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Triangular Surface Patch Based on Bivariate Meyer-König-Zeller Operator

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Abstract: Based on the relationship between probability operators and curve/surface modeling, a new kind of surface modeling method is introduced in this paper. According to a kind of bivariate Meyer-König-Zeller operator, we study the corresponding basis functions called triangular Meyer-König-Zeller basis functions which are defined over a triangular domain. The main properties of the basis functions are studied, which guarantee that the basis functions are suitable for surface modeling. Then, the corresponding triangular surface patch called a triangular Meyer-König-Zeller surface patch is constructed. We prove that the new surface patch has the important properties of surface modeling, such as affine invariance, convex hull property and so on. Finally, based on given control vertices, whose number is finite, a truncated triangular Meyer-König-Zeller surface and a redistributed triangular Meyer-König-Zeller surface are constructed and studied.

Keywords: Surface modeling, Meyer-König-Zeller operator, basis function, triangular surface

1 Introduction

In computer aided geometric design (CAGD), representing a parametric curve or surface with shape preserving is important. Essentially, the shape preserving is guaranteed by the partition of unity and non-negativity of the basis functions which are used to construct the parametric curve or surface. As we all know, shape preserving is the main property of probability operators. Thus, the shape preserving construction methods of parametric curves or surfaces have certain correlations with some probability operators. It is easy to realize that the classical Bézier curve [1] constructed by Bernstein basis functions is related to the Bernstein operator B_n [2] defined for any function $f \in C[0, 1]$,

$$B_n(f; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(\frac{k}{n}), \quad x \in [0, 1]. \tag{1}$$

In recent years, based on the Phillips q-Bernstein operator [3], which is a generalization of the Bernstein operator, generalized Bézier curves and surfaces have been introduced in [4–6]. In [4], Oruç and Phillips constructed q-Bézier curves by the basis functions of Phillips q-Bernstein operator. Dişibüyük and Oruç [5, 6] defined the q-generalization of rational Bernstein-Bézier curves and tensor product q-Bernstein-Bézier surfaces. Moreover, Simeonov et al. [7] introduced a new variant of the blossom, the q-blossom, which is specifically adapted to developing identities and algorithms for q-Bernstein bases and q-Bézier curves. In

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2014, Han et al. [8] constructed a new generalization of Bézier curves and its corresponding tensor product surfaces based on Lupaş q-analogue of the Bernstein operator [9].

We realized directly the relationship between several probability operators and some curve modeling methods. For example, the rational Bézier curve of negative degree [10] constructed by Bernstein basis functions of negative degree is related to Baskakov operators $\mathcal{B}_n$ [11] defined for any function $f \in C[0, \infty)$

$$\mathcal{B}_n(f;t) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} f(\frac{k}{n}), \quad x \in [0, \infty). \tag{2}$$

The Poisson curve [12] constructed by Poisson basis functions is related to Szász-Mirakyan operators M_n [13] defined for any function $f \in C[0, \infty)$

$$M_n(f,x) = \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} e^{-nx}, \quad x \in [0, \infty). \tag{3}$$

Goldman introduced the connection between probability theory and computer-aided geometric design in [14–16]. Fan and Zeng [17] presented a class of discrete distributions called S-λ distributions and constructed the corresponding S-λ basis functions from these distributions. Zhou et al. [18] extended the work of Fan and Zeng to surface modeling and constructed the tensor product S-λ basis functions and triangular S-λ basis function.

Therefore, we can construct a new modeling method based on the connection between probability operators and computer-aided geometric design. In 1960, Meyer König and Zeller [19] presented a univariate operator

$$M_n(f;x) = \begin{cases} f(1), & x = 1; \\ \sum\limits_{k=0}^{\infty} m_{n,k}(x) f(\frac{k}{n+k}), & 0 \le x < 1; \end{cases} \tag{4}$$

where $m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}$, $f \in C[0,1]$, called Meyer-König-Zeller operator. Xiong and Yang [20] introduced a kind of bivariate Meyer-König-Zeller operator which is defined over a triangular domain. For any function $f \in C[\Delta]$, $\Delta = \{(x,y) | 0 \le y \le x \le 1\}$,

$$M_n(f;x,y) = \begin{cases} f(1,y) & x = 1; \\ \sum\limits_{k=0}^{\infty} \sum\limits_{l=0}^{k} P_{n,k,l}(x,y) f(\frac{k}{n+k}, \frac{l}{n+k}), & x \neq 1; \end{cases} \tag{5}$$

where, $(x,y) \in \Delta$, $P_{n,k,l}(x,y) = \binom{n+k}{k}\binom{k}{l}(1-x)^{n+1} y^l (x-y)^{k-l}$. In this paper, we introduce a new surface modeling method by the bivariate Meyer-König-Zeller operator.

The remainder of this paper is organized as follows. In Section 2, we present triangular Meyer-König-Zeller basis functions by the bivariate Meyer-König-Zeller operator defined over a triangular domain (5), and study their main properties. In Section 3, we construct a triangular Meyer-König-Zeller surface patch by the triangular Meyer-König-Zeller basis functions, and prove that the new surface has the main surface modeling properties. For given control vertices, whose number is finite, we introduce a truncated triangular Meyer-König-Zeller surface and a redistributed triangular Meyer-König-Zeller surface in Section 4.

2 Triangular Meyer-König-Zeller basis functions

In this section, the definition and several main properties of bivariate Meyer-König-Zeller basis functions over a triangular domain will be given.

Firstly, we given the barycentric coordinates $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, $\mu_i \ge 0\,(i = 1, 2, 3)$ and $\mu_1 + \mu_2 + \mu_3 = 1$.

Definition 2.1. *Bivariate Meyer-König-Zeller basis functions over a triangle called triangular Meyer-König-Zeller basis functions are defined by*

$$P_{n,k,l}(\boldsymbol{\mu}) = \binom{n+k}{k}\binom{k}{l}\mu_1^{n+1}\mu_2^{l}\mu_3^{k-l}; \tag{6}$$

where $\boldsymbol{\mu} \in D = \{\mu_1 > 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_1 + \mu_2 + \mu_3 = 1\}$ *and* $k,\ l$ *are natural numbers.*

Several illustrations of bivariate Meyer-König-Zeller basis functions with $n = 0, 2, 3$ and $k = 3$ are shown in Figure 1. Triangular Meyer-König-Zeller basis functions have the properties which are similar to the bivariate Bernstein basis functions over a triangle.

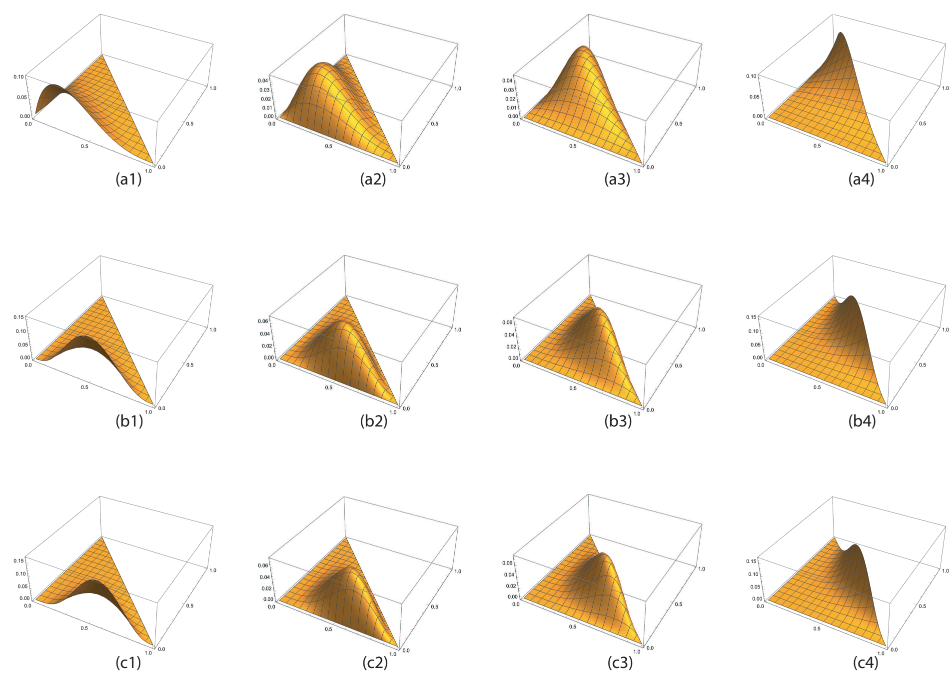

Figure 1: Triangular Meyer-König-Zeller basis functions with $n = 0, 2, 3$ and $k = 3$ which are $P_{0,3,0}(\boldsymbol{\mu})$ (a1), $P_{0,3,1}(\boldsymbol{\mu})$ (a2), $P_{0,3,2}(\boldsymbol{\mu})$ (a3), $P_{0,3,3}(\boldsymbol{\mu})$ (a4), $P_{2,3,0}(\boldsymbol{\mu})$ (b1), $P_{2,3,1}(\boldsymbol{\mu})$ (b2), $P_{2,3,2}(\boldsymbol{\mu})$ (b3), $P_{2,3,3}(\boldsymbol{\mu})$ (b4), $P_{3,3,0}(\boldsymbol{\mu})$ (c1), $P_{3,3,1}(\boldsymbol{\mu})$ (c2), $P_{3,3,2}(\boldsymbol{\mu})$ (c3), $P_{3,3,3}(\boldsymbol{\mu})$ (c4).

It is obvious from the definition of $P_{n,k,l}(\boldsymbol{\mu})$, that it has non-negative properties.

Proposition 2.1 (Non-negative)**.**

$$P_{n,k,l}(\boldsymbol{\mu}) \geq 0. \tag{7}$$

Proposition 2.2 (Partition of Unity)**.**

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) = 1. \tag{8}$$

Proof. From Binomial expansion, it is obvious that

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) &= \sum_{k=0}^{\infty}\binom{n+k}{k}\mu_1^{n+1}\sum_{l=0}^{k}\binom{k}{l}\mu_2^{l}\mu_3^{k-l}\\ &= \sum_{k=0}^{\infty}\binom{n+k}{k}\mu_1^{n+1}(\mu_2+\mu_3)^{k}\end{aligned}$$

$$= \sum_{k=0}^{\infty} \binom{n+k}{k} \mu_1^{n+1}(1-\mu_1)^k.$$

Let $\mu_1 = \frac{1}{1+t}$, $t \in [0, \infty)$, then

$$\begin{aligned}\sum_{k=0}^{\infty} \binom{n+k}{k} \mu_1^{n+1}(1-\mu_1)^k &= \sum_{k=0}^{\infty} \binom{n+k}{k} (1+t)^{-n-k-1} t^k \\ &= \sum_{k=0}^{\infty} \binom{-n-1}{k} (1+t)^{-n-1-k} (-t)^k \\ &= (1+t-t)^{-n-1} = 1,\end{aligned}$$

where $\binom{-n-1}{k} = (-1)^k \binom{n+k}{k}$. Thus, the sum of all triangular Meyer-König-Zeller basis functions equals to 1. □

Proposition 2.3 (Interpolation)**.**

$$P_{n,k,l}(1, 0, 0) = \delta_{0,k}\delta_{k,l}. \tag{9}$$

where $\delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$ *is the Kronecker delta.*

Proof. Since

$$P_{n,k,l}(1, 0, 0) = \binom{n+k}{k} \binom{k}{l} 1^{n+1} 0^l 0^{k-l},$$

when $l = 0$ and $k = l = 0$, $P_{n,0,0}(1, 0, 0) = 1$. It is obvious that $P_{n,k,l}(1, 0, 0) = 0$ in other cases. □

Proposition 2.4 (Linear independence)**.**

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} c_{k,l} P_{n,k,l}(\boldsymbol{\mu}) = 0 \Longleftrightarrow c_{k,l} = 0. \tag{10}$$

Proof. Since $0 < \mu_1 \leq 1$ and

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} c_{k,l} P_{n,k,l}(\boldsymbol{\mu}) = \sum_{k=0}^{\infty} \binom{n+k}{k} \mu_1^{n+1} \sum_{l=0}^{k} c_{k,l} \binom{k}{l} \mu_2^l \mu_3^{k-l},$$

we get

$$\sum_{k=0}^{\infty} \sum_{l=0}^{k} c_{k,l} P_{n,k,l}(\boldsymbol{\mu}) = 0 \Longleftrightarrow \sum_{l=0}^{k} c_{k,l} \binom{k}{l} \mu_2^l \mu_3^{k-l} = 0,$$

by fixing the value of μ_1. As we know, $\binom{k}{l}\mu_2^l\mu_3^{k-l}$ is a univariate Bernstein polynomial over $[0, 1-\mu_1]$. Thus,

$$\sum_{l=0}^{k} c_{k,l} \binom{k}{l} \mu_2^l \mu_3^{k-l} = 0 \Longleftrightarrow c_{k,l} = 0.$$

□

Proposition 2.5 (Boundary)**.** *When* $\mu_2 = 0$ *or* $\mu_3 = 0$, *triangular Meyer-König-Zeller basis functions transform to corresponding univariate Bernstein basis functions of negative degree [10].*

Proof. When $\mu_2 = 0$,

$$P_{n,k,0}(\mu_1, 0, \mu_3) = \binom{n+k}{k} \binom{k}{0} \mu_1^{n+1} \mu_3^k = \binom{n+k}{k} \mu_1^{n+1} (1-\mu_1)^k.$$

Let $\mu_1 = \dfrac{1}{1-t}$, $t \in (-\infty, 0]$, then

$$\begin{aligned}P_{n,k,0}(\mu_1, 0, \mu_3) &= \binom{n+k}{k}(1-t)^{-n-1-k}(-t)^k \\ &= \binom{-n-1}{k}(1-t)^{-n-1-k}t^k = B_k^{-n-1}(t),\end{aligned}$$

where, $B_k^{-n-1}(t)$ is a univariate Bernstein basis function of negative degree [10]. Similarly, $P_{n,k,k}(\mu_1, \mu_2, 0)$ is also a univariate Bernstein basis function of negative degree. □

Proposition 2.6 (Symmetry)**.**

$$P_{n,k,l}(\mu_1, \mu_2, \mu_3) = P_{n,k,k-l}(\mu_1, \mu_3, \mu_2).$$

Proof. Since $\binom{k}{l} = \binom{k}{k-l}$,

$$\begin{aligned}P_{n,k,l}(\boldsymbol{\mu}) &= \binom{n+k}{k}\binom{k}{l}\mu_1^{n+1}\mu_2^l\mu_3^{k-l} \\ &= \binom{n+k}{k}\binom{k}{k-l}\mu_1^{n+1}\mu_3^{k-l}\mu_2^{k-(k-l)} \\ &= P_{n,k,k-l}(\mu_1, \mu_3, \mu_2).\end{aligned}$$

□

Proposition 2.7 (Degree Reduction)**.**

$$P_{n,k,l}(\boldsymbol{\mu}) = \mu_1 P_{n-1,k,l}(\boldsymbol{\mu}) + \mu_2 P_{n,k-1,l-1}(\boldsymbol{\mu}) + \mu_3 P_{n,k-1,l}(\boldsymbol{\mu}), \tag{11}$$

where $P_{n,k,l}(\boldsymbol{\mu}) \equiv 0$, if one of n, k, l is negative or $k < l$.

Proof. We observe that

$$\binom{n+k}{k}\binom{k}{l} = \frac{(n+k)!}{n!k!}\frac{k!}{l!(k-l)!} = \frac{(n+k)!}{n!\,l!\,(k-l)!}.$$

By the definition of triangular Meyer-König-Zeller basis functions,

$$\begin{aligned}P_{n,k,l}(\boldsymbol{\mu}) &= \frac{(n+k-1)!(n+l+k-l)}{n!\,l!\,(k-l)!}\mu_1^{n+1}\mu_2^l\mu_3^{k-l} \\ &= \mu_1\frac{(n-1+k)!}{(n-1)!\,l!\,(k-l)!}\mu_1^n\mu_2^l\mu_3^{k-l} + \mu_2\frac{(n+k-1)!}{n!\,(l-1)!\,(k-l)!}\mu_1^{n+1}\mu_2^{l-1}\mu_3^{k-l} + \mu_3\frac{(n+k-1)!}{n!\,l!\,(k-1-l)!}\mu_1^{n+1}\mu_2^l\mu_3^{k-1-l} \\ &= \mu_1 P_{n-1,k,l}(\boldsymbol{\mu}) + \mu_2 P_{n,k-1,l-1}(\boldsymbol{\mu}) + \mu_3 P_{n,k-1,l}(\boldsymbol{\mu}).\end{aligned}$$

□

Proposition 2.8 (Degree Elevation)**.**

$$P_{n,k,l}(\boldsymbol{\mu}) = \frac{n+1}{n+k+1}P_{n+1,k,l}(\boldsymbol{\mu}) + \frac{l+1}{n+k+1}P_{n,k+1,l+1}(\boldsymbol{\mu}) + \frac{k-l+1}{n+k+1}P_{n,k+1,l}(\boldsymbol{\mu}) \tag{12}$$

Proof. Since $\mu_1 + \mu_2 + \mu_3 = 1$,

$$\begin{aligned}P_{n,k,l}(\boldsymbol{\mu}) &= \binom{n+k}{k}\binom{k}{l}\mu_1^{n+1}\mu_2^l\mu_3^{k-l}(\mu_1 + \mu_2 + \mu_3) \\ &= \binom{n+k}{k}\binom{k}{l}\left(\mu_1^{n+2}\mu_2^l\mu_3^{k-l} + \mu_1^{n+1}\mu_2^{l+1}\mu_3^{k-l} + \mu_1^{n+1}\mu_2^l\mu_3^{k-l+1}\right)\end{aligned}$$

$$= \frac{n+1}{n+k+1}\binom{n+1+k}{k}\binom{k}{l}\mu_1^{n+2}\mu_2^l\mu_3^{k-l} + \frac{l+1}{n+k+1}\binom{n+k+1}{k+1}\binom{k+1}{l+1}\mu_1^{n+1}\mu_2^{l+1}\mu_3^{k-l}$$
$$+\frac{k+1-l}{n+k+1}\binom{n+k+1}{k+1}\binom{k+1}{l}\mu_1^{n+1}\mu_2^l\mu_3^{k+1-l}$$
$$= \frac{n+1}{n+k+1}P_{n+1,k,l}(\boldsymbol{\mu}) + \frac{l+1}{n+k+1}P_{n,k+1,l+1}(\boldsymbol{\mu}) + \frac{k+1-l}{n+k+1}P_{n,k+1,l}(\boldsymbol{\mu}).$$

□

Proposition 2.9 (Integration)**.**

$$\iint_D P_{n,k,l}(\boldsymbol{\mu})d\sigma = \frac{n+1}{(n+k+1)(n+k+2)(n+k+3)}. \tag{13}$$

Proof. The integral area D can be described as $\{(\mu_1,\mu_2)|0<\mu_1\le 1, 0\le\mu_2\le 1-\mu_1\}$, thus

$$\iint_D P_{n,k,l}(\boldsymbol{\mu})d\sigma = \int_0^1 \binom{n+k}{k}\mu_1^{n+1}d\mu_1 \int_0^{1-\mu_1}\binom{k}{l}\mu_2^l(1-\mu_1-\mu_2)^{k-l}d\mu_2$$

Let $t=\frac{\mu_2}{1-\mu_1}$, then

$$\int_0^{1-\mu_1}\binom{k}{l}\mu_2^l(1-\mu_1-\mu_2)^{k-l}d\mu_2 = (1-\mu_1)^{k+1}\int_0^1\binom{k}{l}t^l(1-t)^{k-l}dt.$$

Moreover, from the integration of a univariate Bernstein polynomial, we get

$$\int_0^1\binom{k}{l}t^l(1-t)^{k-l}dt = \frac{1}{k+1}.$$

Therefore,

$$\iint_D P_{n,k,l}(\boldsymbol{\mu})d\sigma = \frac{1}{k+1}\int_0^1\binom{n+k}{k}\mu_1^{n+1}(1-\mu_1)^{k+1}d\mu_1$$
$$= \frac{n+1}{(n+k+1)(n+k+2)}\int_0^1\binom{n+k+2}{k+1}\mu_1^{n+1}(1-\mu_1)^{k+1}d\mu_1$$
$$= \frac{n+1}{(n+k+1)(n+k+2)(n+k+3)}.$$

□

Proposition 2.10 (Differentiation)**.**

$$\frac{\partial}{\partial\mu_1}P_{n,k,l}(\boldsymbol{\mu}) = \frac{(n+1)(n+k)}{n}P_{n-1,k,l}(\boldsymbol{\mu}); \tag{14}$$
$$\frac{\partial}{\partial\mu_2}P_{n,k,l}(\boldsymbol{\mu}) = (n+k)P_{n,k-1,l-1}(\boldsymbol{\mu}); \tag{15}$$
$$\frac{\partial}{\partial\mu_3}P_{n,k,l}(\boldsymbol{\mu}) = (n+k)P_{n,k-1,l}(\boldsymbol{\mu}). \tag{16}$$

Proof.

$$\frac{\partial}{\partial\mu_1}P_{n,k,l}(\boldsymbol{\mu}) = (n+1)\frac{n+k}{k}\binom{k}{l}\mu_1^n\mu_2^l\mu_3^{k-l}$$

$$= \frac{(n+1)(n+k)}{n} \binom{n-1+k}{k} \binom{k}{l} \mu_1^n \mu_2^l \mu_3^{k-l}$$
$$= \frac{(n+1)(n+k)}{n} P_{n-1,k,l}(\boldsymbol{\mu}).$$

Similarly,

$$\frac{\partial}{\partial \mu_2} P_{n,k,l}(\boldsymbol{\mu}) = (n+k) \binom{n+k-1}{k-1} \binom{k-1}{l-1} \mu_1^{n+1} \mu_2^{l-1} \mu_3^{k-l}$$
$$= (n+k) P_{n,k-1,l-1}(\boldsymbol{\mu});$$

$$\frac{\partial}{\partial \mu_3} P_{n,k,l}(\boldsymbol{\mu}) = (n+k) \binom{n+k-1}{k-1} \binom{k-1}{l} \mu_1^{n+1} \mu_2^{l} \mu_3^{k-l-1}$$
$$= (n+k) P_{n,k-1,l}(\boldsymbol{\mu}).$$

□

3 Triangular Meyer-König-Zeller Surface

In this section, we will introduce a method that constructs a kind of surface called a triangular Meyer-König-Zeller surface by triangular Meyer-König-Zeller basis functions $P_{n,k,l}(\boldsymbol{\mu})$.

Definition 3.1. *For given control vertices $\{V_{k,l} \in \boldsymbol{R}^3\}$, $k \geq 0, 0 \leq l \leq k$, we construct a surface denoted by*

$$S(\boldsymbol{\mu}) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) V_{k,l}, \qquad \boldsymbol{\mu} \in D, \tag{17}$$

*which is called a **triangular Meyer-König-Zeller surface**. The mesh constituted by the line segments of $V_{k,l}V_{k+1,l}$, $V_{k,l}V_{k+1,l+1}$, $V_{k+1,l}V_{k+1,l+1}$ is called a **control mesh**.*

Theorem 3.1. *From the properties of triangular Meyer-König-Zeller basis functions, we can derive the geometric properties of a triangular Meyer-König-Zeller surface as follows:*
*(1) **Affine Invariance***
*(2) **Convex Hull Property***
*(3) **Interpolative control vertex** $V_{0,0}$*
*(4) **Non-Degenerate***
*(5) **The boundary curves are rational Bézier curves in terms of the Bernstein basis functions of negative degree***

Proof. (1) The affine invariance is one of the most important geometric properties for curves and surfaces modeling in Computer-aided Geometric Design. For a given triangular Meyer-König-Zeller surface $S(\boldsymbol{\mu})$, an affine transformation operator $\mathcal{A}$ is acting on it, i.e.

$$\mathcal{A}(S(\boldsymbol{\mu})) = \mathcal{A}\left(\sum_{k=0}^{\infty} \sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) V_{k,l} \right).$$

Since the properties of non-negativity and partition of unity of triangular Meyer-König-Zeller basis functions hold, we have,

$$\mathcal{A}(S(\boldsymbol{\mu})) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) \mathcal{A}(V_{k,l}). \tag{18}$$

Thus, the triangular Meyer-König-Zeller surface is affine invariant.

(2) The properties of non-negativity and partition of unity of triangular Meyer-König-Zeller basis functions further guarantee that the triangular Meyer-König-Zeller surface is included within the convex hull of the control mesh which is constituted by $\{V_{k,l}\}$.

(3) By the property of interpolation of triangular Meyer-König-Zeller basis functions, it is obvious that $S(1,0,0) = V_{0,0}$, i.e. the triangular Meyer-König-Zeller surface interpolates the corner control vertex.

(4) Suppose that $S(\boldsymbol{\mu})$ collapses to a vertex $\boldsymbol{Q} \in \boldsymbol{R}^3$, then

$$0 = S(\boldsymbol{\mu}) - \boldsymbol{Q} = \sum_{k=0}^{\infty}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})(V_{k,l} - \boldsymbol{Q}).$$

For any $\boldsymbol{v} \in \boldsymbol{R}^3$, we have

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})(V_{k,l} - \boldsymbol{Q}) \bullet v = 0,$$

where • means the inner product. Since the triangular Meyer-König-Zeller basis functions are linearly independent,

$$(V_{k,l} - \boldsymbol{Q}) \bullet v = 0.$$

According to the arbitrariness of $\boldsymbol{v}$, $V_{k,l} = \boldsymbol{Q}$. Hence, the triangular Meyer-König-Zeller surface is non-degenerate, only if all control vertices were the same vertex.

(5) The boundary curves of a triangular Meyer-König-Zeller surface are $S(\mu_1, 0, \mu_3)$ and $S(\mu_1, \mu_2, 0)$. By the boundary property of triangular Meyer-König-Zeller basis functions, we get

$$S(\mu_1, 0, \mu_3) = \sum_{k=0}^{\infty} P_{n,k,0}(\mu_1, 0, \mu_3)V_{k,0},$$

which is a rational Bézier curve in terms of the Bernstein basis functions of negative degree with control vertices $\{V_{k,0}\}$. Similarly,

$$S(\mu_1, \mu_2, 0) = \sum_{k=0}^{\infty} P_{n,k,0}(\mu_1, \mu_2, 0)V_{k,k}$$

also is a rational Bézier curve in terms of the Bernstein basis functions of negative degree with control vertices $\{V_{k,k}\}$. □

4 Implementation

In this section, based on given control vertices some practical examples of triangular Meyer-König-Zeller surfaces will be shown.

Given control vertices $\{V_{k,l}\}$, $0 \le k \le m,\ 0 \le l \le k$, where k, l and m are natural numbers. m is the number of layers of the triangular control mesh. k and l are the indices of the control vertices. In Figure 2, the topology of the triangular control mesh with $m = 3$ is shown.

Now, we face a difficulty in that the number of triangular Meyer-König-Zeller basis functions is infinite but the number of control vertices is finite. We solve this difficulty by discarding or redistributing redundant basis functions.

4.1 Truncated Triangular Meyer-König-Zeller Surface

Definition 4.1. *For the given control vertices $\{V_{k,l}\}$, $0 \le k \le m,\ 0 \le l \le k$, we have*

$$\bar{S}(\boldsymbol{\mu}) = \sum_{k=0}^{m}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})V_{k,l}, \qquad \boldsymbol{\mu} \in D \tag{19}$$

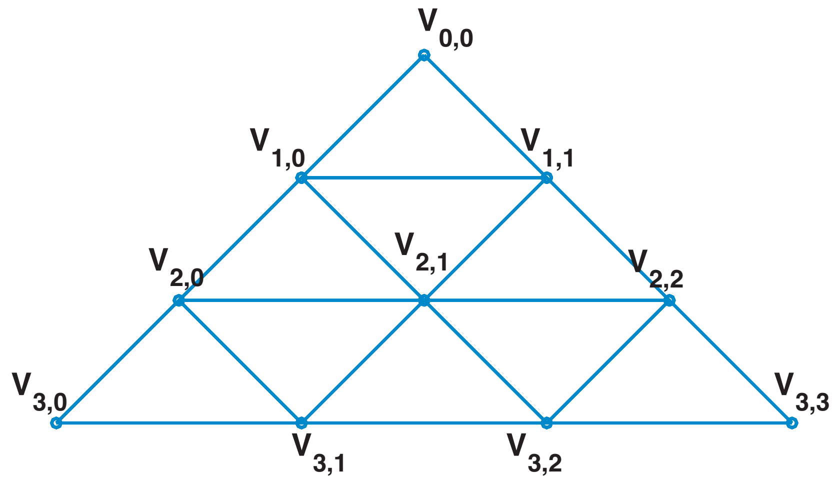

Figure 2: The topology of control mesh with $m = 3$.

called a truncated triangular Meyer-König-Zeller surface.

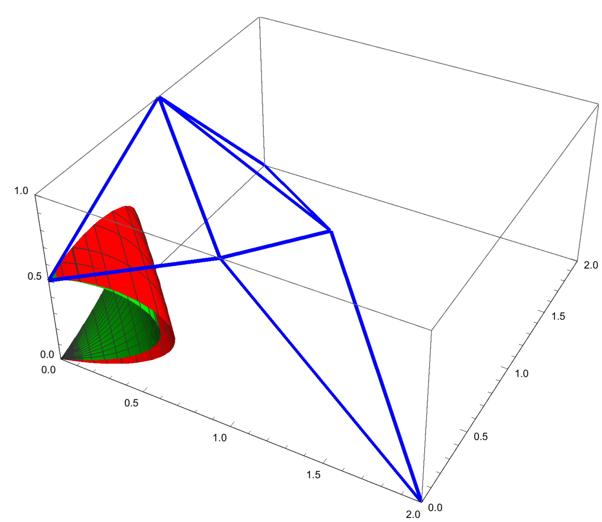

Figure 3: The graphs of truncated triangular Meyer-König-Zeller surfaces: the green surface with $m = 2,\ n = 1$, the red surface with $m = 2,\ n = 3$.

Figure 3 shows the graphs of truncated triangular Meyer-König-Zeller surfaces with $m = 2,\ n = 1$ (the blue one) and $m = 2,\ n = 3$ (the red one). It is obvious that $\lim_{\mu_1 \to 0} \bar{S}(\boldsymbol{\mu}) = \mathbf{0}$. Therefore, we can define $\bar{S}(0, \mu_2, \mu_3) = \mathbf{0}$, i.e. the truncated triangular Meyer-König-Zeller surface interpolates the coordinate origin point.

Remark 4.1. *The truncated triangular Meyer-König-Zeller surface is equivalent to the triangular Meyer-König-Zeller surface with $V_{k,l} = \{0,\ 0,\ 0\}$, $k > m,\ 0 \le l \le k$, i.e.*

$$\bar{S}(\boldsymbol{\mu}) = \sum_{k=0}^{m} \sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) V_{k,l} + \bar{P}_{n,m}(\boldsymbol{\mu}) \bar{V},$$

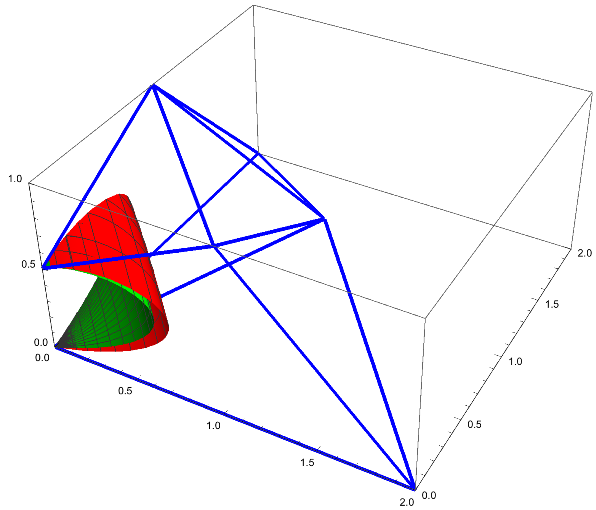

Figure 4: The actual control mesh of truncated triangular Meyer-König-Zeller surfaces with $m = 2, \ n = 1$ (the green one) and $m = 2, \ n = 3$ (the red one).

where $\bar{P}_{n,m}(\boldsymbol{\mu}) = 1 - \sum_{k=0}^{m} \sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})$ *and* $\bar{V} = \{0, 0, 0\}$*. Therefore,* $\bar{S}(\boldsymbol{\mu})$ *is a special kind of triangular Meyer-König-Zeller surface with the control mesh as shown in Figure 4.*

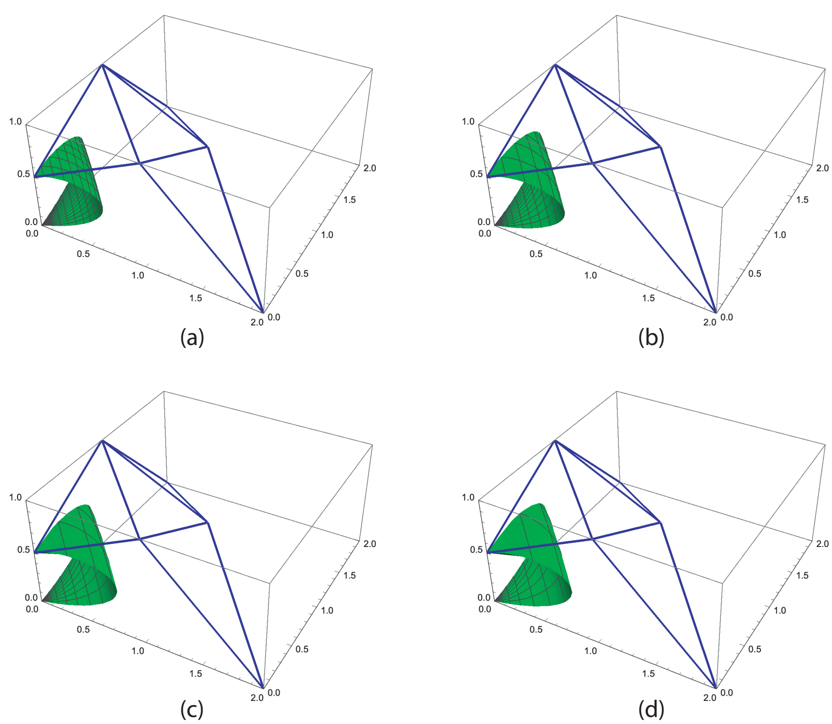

Figure 5: The truncated triangular Meyer-König-Zeller surfaces with $(m = 2, \ n = 1)$(a), $(m = 2, \ n = 3)$(b), $(m = 2, \ n = 6)$(c), $(m = 2, \ n = 9)$(d).

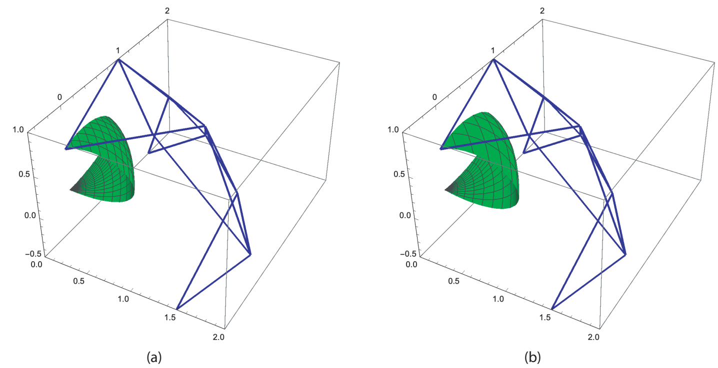

Figure 6: The truncated triangular Meyer-König-Zeller surfaces with $(m = 3,\ n = 1)$(a) and $(m = 3,\ n = 3)$(b).

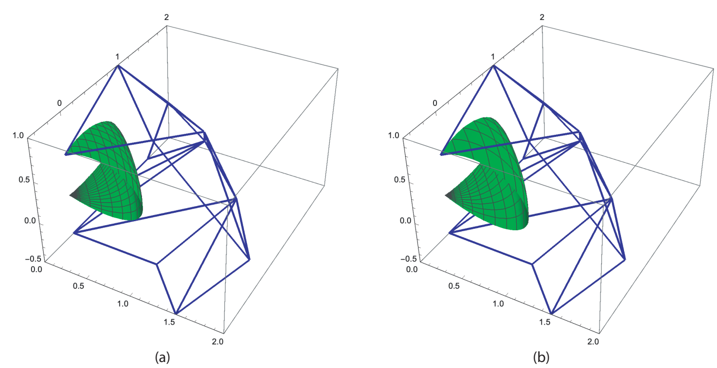

Figure 7: The truncated triangular Meyer-König-Zeller surfaces with $(m = 4,\ n = 1)$(a) and $(m = 4,\ n = 3)$(b).

Figure 5 shows the graphs of the truncated triangular Meyer-König-Zeller surfaces with $(m = 2,\ n = 1)$ Figure 5(a), $(m = 2,\ n = 3)$ Figure 5(b), $(m = 2,\ n = 6)$ Figure 5(c), $(m = 2,\ n = 9)$ Figure 5(d). These graphs indicate that the surfaces intuitively approximate the control mesh with increasing n. We conduct some numerical experiments to verify this conclusion. We define the extent of the surface approximating control mesh as

$$D(\bar{S}(\boldsymbol{\mu}), L(\boldsymbol{\mu})) = \left| \max_{\boldsymbol{\mu}}\{\|\bar{S}(\boldsymbol{\mu})\|\} - \max_{\boldsymbol{\mu}}\{\|L(\boldsymbol{\mu})\|\} \right| \tag{20}$$

where $L(\boldsymbol{\mu})$ is the linear interpolation of the control mesh. Table 1 shows the values of $D(\bar{S}(\boldsymbol{\mu}), L(\boldsymbol{\mu}))$ with $m = 2$ and different n.

Table 1: The values of $D(\bar{S}(\boldsymbol{\mu}), L(\boldsymbol{\mu}))$ with $m = 2$ and different n.

n	1	3	6	9	30	40
$D(\bar{S}, L)$	1.2086	1.1434	1.1076	1.0915	1.0635	1.0608

Figure 6 shows the graphs of the truncated triangular Meyer-König-Zeller surfaces with $(m = 3,\ n = 1)$(a) and $(m = 3,\ n = 3)$(b). Figure 7 shows the graphs of the truncated triangular Meyer-König-Zeller surfaces with $(m = 4,\ n = 1)$(a) and $(m = 4,\ n = 3)$(b).

4.2 Redistributed Triangular Meyer-König-Zeller Surface

We construct a kind of triangular Meyer-König-Zeller surface by redistributing redundant basis functions.

Definition 4.2. *For given weight sequence* $\{\omega_{m,l}\}_{l=0}^{m}$, $\omega_{m,l} \geq 0$ *and* $\sum_{l=0}^{m} \omega_{m,l} = 1$, *we derive a triangular Meyer-König-Zeller surface*

$$\hat{S}(\boldsymbol{\mu}) = \sum_{k=0}^{m} \sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) V_{k,l} + \sum_{l=0}^{m} \omega_{m,l} \bar{P}_{n,m}(\boldsymbol{\mu}) V_{m,l}. \tag{21}$$

called a ***redistributed triangular Meyer-König-Zeller surface****.*

It is obvious that

$$\lim_{\mu_1 \to 0} S(\boldsymbol{\mu}) = \sum_{l=0}^{m} \omega_{m,l} \lim_{\mu_1 \to 0} \bar{P}_{n,m}(\boldsymbol{\mu}) V_{m,l} = \sum_{l=0}^{m} \omega_{m,l} V_{m,l}.$$

Thus, let $\hat{S}(0, \mu_2, \mu_3) = \sum_{l=0}^{m} \omega_{m,l} V_{m,l}$. Figure 8 shows the graphs of the surface $\hat{S}(\boldsymbol{\mu})$ with $m = 2$, $n = 3$ and $(\omega_{2,l} = 1/3,\ l = 0, 1, 2)$(a), $(\omega_{2,0} = \omega_{2,2} = 1/2,\ \omega_{2,1} = 0)$(b), $(\omega_{2,0} = 0.7,\ \omega_{2,1} = 0.2,\ \omega_{2,2} = 0.1)$(c), $(\omega_{2,0} = 0.1,\ \omega_{2,1} = 0.2,\ \omega_{2,2} = 0.7)$(d).

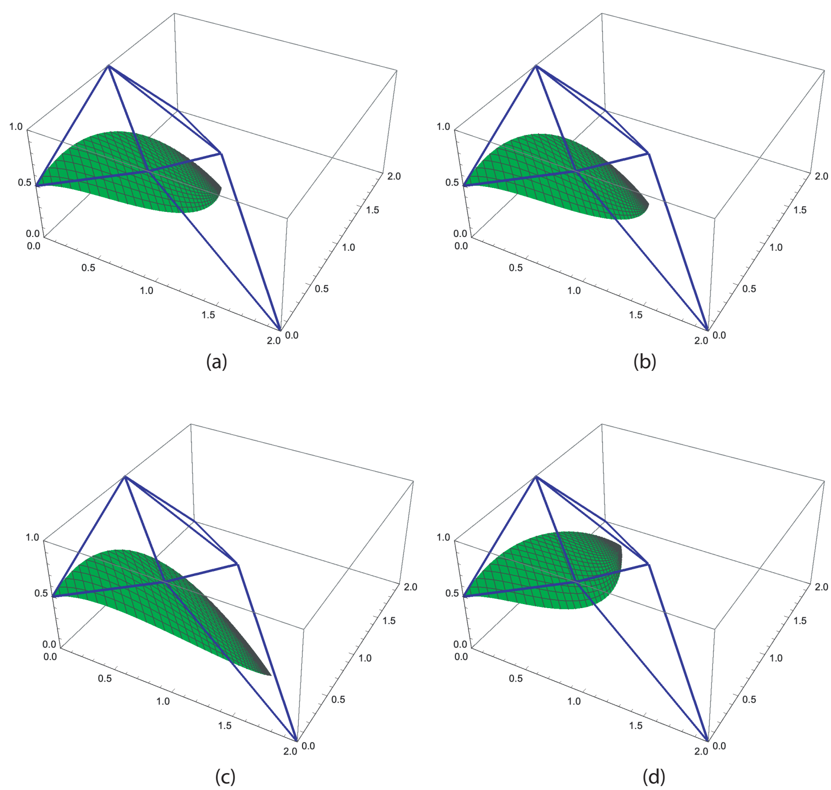

Figure 8: The graphs of redistributed triangular Meyer-König-Zeller surfaces with $m = 2$, $n = 3$ and different $\{\omega_{2,l}\}$.

Theorem 4.1. *Redistributed triangular Meyer-König-Zeller surfaces retain the properties of triangular Meyer-König-Zeller surfaces.*
(1) ***Affine Invariance****.*
(2) ***Convex Hull Property****.*

(3) ***Interpolative control vertex*** $V_{0,0}$.
(4) ***Non-Degenerate***.
(5) ***The boundary curves are rational Bézier curves in terms of the Bernstein basis functions of negative degree***.

Proof. (1) Let $\mathcal{A}$ be an affine transformation operator, then

$$\mathcal{A}(\hat{S}(\boldsymbol{\mu})) = \mathcal{A}\left(\sum_{k=0}^{m}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})V_{k,l} + \sum_{l=0}^{m}\omega_{m,l}\bar{P}_{n,m}(\boldsymbol{\mu})V_{m,l}\right).$$

Since

$$\sum_{k=0}^{m}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu}) + \sum_{l=0}^{m}\omega_{m,l}\bar{P}_{n,m}(\boldsymbol{\mu}) = 1, \tag{22}$$

$$\mathcal{A}(\hat{S}(\boldsymbol{\mu})) = \left(\sum_{k=0}^{m}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})\mathcal{A}V_{k,l} + \sum_{l=0}^{m}\omega_{m,l}\bar{P}_{n,m}(\boldsymbol{\mu})\mathcal{A}V_{m,l}\right). \tag{23}$$

Thus, the redistributed triangular Meyer-König-Zeller surface is affine invariant.

(2) It is obvious that $\omega_{m,l}\bar{P}_{n,m}(\boldsymbol{\mu})$ $(l = 0, \dots, m)$ are non-negative. Moreover, according to equation (22), the redistributed triangular Meyer-König-Zeller surface that we obtain is included within the convex hull of the control mesh which is constituted by $\{V_{k,l}\}$, $(k = 0, \dots, m; l = 0, \dots, k)$.

(3)It is obvious that $\hat{S}(1, 0, 0) = V_{0,0}$, i.e. the redistributed triangular Meyer-König-Zeller surface interpolates the corner control vertex.

(4) Suppose that $\hat{S}(\boldsymbol{\mu})$ collapses to a vertex $\boldsymbol{Q} \in \boldsymbol{R}^3$, then

$$\boldsymbol{0} = \hat{S}(\boldsymbol{\mu}) - \boldsymbol{Q} = \sum_{k=0}^{m}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})(V_{k,l} - \boldsymbol{Q}) + \sum_{l=0}^{m}\omega_{m,l}\bar{P}_{n,m}(\boldsymbol{\mu})(V_{m,l} - \boldsymbol{Q}).$$

For any $\boldsymbol{v} \in \boldsymbol{R}^3$, we have

$$\sum_{k=0}^{m}\sum_{l=0}^{k} P_{n,k,l}(\boldsymbol{\mu})(V_{k,l} - \boldsymbol{Q}) \bullet v + \sum_{l=0}^{m}\omega_{m,l}\bar{P}(\boldsymbol{\mu})(V_{m,l} - \boldsymbol{Q}) \bullet v = 0.$$

By linear independence of triangular Meyer-König-Zeller basis functions, we get

$$(V_{k,l} - \boldsymbol{Q}) \bullet v = 0,\ (k = 0, \dots, m;\ l = 0, \dots, k).$$

Hence, a redistributed triangular Meyer-König-Zeller surface is non-degenerate.

(5) For boundary curves of a redistributed triangular Meyer-König-Zeller surface $\hat{S}(\boldsymbol{\mu})$ are $\hat{S}(\mu_1, 0, \mu_3)$ and $\hat{S}(\mu_1, \mu_2, 0)$. By the boundary property of triangular Meyer-König-Zeller basis functions, we get

$$\begin{aligned}\hat{S}(\mu_1, 0, \mu_3) &= \sum_{k=0}^{m} P_{n,k,0}(\mu_1, 0, \mu_3)V_{k,0} + \sum_{l=0}^{m}\omega_{m,l}\bar{P}_{n,m}(\mu_1, 0, \mu_3)V_{m,l}\\ &= \sum_{k=0}^{m} P_{n,k,0}(\mu_1, 0, \mu_3)V_{k,0} + \left(\sum_{l=0}^{m}\omega_{m,l}V_{m,l}\right)\left(1 - \sum_{k=0}^{m} P_{n,k,0}(\mu_1, 0, \mu_3)\right),\end{aligned} \tag{24}$$

which is a rational Bézier curve in terms of the Bernstein basis functions of negative degree. The control vertices of $\hat{S}(\mu_1, 0, \mu_3)$ are $\{V_{k,0}\}_{k=0}^{\infty}$. For all $k \geq m + 1$, let $V_{k,0} = \sum_{l=0}^{m}\omega_{m,l}V_{m,l}$. Similarly,

$$\hat{S}(\mu_1, \mu_2, 0) = \sum_{k=0}^{m} P_{n,k,k}(\mu_1, \mu_2, 0)V_{k,k} + \left(\sum_{l=0}^{m}\omega_{m,l}V_{m,l}\right)\left(1 - \sum_{k=0}^{m} P_{n,k,k}(\mu_1, \mu_2, 0)\right)$$

is a rational Bézier curve in terms of the Bernstein basis functions of negative degree with control vertices $\{V_{k,k}\}_{k=0}^{\infty}$. For all $k \geq m + 1$, let $V_{k,k} = \sum_{l=0}^{m}\omega_{m,l}V_{m,l}$.

□

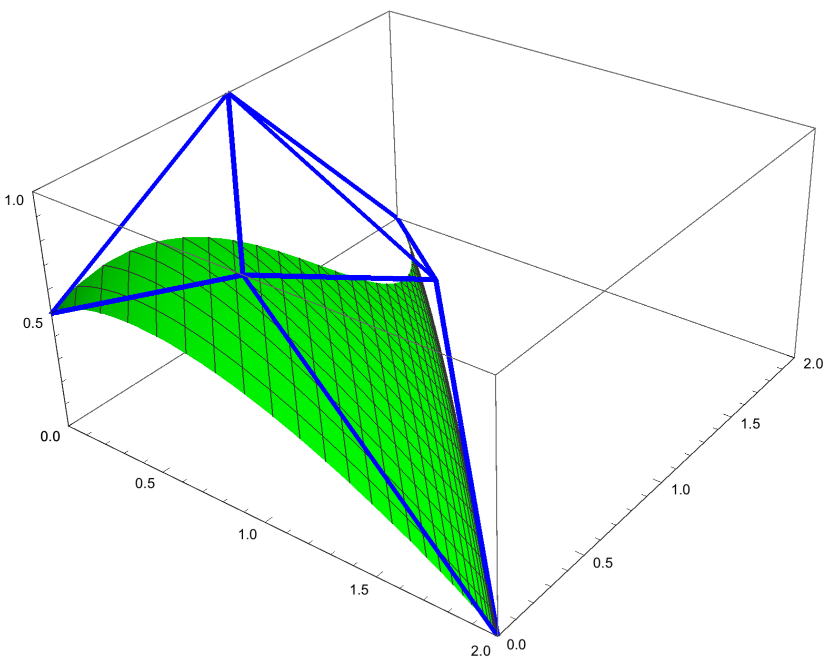

Figure 9: The graph of a redistributed triangular Meyer-König-Zeller surface, where $m = 2$, $n = 3$ and $\omega_{2,l}$ are the Bernstein basis functions of degree 2 with variable μ_2.

Furthermore, we can let $\{\omega_{m,l}\}_{l=0}^{m}$ be a series of basis functions with variable μ_2. Figure 9 shows the graph of a redistributed triangular Meyer-König-Zeller surface with $\omega_{2,l}(\mu_2) = \binom{2}{l}\mu_2^l(1-\mu_2)^{2-l}$ which are the Bernstein basis functions of degree 2. It is obvious that

$$\begin{aligned}\hat{S}(0,\mu_2,\mu_3) &= \sum_{l=0}^{m} \omega_{m,l}(\mu_2)\bar{P}_{n,m}(0,\mu_2,\mu_3)V_{m,l} \\ &= \sum_{l=0}^{m} \omega_{m,l}(\mu_2)V_{m,l}\end{aligned} \tag{25}$$

is a Bézier curve.

Remark 4.2. *When $\{\omega_{m,l}\}_{l=0}^{m}$ is a series of Bernstein basis functions, i.e. $\omega_{m,l} = \binom{m}{l}\mu_2^l(1-\mu_2)^{m-l}$, it is obvious that $\hat{S}(0,0,1) = V_{m,0}$, $\hat{S}(0,1,0) = V_{m,m}$.*

5 Conclusion

We introduced a new kind of surface called a triangular Meyer-König-Zeller surface, which is constructed by triangular Meyer-König-Zeller basis functions. We have studied the main properties of triangular Meyer-König-Zeller basis functions which guarantee that triangular Meyer-König-Zeller surfaces have affine invariance, possess the convex hull property, are non-degenerate, have interpolative control vertex $V_{0,0}$ and the boundary curves are rational Bezier curves in terms of the Bernstein functions of negative degree.

Moreover, for given control vertices, whose number is finite, we presented a truncated triangular Meyer-König-Zeller surface and a redistributed triangular Meyer-König-Zeller surface. We remarked that a truncated triangular Meyer-König-Zeller surface is a special kind of triangular Meyer-König-Zeller surface with special control vertices ($V_{k,l} = \{0, 0, 0\}$, $k > m$, $0 \leq l \leq k$). Theorem 4.1 shows that a redistributed a triangular Meyer-König-Zeller surface retained all the properties of triangular Meyer-König-Zeller surface.

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