C}, A - \{x\} \subseteq C \Rightarrow x \in C\} \\ &= \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in co(F - \{x\})\} \\ &= \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in co(F) \cap (X - F)\}. \end{aligned}$$

(2) $co(A) = d_{\mathcal{C}}(A) \cup A$.

Proof. (1) First, it is trivial that $d_{\mathcal{C}}(A) = \{x \in X \mid \forall C \in \mathcal{C}, A - \{x\} \subseteq C \Rightarrow x \in C\}$. Let

$$K = \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in co(F - \{x\})\}.$$

Now we prove $d_{\mathcal{C}}(A) = K$. Take any $x \in K$. Then there exists $F \in \mathcal{P}_{fin}(A)$ such that $x \in co(F - \{x\})$. For each $C \in \mathcal{C}$, if $A - \{x\} \subseteq C$, then $F - \{x\} \subseteq C$. It follows $x \in co(F - \{x\}) \subseteq C$. This means $x \in d_{\mathcal{C}}(A)$ and thus $K \subseteq d_{\mathcal{C}}(A)$. Conversely, take any $x \in d_{\mathcal{C}}(A)$. Note that

$$A - \{x\} \subseteq \bigcup \{co(F - \{x\}) \mid F \in \mathcal{P}_{fin}(A)\} \in \mathcal{C}.$$

It follows from $co(A - \{x\}) \subseteq \bigcup \{co(F - \{x\}) \mid F \in \mathcal{P}_{fin}(A)\}$. Take any $x \in d_{\mathcal{C}}(A)$. Then

$$x \in \bigcup \{co(F - \{x\}) \mid F \in \mathcal{P}_{fin}(A)\}.$$

It follows $x \in co(F - \{x\})$ for some $F \in \mathcal{P}_{fin}(A)$. This means $x \in K$. We obtain $d_{\mathcal{C}}(A) \subseteq K$. Therefore

$$d_{\mathcal{C}}(A) = \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in co(F - \{x\})\}.$$

Moreover, an easy induction can obtain that

$$K = \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in co(F) \cap (X - F)\}.$$

(2) It suffices to prove $co(A) \subseteq d_{\mathcal{C}}(A) \cup A$. Take any $x \in co(A)$ and $x \notin A$. Then by (AC4) of Proposition 2.3, we obtain $x \in co(F)$ for some $F \in \mathcal{P}_{fin}(A)$. Note that $x \notin A$, which means $x \in co(F) \cap (X - F)$. By the conclusion of (1), we obtain $x \in d_{\mathcal{C}}(A)$. \square

Next, we will verify that the operator $d_{\mathcal{C}}$ is a c-derived operator on X . Before proving this, the following lemma is necessary.

Lemma 4.3. *Let (X, \mathcal{C}) be a convex space and let $d_{\mathcal{C}}$ be the operator induced by Definition 4.1. Then the following statements hold for any $A \in \mathcal{P}(X)$.*

- (1) *If $F \subseteq d_{\mathcal{C}}(A)$ is finite, then $F \subseteq co(G)$ for some $G \in \mathcal{P}_{fin}(A)$.*
- (2) *$d_{\mathcal{C}}(d_{\mathcal{C}}(A)) \subseteq d_{\mathcal{C}}(A) \cup A$.*

Proof. (1) For each $x \in F$, we have $x \in d_{\mathcal{C}}(A)$ which means $x \in co(F_x)$ for some $F_x \in \mathcal{P}_{fin}(A)$. Let $G = \bigcup_{x \in F} F_x$. Obviously, G is a finite subset of A satisfying $F \subseteq co(G)$.

(2) Assume $x \notin d_{\mathcal{C}}(A) \cup A$ and $x \in d_{\mathcal{C}}(d_{\mathcal{C}}(A))$. Then there exists a finite subset $F \subseteq d_{\mathcal{C}}(A)$ satisfying $x \in co(F)$. By (1), there exists $G \in \mathcal{P}_{fin}(A)$ satisfying $co(F) \subseteq co(G)$, which means $x \in co(G - \{x\})$. This shows that $x \in d_{\mathcal{C}}(A)$, a contradiction. \square

Proposition 4.4. *For a convex space (X, \mathcal{C}) , the operator $d_{\mathcal{C}}$ is a c-derived operator on X .*

Proof. The verification of (CD1) is trivial. It suffices to verify (CD2)–(CD4).

(CD2) Take any $x \in d_{\mathcal{C}}(A)$. By Proposition 4.2, we have $x \in co(F - \{x\})$ for some $F \in \mathcal{P}_{fin}(A)$. Let $E = F - \{x\}$. It follows that $E \in \mathcal{P}_{fin}(A - \{x\})$ and $x \in co(E) = co(E - \{x\})$. Hence $x \in d_{\mathcal{C}}(A - \{x\})$.

(CD3) Take any $x \in d_{\mathcal{C}}(d_{\mathcal{C}}(A) \cup A)$ and $x \notin A$. Then by Proposition 4.2, there exists a finite $F \subseteq d_{\mathcal{C}}(A) \cup A$ such that $x \in co(F) \cap (X - F)$. If $F \subseteq A$, then $x \in d_{\mathcal{C}}(A)$. The proof is completed. Otherwise, let $F_1 = F \cap d_{\mathcal{C}}(A)$ and let $F_2 = F \cap A$. Then F_1 is a nonempty finite set and $F = F_1 \cup F_2$. Since $F_1 \subseteq d_{\mathcal{C}}(A)$ and Lemma 4.3, there exists $G \in \mathcal{P}_{fin}(A)$ satisfying $F_1 \subseteq co(G)$. It follows that $x \in co(F) \subseteq co(co(G) \cup F_2) = co(G \cup F_2)$. Note that $G \cup F_2 \in \mathcal{P}_{fin}(A)$ and $x \notin G \cup F_2$, which means $x \in d_{\mathcal{C}}(A)$.

(CD4) On one hand, the inequality

$$\bigcup \{d_{\mathcal{C}}(F) \mid F \in \mathcal{P}_{fin}(A)\} \subseteq d_{\mathcal{C}}(A)$$

holds obviously. On the other hand, take any $x \in d_{\mathcal{C}}(A)$. Then there exists $E \in \mathcal{P}_{fin}(A)$ satisfying $x \in co(E - \{x\})$. This implies $x \in d_{\mathcal{C}}(E) \subseteq \bigcup \{d_{\mathcal{C}}(F) \mid F \in \mathcal{P}_{fin}(A)\}$. \square

Proposition 4.5. *Let $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ be a CP mapping between c-derived spaces. Then $f : (X, d_{\mathcal{C}_X}) \rightarrow (Y, d_{\mathcal{C}_Y})$ is a DP mapping.*

Proof. It suffices to prove $f(d_X(A)) \subseteq f(A) \cup d_Y(f(A))$ for all $A \subseteq X$. Since $f : (X, \mathcal{C}_X) \rightarrow (Y, \mathcal{C}_Y)$ is CP and Theorem 2.9, we have $f(co_X(A)) \subseteq co_Y(f(A))$. It follows that

$$f(d_X(A)) \subseteq f(d_X(A) \cup A) = f(co_X(A)) \subseteq co_Y(f(A)) = f(A) \cup d_Y(f(A)).$$

The proof is completed. \square

By Proposition 4.4 and Proposition 4.5, we obtain a functor $\mathbb{G} : \mathbf{Conv} \rightarrow \mathbf{Deri}$ defined by

$$\mathbb{G}(X, \mathcal{C}) = (X, d_{\mathcal{C}}) \text{ and } \mathbb{G}(f) = f.$$

Lemma 4.6. *Let (X, \mathcal{C}) be a convex space and $(X, d_{\mathcal{C}})$ be the c -derived space generated by \mathcal{C} . Then we have*

$$d_{\mathcal{C}}(A) \subseteq A \text{ if and only if } co(A) \subseteq A.$$

Proof. Assume $co(A) \subseteq A$. Take any $x \in d_{\mathcal{C}}(A)$. Then there exists $F \in \mathcal{P}_{fin}(A)$ satisfying $x \in co(F) \subseteq co(A) \subseteq A$. Conversely, Assume $d_{\mathcal{C}}(A) \subseteq A$. Then for each $x \in co(A) \cap (X - A)$, there exists $F \in \mathcal{P}_{fin}(A)$ satisfying $x \in co(F) \cap (X - F)$. This shows that $x \in d_{\mathcal{C}}(A) \subseteq A$, a contradiction. \square

Now we give the main result in this section.

Theorem 4.7. *The category \mathbf{Deri} is isomorphic to \mathbf{Conv} .*

Proof. We need to prove $\mathbb{G} \circ \mathbb{F} = \mathbb{I}_{\mathbf{Deri}}$ and $\mathbb{F} \circ \mathbb{G} = \mathbb{I}_{\mathbf{Conv}}$. It suffices to verify (1) $d_{\mathcal{C}_d} = d$ and (2) $\mathcal{C}_{d_{\mathcal{C}}} = \mathcal{C}$ for any c -derived space (X, d) and convex space (X, \mathcal{C}) .

For (1), take any $A \subseteq X$. Since $co_d(A) = d(A) \cup A$, we have

$$\begin{aligned} d_{\mathcal{C}_d}(A) &= \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in co_d(F - \{x\})\} \\ &= \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in d(F - \{x\}) \cup (F - \{x\})\} \\ &= \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in d(F - \{x\})\} \\ &= \{x \in X \mid \exists F \in \mathcal{P}_{fin}(A), x \in d(F)\} \\ &= \bigcup \{d(F) \mid F \in \mathcal{P}_{fin}(A)\} = d(A). \end{aligned}$$

For (2), let co be the hull operator on (X, \mathcal{C}) . By Lemma 4.6, we have

$$\mathcal{C}_{d_{\mathcal{C}}} = \{C \subseteq X \mid d_{\mathcal{C}}(C) \subseteq C\} = \{C \subseteq X \mid co(C) \subseteq C\} = \mathcal{C}.$$

\square

Proposition 4.8. *Let (X, \mathcal{C}) be a convex space and let $A \in \mathcal{P}(X)$. Then $x \in d_{\mathcal{C}}(A)$ if and only if $x \in co(A)$ and $co(A) = co(A - \{x\})$.*

Proof. Necessity. Assume that $x \in d_{\mathcal{C}}(A)$. Clearly, $x \in co(A)$. It remains to show $A \cup d_{\mathcal{C}}(A) \subseteq co(A - \{x\})$. Take any $y \in A \cup d_{\mathcal{C}}(A)$. If $y = x$, then $y \in d_{\mathcal{C}}(A)$ and hence $y \in d_{\mathcal{C}}(A - \{y\}) \subseteq co(A - \{y\}) = co(A - \{x\})$. If $y \neq x$ and $y \in A$, then $y \in A - \{x\} \subseteq co(A - \{x\})$. Now assume $y \neq x$, $y \in d_{\mathcal{C}}(A)$ and $y \notin A$. It follows from $y \in d_{\mathcal{C}}(A)$ that there exists $F \in \mathcal{P}_{fin}(A)$ such that $y \in co(F)$. If $x \notin F$, then $F \subseteq A - \{x\}$. This means $y \in co(F - \{x\}) \subseteq co(A - \{x\})$. If $x \in F$, then by $x \in d_{\mathcal{C}}(A)$, there exists $G \in \mathcal{P}_{fin}(A)$ such that $x \in co(G - \{x\})$. Furthermore, we obtain

$$\begin{aligned} y &\in co((F - \{x\}) \cup \{x\}) \subseteq co((F - \{x\}) \cup co(G - \{x\})) \\ &= co((F - \{x\}) \cup (G - \{x\})) = co((F \cup G) - \{x\}) \subseteq co(A - \{x\}). \end{aligned}$$

Therefore $co(A) = A \cup d_{\mathcal{C}}(A) \subseteq co(A - \{x\})$.

Sufficiency Assume that $x \in co(A)$ and $co(A) = co(A - \{x\})$. Then $x \in co(A - \{x\})$, and hence by Definition 4.1 we have $x \in d_{\mathcal{C}}(A)$. \square

Theorem 4.9. *Let (X, \mathcal{C}) be a convex space and let A be a nonempty set of X . Then A is convexly independent (i.e., $x \notin co(A - \{x\})$ for all $x \in A$) if and only if $A \cap d_{\mathcal{C}}(A) = \emptyset$.*

Proof. It is straightforward by Proposition 4.8. \square

Before proceeding to the next section, we show some examples.

Example 4.10. (1) Let P be a poset. A subset C is called *order convex* provided

$$z \in C \text{ whenever } x \leq z \leq y \text{ and } x, y \in C.$$

Then the family $\mathcal{C}(P)$ of all order convex sets forms a convex structure on P , and it is trivial to check that for every subset A of P , the c -derived set

$$d(A) = co(A) - \min(A) \cup \max(A),$$

where $\min(A)$ is the minimal element of A and $\max(A)$ is the maximal element of A respectively. As $\min(A) \subseteq A$ and $\max(A) \subseteq A$, it holds that $co(A) = A \cup d(A)$.

(2) Let $X = \{a_i \mid i = 1, 2, 3, 4, 5\}$ be a metric space, and the metric δ on X is defined as figure 1. Note that $\delta(a_1, a_5) = \delta(a_2, a_4) = 3$. A subset C of X is called *geodesically convex* provided

$$x \in C \text{ whenever } \delta(a, x) + \delta(x, b) = \delta(a, b) \text{ and } a, b \in C.$$

Consider $A = \{a_1, a_5\}$, it is trivial to check that $co(A) = X$, $d(A) = X - A$ and thus $co(A) = A \cup d(A)$.

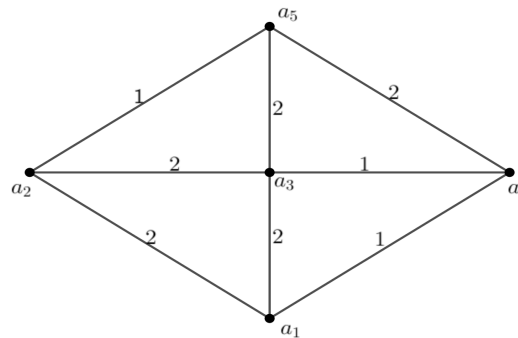


Figure 1: The metric space (X, δ) .

(3) Let V be a vector space over a field \mathbb{K} and let V_0 be the set V minus the zero vector 0 . A nonempty set $C \subseteq V_0$ is called *linear convex* provided

$$s \cdot p + t \cdot q \in C \text{ whenever } p, q \in C \text{ and } s, t \in \mathbb{K}.$$

Then the family of all linear convex sets forms a convex structure on V_0 , and it is trivial to check that for any subset A of V_0 ,

$$co(A) = \left\{ \sum_{i=1}^n t_i \cdot p_i \mid p_1, p_2, \dots, p_n \in A, t_1, t_2, \dots, t_n \in \mathbb{K} \right\}, \text{ and hence} \\ x \in d(A) \text{ if and only if } x = \sum_{i=1}^n t_i \cdot p_i \text{ for some } p_1, p_2, \dots, p_n \in A - \{p\}, t_1, t_2, \dots, t_n \in \mathbb{K}.$$

5 Restricted c -derived operators

In convexity theory, a notable result is that every convex structure can be completely determined by the polytopes (the hull of finite sets). This property entails a new operator, called *restricted hull operator*, by restricting the hull operator to the family of all finite sets. Motivated by this, we present the notion of restricted c -derived operators, and establish its relationship to convex structures.

A *restricted hull operator* [11] is a mapping $h : \mathcal{P}_{fin}(X) \rightarrow \mathcal{P}(X)$ satisfying the following conditions:

- (H1) $h(\emptyset) = \emptyset$;
 (H2) for any $F \in \mathcal{P}_{fin}(X)$, $F \subseteq h(F)$;
 (H3) for any $F, G \in \mathcal{P}_{fin}(X)$, $G \subseteq h(F)$ implies $h(G) \subseteq h(F)$.

It is known that every restricted hull operator can uniquely determine a convex structure [11].

Definition 5.1. A mapping $\bar{d} : \mathcal{P}_{fin}(X) \rightarrow \mathcal{P}(X)$ is called a *restricted c-derived operator* provided for any $F, G \in \mathcal{P}_{fin}(X)$, it satisfies the following conditions:

- (RD1) $\bar{d}(\emptyset) = \emptyset$;
 (RD2) $F \subseteq G$ implies $\bar{d}(F) \subseteq \bar{d}(G)$;
 (RD3) $x \in \bar{d}(F)$ implies $x \in \bar{d}(F - \{x\})$;
 (RD4) $G \subseteq \bar{d}(F)$ implies $\bar{d}(G \cup F) \subseteq \bar{d}(F) \cup F$.

Proposition 5.2. Let $\bar{d} : \mathcal{P}_{fin}(X) \rightarrow \mathcal{P}(X)$ be a restricted c-derived operator. Then the mapping $h_{\bar{d}} : \mathcal{P}_{fin}(X) \rightarrow \mathcal{P}(X)$ defined by

$$\forall F \in \mathcal{P}_{fin}(X), \quad h_{\bar{d}}(F) := F \cup \bar{d}(F)$$

is a restricted hull operator.

Proof. The verifications of (H1) and (H2) are trivial. For (H3), Assume that $F, G \in \mathcal{P}_{fin}(X)$ and $G \subseteq h_{\bar{d}}(F)$. Let $G_1 = G \cap F$ and $G_2 = G - G_1$. Since $G \subseteq h_{\bar{d}}(F) = \bar{d}(F) \cup F$, then $G_2 \subseteq \bar{d}(F)$. By (RD4), we know $\bar{d}(G_2 \cup F) \subseteq \bar{d}(F) \cup F$. Furthermore by (RD2), we have

$$\bar{d}(G) = \bar{d}(G_1 \cup G_2) \subseteq \bar{d}(F \cup G_2) \subseteq \bar{d}(F) \cup F = h_{\bar{d}}(F).$$

Hence, $h_{\bar{d}}(G) = \bar{d}(G) \cup G \subseteq h_{\bar{d}}(F)$. □

Lemma 5.3. Every restricted hull operator h is order-preserving, that is, for any $F, G \in \mathcal{P}_{fin}(X)$,

$$F \subseteq G \Rightarrow h(F) \subseteq h(G).$$

Proposition 5.4. Let h be a restricted hull operator on X . Then the mapping $\bar{d}_h : \mathcal{P}_{fin}(X) \rightarrow \mathcal{P}(X)$ defined by

$$\forall F \in \mathcal{P}_{fin}(X), \quad \bar{d}_h(F) := \{x \mid x \in h(F - \{x\})\}$$

is a restricted c-derived operator.

Proof. By (H1), (H2) and Lemma 5.3, the verifications of (RD1)–(RD3) are straightforward. For (RD4), assume that $G \subseteq \bar{d}_h(F)$ and $x \in \bar{d}_h(G \cup F)$. If $x \in F$, then we are done. So assume $x \notin F$. Since $G \subseteq \bar{d}_h(F)$, we have $g \in h(F - \{g\})$ for all $g \in G$. It follows that $G \subseteq h(F)$. By (H2) and (H3), $F \cup G \subseteq h(F)$ and hence $h(F \cup G) \subseteq h(F)$. Note that $x \notin F$, which shows $F = F - \{x\}$. Therefore, we have $x \in h(F \cup G) \subseteq h(F) = h(F - \{x\})$. This implies $x \in \bar{d}_h(F)$. □

Theorem 5.5. The relationship between restricted c-derived operators and restricted hull operators is bijective.

Proof. Let (X, \bar{d}) be a c-derived space and let (X, h) be a restricted hull space. It suffices to prove (1) $\bar{d}_{h_{\bar{d}}} = \bar{d}$ and (2) $h_{\bar{d}_h} = h$.

(1) For each $F \in \mathcal{P}_{fin}(X)$, we have

$$\begin{aligned} \bar{d}_{h_{\bar{d}}}(F) &= \{x \mid x \in h_{\bar{d}}(F - \{x\})\} = \{x \mid x \in F - \{x\} \cup \bar{d}(F - \{x\})\} \\ &= \{x \mid x \in \bar{d}(F - \{x\})\} = \{x \mid x \in \bar{d}(F)\} = \bar{d}(F). \end{aligned}$$

(2) Take any $F \in \mathcal{P}_{fin}(X)$. On one hand, we have

$$h_{\bar{d}_h}(F) = F \cup \bar{d}_h(F) = \{x \mid x \in F \cup h(F - \{x\})\} \subseteq h(F).$$

On the other hand, take any $x \in h(F)$. If $x \in F$, then we are done. So assume $x \notin F$, and $x \in h(F) = h(F - \{x\}) = \bar{d}_h(F) \subseteq h_{\bar{d}_h}(F)$. Hence, $h_{\bar{d}_h}(F) = h(F)$. \square

Theorem 5.6. *Let \bar{d} be a restricted c -derived operator on X . Then there is precisely one convex structure on X with a c -derived operator equal to \bar{d} on $\mathcal{P}_{fin}(X)$. Conversely, the c -derived operator of any convex structure on X satisfies the conditions (RD1)-(RD4).*

Proof. Let

$$\mathcal{C}_{\bar{d}} = \left\{ C \mid F \in \mathcal{P}_{fin}(C) \Rightarrow \bar{d}(F) \subseteq C \right\}.$$

It is easy to check $\mathcal{C}_{\bar{d}}$ is a convex structure on X . We prove the conclusion in three steps:

Step 1. We prove $co(A) = \bigcup \left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\}$ (specially, $co(A) = \bar{d}(A) \cup A$ whenever A is finite). Note that $\left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\}$ is directed. Then for any finite set $G \subseteq \bigcup \left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\}$, there exists $H \in \mathcal{P}_{fin}(A)$ satisfying $G \subseteq \bar{d}(H) \cup H$. It follows that

$$\bar{d}(G) \subseteq \bar{d}(\bar{d}(H) \cup H) \subseteq \bar{d}(H) \cup H \subseteq \bigcup \left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\},$$

which means $\bigcup \left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\} \in \mathcal{C}_{\bar{d}}$. Therefore, $co(A) \subseteq \bigcup \left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\}$. The converse $\bigcup \left\{ \bar{d}(F) \cup F \mid F \in \mathcal{P}_{fin}(A) \right\} \subseteq co(A)$ is trivial.

Step 2. For any $F \in \mathcal{P}_{fin}(X)$, we have

$$\begin{aligned} d_{\mathcal{C}_{\bar{d}}}(F) &= \{x \in X \mid x \in co(F - \{x\})\} = \left\{ x \in X \mid x \in \bar{d}(F - \{x\}) \cup (F - \{x\}) \right\} \\ &= \left\{ x \in X \mid x \in \bar{d}(F - \{x\}) \right\} = \left\{ x \in X \mid x \in \bar{d}(F) \right\} = \bar{d}(F). \end{aligned}$$

Step 3. We prove the uniqueness of $\mathcal{C}_{\bar{d}}$. If there exists another convex structure \mathcal{C}^* on X with a c -derived operator d^* equal to \bar{d} on $\mathcal{P}_{fin}(X)$. Then for any $A \subseteq X$, we have

$$d^*(A) = \bigcup \left\{ d^*(F) \mid F \in \mathcal{P}_{fin}(A) \right\} = \bigcup \left\{ \bar{d}(F) \mid F \in \mathcal{P}_{fin}(A) \right\} = \bigcup \left\{ d(F) \mid F \in \mathcal{P}_{fin}(A) \right\} = d(A).$$

Then by Theorem 4.7, $\mathcal{C}_{\bar{d}} = \mathcal{C}^*$. \square

6 An application to anti-matroids

In this section, we will investigate the relationship between c -derived operators and Shi's m -derived operators on matroids, and present an application of c -derived operators to anti-maroids.

In a vector space V , let $E \in \mathcal{P}_{fin}(V)$ and let d be a mapping on $\mathcal{P}(E)$ defined by

$$d(A) = \{x \in E \mid x \text{ is the linear combination of } A - \{x\}\}.$$

for all $A \subseteq E$. Xin and Shi [14] generalized this operator as follows.

Let E be a finite set. A mapping d on $\mathcal{P}(E)$ is called a *matroid derived operator* (*m-derived operator* for short) [14] if d satisfies the following conditions:

- (D1) $d(\emptyset) = \emptyset$;
- (D2) $A \subseteq B \Rightarrow d(A) \subseteq d(B)$;
- (D3) $x \in d(A) \Rightarrow x \in d(A - \{x\})$;
- (D4) $d(d(A) \cup A) \subseteq d(A) \cup A$;
- (D5) $y \in d(A) - d(A - \{x\}) \Rightarrow x \in d((A - \{x\}) \cup \{y\})$.

Remark 6.1. Since E is a finite set, the Algebraic Law (see (CD4) in Definition 3.1) always holds. That is to say every m -derived operator on a finite set is a c -derived operator.

Next we will show that the m -derived operators are exactly the c -derived operators satisfying the Exchange Law.

Theorem 6.2. *A c -derived operator d is an m -derived operator on a finite set E if and only if it satisfies the following Exchange Law:*

(CD5) *If $p \notin C \cup d(C)$, then $p \in d(C \cup \{q\})$ implies $q \in d(C \cup \{p\})$.*

Proof. Necessity. It suffices to verify (CD5). Since $q \notin C$ (otherwise $p \in d(C)$), we have $C = C \cup \{q\} - \{q\}$, implying that $p \in d(C \cup \{q\}) - d((C \cup \{q\}) - \{q\})$. Hence by (D5), we obtain $q \in d((C \cup \{q\} - \{q\}) \cup \{p\}) = d(C \cup \{p\})$.

Sufficiency. It suffices to verify (D5). Take any $y \in d(A) - d(A - \{x\})$. By (D3), we have $y \in d(A - \{y\}) - d(A - \{y, x\})$. Since $y \notin A - \{y, x\}$, by (CD5) we obtain $x \in d((A - \{x, y\}) \cup \{y\}) = d((A - \{x\}) \cup \{y\})$. \square

Theorem 6.3. *Let (X, \mathcal{C}) be a convex space and let d be the induced c -derived operator. Then d satisfies condition (CD5) if and only if \mathcal{C} satisfies the Exchange Law:*

if $p, q \notin co(A)$, then $p \in co(A \cup \{q\})$ implies $q \in co(A \cup \{p\})$.

Proof. Necessity. Note that $co(A) = d(A) \cup A$. It's trivial if $p = q$. Now assume $p \neq q$. Since $p \in co(A \cup \{q\}) = A \cup \{q\} \cup d(A \cup \{q\})$ and $p \notin A \cup \{q\}$, we know $p \in d(A \cup \{q\})$. By (CD5), we have $q \in d(A \cup \{p\}) \subseteq co(A \cup \{p\})$.

Sufficiency. Take any $p \in d(C \cup \{q\}) - (C \cup d(C))$. If $q \in d(C)$, then we are done. So assume $q \notin d(C)$. If $q \in C$, then $p \in d(C \cup \{q\}) = d(C)$, a contradiction. This shows $q \notin C \cup d(C)$. Since $p \in d(C \cup \{q\}) \subseteq co(C \cup \{q\})$, by the Exchange Law, we obtain $q \in co(C \cup \{p\}) = C \cup \{p\} \cup d(C \cup \{p\})$. Since $q \notin C$ and $p \neq q$ (otherwise by (CD2), $p \in d(C \cup \{q\})$ implies $p \in d(C)$), we have $q \in d(C \cup \{p\})$. \square

Corollary 6.4. *Let (X, \mathcal{C}) be a convex space and let $d_{\mathcal{C}}$ be the induced c -derived operator. Then $d_{\mathcal{C}}$ satisfies (CD5) if and only if \mathcal{C} is a matroid.*

Theorem 6.5. *Let (E, \mathcal{C}) be an anti-matroid. Then $co(A) = co(A - d(A))$. In particular, $co(C - d(C)) = C$ for all $C \in \mathcal{C}$.*

Proof. Take any $x \in d(A)$. By Proposition 4.8, we have $co(A - \{x\}) = co(A)$. Further, by Proposition 2.5 $co(A) = co\left(\bigcap_{x \in d(A)} (A - \{x\})\right) = co(A - d(A))$. \square

Proposition 6.6. *Let (E, \mathcal{C}) be anti-matroid. Then for any $A \subseteq E$ and $C \in \mathcal{C}$, the following statements hold.*

- (1) $C - d(C)$ is the generator of C .
- (2) $A - d(A)$ is the generator of $co(A)$.
- (3) $A - d(A) = co(A) - d(co(A))$.

Proof. (1) By Theorem 6.5, we know $co(C - d(C)) = C$. This means that $C - d(C)$ generates C . Furthermore, assume $co(B) = C$ and $x \in C - d(C)$. If $x \notin B$, then $x \in d(B) \subseteq d(C)$, a contradiction. Thus $x \in B$, implying that $C - d(C) \subseteq B$. Therefore, $C - d(C)$ is the generator of C .

(2) Assume that $x \in A - d(A)$ satisfying

$$co(A) = co((A - d(A)) - \{x\}) = co(A - (d(A) \cup \{x\})).$$

Note that

$$x \notin (A - (d(A) \cup \{x\})) \cup d(A - (d(A) \cup \{x\})) = co(A - (d(A) \cup \{x\})) = co(A),$$

a contradiction to $x \in co(A)$. It follows that $co(F) \neq co(A)$ for all $F \subsetneq A - d(A)$. Now assume $co(B) = co(A)$. Note that $co(A - d(A)) = co(A)$ by Theorem 6.5, and thus $co((A - d(A)) \cap B) = co(A)$ by Proposition 2.5. It follows that $A - d(A) = (A - d(A)) \cap B$, and hence $A - d(A) \subseteq B$. Therefore $A - d(A)$ is the generator of $co(A)$.

(3) It's trivial by the result (1), (2) and Proposition 2.7. □

The following result shows that a convex set is completely determined by the complement of its c -derived set, and the proof is trivial by Proposition 2.7 and Proposition 6.6.

Theorem 6.7. *If \mathcal{C} is a convex structure on a finite set E , then the following statements are equivalent.*

- (1) (E, \mathcal{C}) is anti-matroid.
- (2) (E, \mathcal{C}) is uniquely generated.
- (3) For any $C \in \mathcal{C}$, $co(C - d(C)) = C$.
- (4) For any $C \in \mathcal{C}$, $C - d(C)$ is the generator of C .
- (5) For any $A \in \mathcal{P}(E)$, $co(A - d(A)) = co(A)$.
- (6) For any $A \in \mathcal{P}(E)$, $A - d(A)$ is the generator of $co(A)$.

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