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Periodic solution for ϕ-Laplacian neutral differential equation

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Abstract: This paper is devoted to the existence of a periodic solution for ϕ-Laplacian neutral differential equation as follows

$$(\phi(x(t) - cx(t - \tau))')' = f(t, x(t), x'(t)).$$

By applications of an extension of Mawhin's continuous theorem due to Ge and Ren, we obtain that given equation has at least one periodic solution. Meanwhile, the approaches to estimate a priori bounds of periodic solutions are different from the corresponding ones of the known literature.

Keywords: neutral operator; ϕ-Laplacian; periodic solution; extension of Mawhin's continuation theorem

MSC: 34C25, 34K14

1 Introduction

In this paper, we consider a kind of second order ϕ-Laplacian neutral differential equation as follows

$$(\phi(x(t) - cx(t - \tau))')' = f(t, x(t), x'(t)), \tag{1.1}$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is continuous function with $f(t + T, \cdot, \cdot) \equiv f(t, \cdot, \cdot)$; $c,\ \tau$ are constants. $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\phi(0) = 0$ which satisfies

(A_1) $(\phi(x_1) - \phi(x_2))(x_1 - x_2) > 0$ for $\forall\, x_1 \neq x_2,\ \ x_1,\ \ x_2 \in \mathbb{R}$;

(A_2) There exists a function $\alpha : [0, +\infty] \to [0, +\infty],\ \ \alpha(s) \to +\infty$ as $s \to +\infty$, such that $\phi(x) \cdot x \geq \alpha(|x|)|x|$ for $\forall\, x \in \mathbb{R}$.

It is easy to see that ϕ represents a large class of nonlinear operator, including $\phi_p : \mathbb{R} \to \mathbb{R}$ is a p-Laplacian, i.e., $\phi_p(x) = |x|^{p-2}x$ for $x \in \mathbb{R}$.

The study of p-Laplacian neutral differential equations began with the paper of Zhu and Lu. In 2007, Zhu and Lu [1] discussed the existence of a periodic solution for a kind of p-Laplacian neutral differential equation as follows

$$(\phi_p(x(t) - cx(t - \tau))')' + g(t, x(t - \delta(t))) = p(t),$$

where c is a constant and $|c| \neq 1$. Since $(\phi_p(x'(t)))'$ is nonlinear (i.e. quasilinear), Mawhin's continuous theorem [2] can not be apply directly. In order to get around this difficulty, Zhu and Lu translated the p-Laplacian neutral differential equation into a two-dimensional system

$$\begin{cases} (x_1(t) - cx_1(t - \tau))'(t) = \phi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t) \\ x_2'(t) = -g(t, x_1(t - \delta(t))) + p(t), \end{cases}$$

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where $\frac{1}{p}+\frac{1}{q}=1$, for which Mawhin's continuation theorem can be applied. Zhu and Lu's work attracted the attention of many scholars in neutral differential equation and they have contributed to the research of p-Laplacian neutral differential equation (see [3]-[12]). Besides, a good deal of work has been performed on the existence of periodic solutions to ϕ-Laplacian differential equation. Manásevich and Mawhin [13] in 1998 investigated ϕ-Laplacian differential equation

$$(\phi(x(t)'))' = f(t, x(t), x'(t)).$$

Applying Leray-Schauder degree theory, the authors proved that the above equation has at least one periodic solution.

All the aforementioned results are related to p-Laplacian neutral equations [1], [3]-[12] or ϕ-Lpalacian differential equation [13]. Naturally, a new question arises: how neutral differential equation works on ϕ-Laplacian operator? Besides practical interests, the topic has obvious intrinsic theoretical significance. To answer this question, in this paper, we try to fill the gap and establish the existence of periodic solutions of (1.1) by employing the extension of Mawhin's continuation theorem due to Ge and Ren. The obvious difficulty lies in the following two aspects. The first is that since the leading term contains a ϕ-Laplacian neutral operator, the operator is much more than the corresponding p-Laplacian neutral operator; the second is that *a priori bounds* of periodic solutions are not easy to estimate. For example, the key step for ϕ_p to get the priori bounds of periodic solution, $\int_0^T(\phi_p'(x(t)))'x(t)dt = -\int_0^T|x'(t)|^p dt$, is no longer available for general ϕ-Laplacian. So we need to find a new method to solve that problem.

The remaining part of the paper is organized as follows. In section 2, we give some preliminary lemmas. In Section 3, by employing the extension of Mawhin's continuation theorem, we state and prove the existence of periodic solution for (1.1) in $|c|\neq 1$ and $|c|=1$ (critical) cases. In Section 4, we investigate the existence of the result for a kind of ϕ-Laplacian neutral Liénard equation in $|c|\neq 1$ case by applications of the Theorem 3.1. In Section 5, we consider the existence of periodic solution for a kind of p-Laplacian neutral Liénard equation in $|c|\neq 1$ and $|c|=1$ cases by applications of the Theorem 3.1. In Section 6, two numerical examples demonstrate the validity of the method.

Throughout this paper, we will denote by Z the set of integers, Z_1 the set of odd integers, Z_2 the set of even integers, N the set of positive integers, N_1 the set of odd positive integers and N_2 the set of even positive integers. Let $C_T := \{x|x \in C(\mathbb{R},\mathbb{R}),\ x(t+T)-x(t)\equiv 0,\ \forall\ t\in\mathbb{R}\}$, $C_T^1 := \{x|x \in C(\mathbb{R},\mathbb{R}),\ x(t+T)-x(t)\equiv 0,\ \forall\ t\in\mathbb{R}\}$. $L_{2\pi}^2 := \{x:\ x(t+2\pi)-x(t)\equiv 0,\ t\in\mathbb{R}$ and $\int_0^{2\pi}|x(s)|^2ds < +\infty\}$, under the norm $|\varphi|_2 = \left(\int_0^{2\pi}|\varphi(t)|^2dt\right)^{\frac{1}{2}}$, $L_{2\pi}^{2-} = \{x : x\in L_{2\pi}^2,\ x(t+\pi)+x(t)\equiv 0\}$ and $L_{2\pi}^{2+} = \{x : x\in L_{2\pi}^2,\ x(t+\pi)-x(t)\equiv 0\}$ with the norm $|\cdot|_2$. Clearly, $L_{2\pi}^2$, $L_{2\pi}^{2-}$ and $L_{2\pi}^{2+}$ are all Banach spaces.

2 Preliminaries

In order to use the extension of Mawhin's continuous theorem [14] due to Ge and Ren, we first recall it.

Let X and Z be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. A continuous operator M : $X\bigcap \mathrm{dom}M \to Z$ is said to be $quasi-linear$ if

(a) $\mathrm{Im}M := M(X\bigcap \mathrm{dom}M)$ is a closed subset of Z;

(b) $\ker M := \{x\in X\bigcap \mathrm{dom}M :\ Mx = 0\}$ is a subspace of X with $\dim\ker M < +\infty$.

Let $X_1 = \ker M$ and X_2 be the complement space of X_1 in X, then $X = X_1\oplus X_2$. On the other hand, Z_1 is a subspace of Z and Z_2 is the complement space of Z_1 in Z, so that $Z = Z_1\oplus Z_2$. Suppose that $P : X\to X_1$ and $Q : Z\to Z_1$ two projects and $\Omega\subset X$ is an open and bounded set with the origin $\theta\in\Omega$.

Let $N_\lambda : \bar{\Omega}\to Z,\ \lambda\in[0,1]$ be a continuous operator. Denote N_1 by N, and let $\sum_\lambda = \{x\in\bar{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be $M-compact$ in $\bar{\Omega}$ if

(c) there is a vector subspace Z_1 of Z with $\dim Z_1 = \dim X_1$ and an operator R : $\bar{\Omega}\times[0,1]\to X_2$ being continuous and compact such that for $\lambda\in[0,1]$,

$$(I-Q)N_\lambda(\bar{\Omega}) \subset \mathrm{Im}M \subset (I-Q)Z, \tag{2.1}$$

$$QN_\lambda x = 0, \quad \lambda \in (0,1) \Leftrightarrow QNx = 0, \tag{2.2}$$

$$R(\cdot, 0) \text{ is the zero operator and } R(\cdot,\lambda)|_{\sum_\lambda} = (I-P)|_{\sum_\lambda}, \tag{2.3}$$

and

$$M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda. \tag{2.4}$$

Let $J : Z_1 \to X_1$ be a homeomorphism with $J(\theta) = \theta$.

Lemma 2.1. ([14]) Let X and Z be Banach spaces with norm $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and $\Omega \subset X$ be an open and bounded set with origin $\theta \in \Omega$. Suppose that $M : X \cap \mathrm{dom}M \to Z$ is a $quasi-linear\ operator$ and

$$N_\lambda : \bar{\Omega} \to Z, \ \lambda \in (0,1)$$

is an *M-compact* mapping. In addition, if
(i) $Mx \neq N_\lambda x, \ \lambda \in (0,1), \ x \in \partial\Omega$,
(ii) $\deg\{JQN, \Omega \cap \ker M, 0\} \neq 0$,
where $N = N_1$, then the abstract equation $Mx = Nx$ has at least one solution in $\bar{\Omega}$.

Lemma 2.2. (see [15]) If $|c| \neq 1$, then the operator $(Ax)(t) := x(t) - cx(t-\tau)$ has a continuous inverse A^{-1} on the space C_T, and satisfying

$$\|A^{-1}\| \leq \frac{1}{|1-|c||}.$$

Lemma 2.3. (see [16, 17]) The follow propositions are true:

($\mathbf{P_1}$) Suppose $c = -1, \ |\tau| = (m/n)\pi$, where $m, \ n$ are coprime positive integers with m even, then $A : L^2_{2\pi} \to L^2_{2\pi}$, has a unique inverse $A^{-1} : L^2_{2\pi} \to L^2_{2\pi}$ satisfying

$$\|A^{-1}\| \leq \frac{1}{\sigma_1},$$

where $\sigma_1 := \inf_{k\in N} |1 - ce^{-ik\tau}| = \inf_{k\in N} \left[2(1+\cos k\tau)\right]^{\frac{1}{2}} > 0$.

($\mathbf{P_2}$) Suppose $c = -1, \ |\tau| = (m/n)\pi$, where $m, \ n$ are coprime odd positive integers, then $A : L^{2+}_{2\pi} \to L^{2+}_{2\pi}$, has a unique inverse $A^{-1} : L^{2+}_{2\pi} \to L^{2+}_{2\pi}$ satisfying

$$\|A^{-1}\| \leq \frac{1}{\sigma_2},$$

where $\sigma_2 := \inf_{k\in N_1} |1 - ce^{-ik\tau}| = \inf_{k\in N_1} \left[2(1+\cos k\tau)\right]^{\frac{1}{2}} > 0$.

($\mathbf{P_3}$) Suppose $c = -1, \ |\tau| = (m/n)\pi$, where $m, \ n$ are coprime positive integers with m odd and n even, then $A : L^{2-}_{2\pi} \to L^{2-}_{2\pi}$, has a unique inverse $A^{-1} : L^{2-}_{2\pi} \to L^{2-}_{2\pi}$ satisfying

$$\|A^{-1}\| \leq \frac{1}{\sigma_3},$$

where $\sigma_3 := \inf_{k\in N_2} |1 - ce^{-ik\tau}| = \inf_{k\in N_2} \left[2(1+\cos k\tau)\right]^{\frac{1}{2}} > 0$.

($\mathbf{P_4}$) Suppose $c = 1, \ |\tau| = (m/n)\pi$, where $m, \ n$ are coprime positive integers with m odd, then $A : L^{2-}_{2\pi} \to L^{2-}_{2\pi}$, has a unique inverse $A^{-1} : L^{2-}_{2\pi} \to L^{2-}_{2\pi}$ satisfying

$$\|A^{-1}\| \leq \frac{1}{\sigma_4},$$

where $\sigma_4 := \inf_{k\in N_1} |1 - ce^{-ik\tau}| = \inf_{k\in N_1} \left[2(1+\cos k\tau)\right]^{\frac{1}{2}} > 0$.

($\mathbf{P_5}$) Suppose $c = 1$, $|\tau| = \pi$, then $A : L^{2-}_{2\pi} \to L^{2-}_{2\pi}$, has a unique inverse $A^{-1} : L^{2-}_{2\pi} \to L^{2-}_{2\pi}$ satisfying

$$\|A^{-1}\| \le \frac{1}{\sigma_5},$$

where $\sigma_5 := \inf_{k \in N_1} |1 - ce^{-ik\tau}| = \inf_{k \in N_1} \left[2(1 + \cos k\tau)\right]^{\frac{1}{2}} = 2 > 0$.

3 Periodic solution for (1.1)

In this section, we will prove the existence of a periodic solution for ϕ-Laplacian neutral operator with $|c| \neq 1$ and $|c| = 1$ by using Lemma 2.1.

Theorem 3.1. Assume that condition (A_1), (A_2) and $|c| \neq 1$, Ω is an open bounded set in C^1_T. Suppose the following conditions hold:
(C_1) For each $\lambda \in (0, 1)$ the equation

$$(\phi(Ax)'(t))' = \lambda f(t, x(t), x'(t)) \tag{3.1}$$

has no solution on $\partial\Omega$;
(C_2) The equation

$$F(a) := \frac{1}{T}\int_0^T f(t, x(t), x'(t))dt = 0,$$

has no solution on $\partial\Omega \cap \mathbb{R}$;
(C_3) The Brouwer degree

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then (1.1) has at least one periodic solution on $\bar{\Omega}$.

Proof. In order to use Lemma 2.1 studying the existence of a periodic solution to (3.1), we set $X := \{x \in C[0, T] : x(0) = x(T)\}$ and $Z := C[0, T]$,

$$M : X \cap \operatorname{dom} M \to Z, \quad (Mx)(t) = (\phi(Ax)'(t))', \tag{3.2}$$

where $\operatorname{dom} M := \{u \in X : \phi(Au)' \in C^1(\mathbb{R}, \mathbb{R})\}$. Then $\ker M = \mathbb{R}$. In fact

$$\begin{aligned}\ker M &= \{x \in X : (\phi(Ax)'(t))' = 0\}\\ &= \{x \in X : \phi(Ax)' \equiv c\}\\ &= \{x \in X : (Ax)' \equiv \phi^{-1}(c) := c_1\}\\ &= \{x \in X : (Ax)(t) \equiv c_1 t + c_2\},\end{aligned}$$

where c, c_1, c_2 are constant in $\mathbb{R}$. Since $(Ax)(0) = (Ax)(T)$, then, we get $\ker M = \{x \in X : (Ax)(t) \equiv c_2\}$. In addition,

$$\operatorname{Im} M = \{y \in Z,\ \text{for } x \in X \cap \operatorname{dom} M,\ (\phi(x'))'(t) = y(t),\ \int_0^T y(t)dt = \int_0^T (\phi(x'))'(t)dt = 0\}.$$

So M is *quasi-linear*. Let

$$\begin{aligned}X_1 &= \ker M, \quad X_2 = \{x \in X : x(0) = x(T) = 0\}.\\ Z_1 &= \mathbb{R}, \qquad Z_2 = \operatorname{Im} M.\end{aligned}$$

Clearly, $\dim X_1 = \dim Z_1 = 1$, and $X = X_1 \oplus X_2$, $P : X \to X_1$, $Q : Z \to Z_1$, are defined by

$$Px = x(0), \qquad Qy = \frac{1}{T}\int_0^T y(s)ds.$$

For $\forall\, \bar{\Omega} \subset X$ define $N_\lambda : \bar{\Omega} \to Z$ by

$$(N_\lambda x)(t) = \lambda f(t, x, x').$$

We claim $(I-Q)N_\lambda(\bar{\Omega}) \subset \mathrm{Im}M = (I-Q)Z$ holds. In fact, for $x \in \bar{\Omega}$, we have

$$\begin{aligned}\int_0^T (I-Q)N_\lambda x(t)dt &= \int_0^T (I-Q)\lambda f(t, x(t), x'(t))dt \\ &= \int_0^T \lambda f(t, x(t), x'(t))dt - \frac{\lambda}{T}\int_0^T\int_0^T f(s, x(s), x'(s))dsdt \\ &= 0.\end{aligned}$$

Hence, we have $(I-Q)N_\lambda(\bar{\Omega}) \subset \mathrm{Im}\, M$.

Moreover, for any $x \in Z$, we have

$$\begin{aligned}\int_0^T (I-Q)x(t)dt &= \int_0^T \left(x(t) - \frac{1}{T}\int_0^T\int_0^T x(t)dt \right) dt \\ &= 0.\end{aligned}$$

So, we have $(I-Q)Z \subset \mathrm{Im}M$. On the other hand, $x \in \mathrm{Im}M$ and $\int_0^T x(t)dt = 0$, then we have $x(t) = x(t) - \int_0^T x(t)dt$. Hence, we can get $x(t) \in (I-Q)Z$. Therefore, $\mathrm{Im}M = (I-Q)Z$.

From $QN_\lambda x = 0$, we can get $\frac{\lambda}{T}\int_0^T f(t, x(t), x'(t))dt = 0$. Since $\lambda \in (0,1)$, then we have $\frac{1}{T}\int_0^T f(t, x(t), x'(t))dt = 0$. Therefore, we can get $QNx = 0$, then, (2.4) also holds.

Let $J : Z_1 \to X_1$, $J(x) = x$, then $J(0) = 0$. Define $R : \bar{\Omega} \times [0,1] \to X_2$, by Lemma 2.2, we know that there exists a continuous inverse operator A^{-1} of neutral operator A such that

$$R(x,\lambda)(t) = \int_0^t A^{-1}\left(\phi^{-1}\left(a + \int_0^s \lambda f(u, x(u), x'(u))\right) du - \frac{\lambda s}{T}\int_0^T f(u, x(u), x'(u))du\right) ds, \tag{3.3}$$

where $a \in R$ is a constant such that

$$\begin{aligned}R(x,\lambda)(T) &= \int_0^T A^{-1}\left(\phi^{-1}\left(a + \int_0^s \lambda f(u, x(u), x'(u))\right) du - \frac{\lambda s}{T}\int_0^T f(u, x(u), x'(u))du\right) ds \\ &= 0.\end{aligned} \tag{3.4}$$

From Lemma 2.3 of [13], we know that a is uniquely defined by

$$a = \tilde{a}(x, \lambda),$$

where $\tilde{a}(x,\lambda)$ is continuous on $\bar{\Omega} \times [0,1]$ and bounded sets of $\bar{\Omega} \times [0,1]$ into bounded sets of $\mathbb{R}$.

From (3), we can find that

$$R : \bar{\Omega} \times [0,1] \to X_2.$$

Now, for any $x \in \sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\} = \{x \in \bar{\Omega} : (\phi(Ax)'(t))' = \lambda f(t, x(t), x'(t))\}$, we have $\int_0^T f(t, x(t), x'(t))dt = 0$, together with (3) gives

$$R(x,\lambda)(t) = \int_0^t A^{-1}\left(\phi^{-1}\left(a + \int_0^s \lambda f(u, x(u), x'(u))du\right)\right) ds$$

$$= \int_0^t A^{-1}\left(\phi^{-1}\left(a + \int_0^s (\phi(Ax)'(u))' du\right)\right) ds$$

$$= \int_0^t A^{-1}\left(\phi^{-1}\left(a + \phi(Ax)'(s) - \phi(Ax)'(0)\right)\right) ds.$$

Take $a = \phi(Ax)'(0)$, from $(Ax)'(t) = (Ax')(t)$, then we can get

$$\begin{aligned} R(x,\lambda)(T) = & \int_0^T A^{-1}(\phi^{-1}(\phi(Ax)'(s)))ds \\ = & x(T) - x(0) \\ = & 0, \end{aligned}$$

where a is unique, we see that

$$a = \tilde{a}(x,\lambda) = \phi(Ax)'(0), \quad \forall \lambda \in [0,1].$$

So, we have

$$\begin{aligned} R(x,\lambda)(t)|_{x\in\sum_\lambda} = & \int_0^t A^{-1}\left(\phi^{-1}\left(\phi(Ax)'(0) + \int_0^s \lambda f(t,u,x(u),x'(u))du\right)\right) ds \\ = & \int_0^t A^{-1}\left(\phi^{-1}(\phi(Ax)'(s))\right) ds \\ = & x(t) - x(0) \\ = & (I-P)x(t), \end{aligned}$$

which yields the second part of (2.4). Meanwhile, if $\lambda = 0$, the
$\sum_\lambda = \{x \in \bar{\Omega} : Mx = N_\lambda x\} = \{x \in \bar{\Omega} : (\phi(Ax)'(t))' = \lambda f(t, x(t), x'(t))\} = c_3$,
where $c_3 \in \mathbb{R}$ is a constant, so by the continuity of $\tilde{a}(x,\lambda)$ with respect to (x,λ), $a = \tilde{a}(x,0) = \phi(Ac)'(0) = \theta$.
So,

$$R(x,0)(t) = \int_0^t A^{-1}\phi^{-1}(\theta)ds = 0, \quad \forall x \in \bar{\Omega},$$

which yields the first part of (2.4). Furthermore, we consider the following equation

$$M(P+R) = (I-Q)N_\lambda.$$

In fact,

$$\frac{d}{dt}\phi(A(P+R))' = (I-Q)N_\lambda. \tag{3.5}$$

Integrating both sides of (3.5) over $[0,s]$, we have

$$\int_0^s \frac{d}{dt}\phi(A(P+R))'ds = \int_0^s (I-Q)N_\lambda ds.$$

Therefore, we have

$$\phi(A(P+R))'(s) - a = \lambda\int_0^s f(u,x(u),x'(u))du - \int_0^s \frac{\lambda}{T}\int_0^T f(u,x(u),x'(u))dudt$$

$$= \int_0^s f(u, x(u), x'(u))du - \frac{\lambda s}{T} \int_0^T f(u, x(u), x'(u))du,$$

where $a := \phi(A(P+R))'(0)$. Then, we can get

$$(A(P+R))'(s) = \phi^{-1}\left(a + \lambda \int_0^s f(u, x(u), x'(u))du - \frac{\lambda s}{T} \int_0^T f(u, x(u), x'(u))du\right).$$

Then, we have

$$(P+R)'(s) = A^{-1}\left(\phi^{-1}\left(a + \lambda \int_0^s f(u, x(u), x'(u))du - \frac{\lambda s}{T} \int_0^T f(u, x(u), x'(u))du\right)\right),$$

since $(A(P+R))'(s) = A(P+R)'(s)$. Hence, we have

$$R(x,\lambda)(t) - R(x,\lambda)(0) = \int_0^t A^{-1}\left(\phi^{-1}\left(a + \lambda \int_0^s f(u, x(u), x'(u))du - \frac{\lambda s}{T} \int_0^T f(u, x(u), x'(u))du\right)\right) ds,$$

since $R(x,\lambda)(0) = 0$. So, we can get

$$R(x,\lambda)(t) = \int_0^t A^{-1}\left(\phi^{-1}\left(a + \lambda \int_0^s f(u, x(u), x'(u))du - \frac{\lambda s}{T} \int_0^T f(u, x(u), x'(u))du\right)\right) ds.$$

Thus, N_λ is M-compact on $\bar{\Omega}$. Obviously, the equation

$$(\phi(Ax)'(t))' = \lambda f(t, x(t), x'(t))$$

can be converted to

$$Mx = N_\lambda x, \quad \lambda \in (0,1),$$

where M and N_λ are defined by (3.2) and (3), respectively. As proved above,

$$N_\lambda : \bar{\Omega} \to Z, \ \ \lambda \in (0,1)$$

is an M-compact mapping. From assumption (C_1), one find

$$Mx \neq N_\lambda x, \quad \lambda \in (0,1), \ x \in \partial\Omega,$$

and assumptions (C_2) and (C_3) imply that $\deg\{JQN, \Omega \cap \ker M, \theta\}$ is valid and

$$\deg\{JQN, \Omega \cap \ker M, \theta\} \neq 0.$$

So by applications of Lemma 2.1, we see that (3.1) has a T-periodic solution. □

In the following, applying Lemma 2.3 and Theorem 3.1, we consider the existence of a periodic solution to (1.1) in the case that $|c| = 1$.

Theorem 3.2. Assume that conditions (A_1), (A_2) (C_1), (C_2) and (C_3) hold, Ω is an open bounded set in $C^1_{2\pi}$. Furthermore, suppose one of the following conditions holds:

(i) $c = -1$ and $|\tau| = (m/n)\pi$, with $m,\ n$ are coprime positive integers with m even;
(ii) $c = -1$ and $|\tau| = (m/n)\pi$, with $m,\ n$ are coprime odd positive integers;
(iii) $c = -1$ and $|\tau| = (m/n)\pi$, with $m,\ n$ are coprime positive integers with m odd and n even;
(iv) $c = 1$ and $|\tau| = (m/n)\pi$, with $m,\ n$ are coprime positive integers with m odd;
(v) $c = 1$ and $|\tau| = \pi$.

Then (1.1) has at least one periodic solution on $\bar{\Omega}$.

Proof. We follow the same strategy and notation as in the proof of Theorem 3.1. Next, we consider $R(x,\lambda)(t)$.

Case (i) $c=-1$ and $|\tau|=(m/n)\pi$, with $m,\ n$ are coprime positive integers with m even. Take $T=2\pi$, from (3.3) and (3.4), applying Lemma 2.3, we know that there exist a continuous inverse operator A^{-1} of neutral operator A in the case that $c=-1$ such that

$$R(x,\lambda)(t)=\int_0^t A^{-1}\left(\phi^{-1}\left(a+\int_0^s \lambda f(u,x(u),x'(u))\right)du-\frac{\lambda s}{2\pi}\int_0^{2\pi} f(u,x(u),x'(u))du\right)ds,$$

where $a\in R$ is a constant such that

$$\begin{aligned}R(x,\lambda)(2\pi)&=\int_0^{2\pi} A^{-1}\left(\phi^{-1}\left(a+\int_0^s \lambda f(u,x(u),x'(u))\right)du-\frac{\lambda s}{2\pi}\int_0^{2\pi} f(u,x(u),x'(u))du\right)ds\\&=0.\end{aligned}$$

Similarly, we can get Case (ii)-Case (v). This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1. □

4 Application of Theorem 3.1: ϕ-Laplacian operator

As an application, we consider the following ϕ-Laplacian neutral Liénard equation

$$(\phi(Ax)'(t))'+f(x(t))x'(t)+g(t,x(t))=e(t), \tag{4.1}$$

where g is a continuous function defined on $\mathbb{R}^2$ and periodic in t with $g(t,\cdot)=g(t+T,\cdot)$, $f\in C(\mathbb{R},\mathbb{R})$, e is a continuous periodic function defined on $\mathbb{R}$ with period T and $\int_0^T e(t)\,dt=0$. Next, by applications of Theorem 3.1, we investigate the existence of a periodic solution for (4.1) in the case that $|c|\neq 1$.

Theorem 4.1. Suppose $|c|\neq 1$, (A_1) and (A_2) hold. Assume that the following conditions hold:

(H_1) There exists a constant $D>0$ such that

$$xg(t,x)>0,\ \ \forall\,(t,x)\in[0,T]\times\mathbb{R},\ \text{with } |x|>D.$$

(H_2) There exist two positive constants $\sigma_\star,\ \sigma^\star$ such that $\sigma_\star\le|f(x(t))|\le\sigma^\star,\ \forall\, t\in\mathbb{R}$.

(H_3) There exist positive constants $a,\ b,\ B$ such that

$$|g(t,x(t))|\le a|x(t)|+b,\quad \text{for } |x(t)|>B \text{ and } t\in\mathbb{R}.$$

Then (4.1) has at least one solution with period T if $\sigma_\star-\left(\sigma^\star|c|+\frac{\sqrt{2}}{2}(1+|c|)aT\right)>0$.

Proof. Consider the homotopic equation

$$(\phi(Ax)'(t))'+\lambda f(x(t))x'(t)+\lambda g(t,x(t))=\lambda e(t). \tag{4.2}$$

Firstly, we will claim that the set of all T-periodic solutions of (4.2) is bounded. Let $x(t)\in C_T^1$ be an arbitrary T-periodic solution of (4.2). As $(Ax)(0)=(Ax)(T)$, there exists a point $t_0\in(0,T)$ such that $(Ax)'(t_0)=0$, while $\phi(0)=0$, and we see

$$\begin{aligned}|\phi(Ax)'(t)|&=\left|\int_{t_0}^t(\phi(Ax)'(s))'ds\right|\\&=\lambda\int_0^T|f(x(t))||x'(t)|dt+\lambda\int_0^T|g(t,x(t))|dt+\lambda\int_0^T|e(t)|dt.\end{aligned}\tag{4.3}$$

Integrating both sides of (4.2) over $[0, T]$, we have

$$\int_0^T g(t, x(t))dt = 0. \tag{4.4}$$

From the mean value theorem, there is a constant $\xi \in (0, T)$ such that

$$g(\xi, x(\xi)) = 0.$$

In view of (H_1), we obtain

$$|x(\xi)| \leq D.$$

Then, we have

$$\begin{aligned}
\|x\| &= \max_{t\in[0,T]} |x(t)| = \max_{t\in[\xi,\xi+T]} |x(t)| \\
&= \frac{1}{2}\max_{t\in[\xi,\xi+T]} \left(|x(t)| + |x(t-T)|\right) \\
&= \frac{1}{2}\max_{t\in[\xi,\xi+T]} \left(\left|x(\xi) + \int_\xi^T x'(s)ds\right| + \left|x(\xi) - \int_{t-T}^\xi x'(s)ds\right|\right) \\
&\leq D + \frac{1}{2}\left(\int_\xi^t |x'(s)|ds + \int_{t-T}^\xi |x'(s)|ds\right) \\
&\leq D + \frac{1}{2}\int_0^T |x'(s)|ds.
\end{aligned} \tag{4.5}$$

Multiplying both sides of (4.2) by $(Ax)'(t)$ and integrating over the interval $[0, T]$, we get

$$\int_0^T (\phi(Ax)'(t))'(Ax)'(t)dt + \lambda\int_0^T f(x(t))x'(t)(Ax)'(t)dt + \lambda\int_0^T g(t, x(t))(Ax)'(t)dt = \lambda\int_0^T e(t)(Ax)'(t)dt. \tag{4.6}$$

Substituting $\int_0^T (\phi(Ax)'(t))'(Ax)'(t)dt = 0$, $\int_0^T f(x(t))x'(t)(Ax)'(t)dt = \int_0^T f(x(t))(x'(t))^2 dt - c\int_0^T f(x(t))x'(t)x'(t-\tau))dt$ into (4.6), we have

$$\int_0^T f(x(t))|x'(t)|^2 dt = c\int_0^T f(x(t))x'(t)x'(t-\tau)dt - \int_0^T g(t, x(t))(Ax)'(t)dt + \int_0^T e(t)(Ax)'(t)dt.$$

Therefore, we have

$$\left|\int_0^T f(x(t))|x'(t)|^2 dt\right| = \left|c\int_0^T f(x(t))x'(t)x'(t-\tau)dt - \int_0^T g(t, x(t))(Ax)'(t)dt + \int_0^T e(t)(Ax)'(t)dt\right|.$$

From (H_2), we have

$$\left|\int_0^T f(x(t))|x'(t)|^2 d\right| = \int_0^T |f(x(t))||x'(t)|^2 dt \geq \sigma_\star \int_0^T |x'(t)|^2 dt.$$

So, we have

$$\sigma_\star \int_0^T |x'(t)|^2 dt \leq |c|\int_0^T |f(x(t))||x'(t)||x'(t-\tau)|dt + \int_0^T |g(t, x(t))||x'(t)|dt + |c|\int_0^T |g(t, x(t))||x'(t-\tau)|dt$$

$$+\int_0^T |e(t)||x'(t)|dt+|c|\int_0^T |e(t)||x'(t-\tau)|dt.$$

Define

$$E_1:=\{t\in[0,T]|\ |x(t)|\le B\},\quad E_2:=\{t\in[0,T]|\ |x(t)|>B\}.$$

From (H_2) and the Hölder inequality, we have

$$\begin{aligned}
\sigma_\star\int_0^T |x'(t)|^2dt \le& |c|\sigma^\star\int_0^T |x'(t)||x'(t-\tau)|dt+\int_{E_1+E_2}|g(t,x(t))||x'(t)|dt+|c|\int_{E_1+E_2}|g(t,x(t))||x'(t-\tau)|dt\\
&+\|e\|\int_0^T |x'(t)|dt+|c|\|e\|\int_0^T |x'(t-\tau)|dt\\
\le& |c|\sigma^\star\left(\int_0^T |x'(t)|^2dt\right)^{\frac12}\left(\int_0^T |x'(t-\tau)|^2dt\right)^{\frac12}+\int_{E_1}|g(t,x(t))||x'(t)|dt+\int_{E_2}|g(t,x(t))||x'(t)|dt\\
&+|c|\int_{E_1}|g(t,x(t))||x'(t-\tau)|dt+|c|\int_{E_2}|g(t,x(t))||x'(t-\tau)|dt\\
&+\|e\|\int_0^T |x'(t)|dt+|c|\|e\|\int_0^T |x'(t-\tau)|dt
\end{aligned}\tag{4.7}$$

where $\|e\|:=\max\limits_{t\in[0,T]}|e(t)|$. Substituting $\int_0^T|x'(t-\tau)|dt=\int_0^T|x'(t)|dt$ into (4.7), and by applications of condition (H_3), we have

$$\begin{aligned}
\sigma_\star\int_0^T |x'(t)|^2dt \le& |c|\sigma^\star\int_0^T |x'(t)|^2dt+(1+|c|)\|g_B\|\int_0^T |x'(t)|dt+\int_{E_2}|g(t,x(t))||x'(t)|dt\\
&+|c|\int_{E_2}|g(t,x(t))||x'(t-\tau)|dt+(1+|c|)\|e\|\int_0^T |x'(t)|dt\\
\le& |c|\sigma^\star\int_0^T |x'(t)|^2dt+(1+|c|)\|g_B\|T^{\frac12}\left(\int_0^T |x'(t)|dt\right)^{\frac12}+a\int_0^T |x(t)||x'(t)|dt+b\int_0^T |x'(t)|dt\\
&+a|c|\int_0^T |x(t)||x'(t-\tau)|dt+b|c|\int_0^T |x'(t-\tau)|dt+(1+|c|)\|e\|T^{\frac12}\left(\int_0^T |x'(t)|dt\right)^{\frac12}\\
\le& |c|\sigma^\star\int_0^T |x'(t)|^2dt+a\left(\int_0^T |x(t)|dt\right)^{\frac12}\left(\int_0^T |x'(t)|dt\right)^{\frac12}+a|c|\left(\int_0^T |x(t)|dt\right)^{\frac12}\\
&\cdot\left(\int_0^T |x'(t-\tau)|dt\right)^{\frac12}+(1+|c|)\|g_B\|T^{\frac12}\left(\int_0^T |x'(t)|dt\right)^{\frac12}\\
&+(1+|c|)bT^{\frac12}\left(\int_0^T |x'(t)|dt\right)^{\frac12}+(1+|c|)\|e\|T^{\frac12}\left(\int_0^T |x'(t)|dt\right)^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&=|c|\sigma^{\star}\int_0^T |x'(t)|^2dt+(1+|c|)a\left(\int_0^T |x(t)|dt\right)^{\frac{1}{2}}\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}+N_1\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}\\
&\le|c|\sigma^{\star}\int_0^T |x'(t)|^2dt+(1+|c|)aT^{\frac{1}{2}}\|x\|^{\frac{1}{2}}\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}+N_1\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}
\end{aligned}
\tag{4.8}
$$

where $\|g_B\| := \max_{|x(t)|\le B}|g(t,x(t))|$ and $N_1 := (1+|c|)(|g_B|+b+\|e\|)T^{\frac{1}{2}}$. Substituting (4.5) into (4.8), we have

$$
\begin{aligned}
\sigma_{\star}\int_0^T |x'(t)|^2dt \le&|c|\sigma^{\star}\int_0^T |x'(t)|^2dt+(1+|c|)aT^{\frac{1}{2}}\left(D+\frac{1}{2}\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}\\
&+N_1\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}\\
&\le|c|\sigma^{\star}\int_0^T |x'(t)|^2dt+\frac{\sqrt{2}}{2}(1+|c|)aT\int_0^T |x'(t)|^2dt+\left((1+|c|)a(TD)^{\frac{1}{2}}+N_1\right)\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}
\end{aligned}
\tag{4.9}
$$

since $(a+b)^k \le a^k+b^k,\ 0<k<1$. From (4.9), we can get

$$
\sigma_{\star}\int_0^T |x'(t)|^2dt \le \left(|c|\sigma^{\star}+\frac{\sqrt{2}}{2}(1+|c|)aT\right)\int_0^T |x'(t)|^2dt+\left((1+|c|)a(TD)^{\frac{1}{2}}+N_1\right)\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{2}}.
$$

Since $\sigma_{\star}-\left(\sigma^{\star}|c|+\frac{\sqrt{2}}{2}(1+|c|)aT\right)>0$, it is easy to see that there exists a constant $M_1'>0$ (independent of λ) such that

$$
\int_0^T |x'(t)|^2dt \le M_1'. \tag{4.10}
$$

From (4.5) and the Hölder inequality, we have

$$
\|x\| \le D+\frac{1}{2}\int_0^T |x'(s)|ds \le D+\frac{1}{2}T^{\frac{1}{2}}\left(\int_0^T |x'(s)|^2ds\right)^{\frac{1}{2}} \le D+\frac{1}{2}T^{\frac{1}{2}}\left(M_1'\right)^{\frac{1}{2}} := M_1. \tag{4.11}
$$

From (4.3), (4.10) and (4.11), we see that

$$
\begin{aligned}
|\phi(Ax)'(t)| &= \left|\int_{t_0}^t (\phi(Ax)'(s))'ds\right|\\
&\le\lambda\int_0^T |f(x(t))||x'(t)|dt+\lambda\int_0^T g(t,x(t))|dt+\lambda\int_0^T |e(t)|dt\\
&\le\left(\|f_{M_1}\|T^{\frac{1}{2}}\left(\int_0^T |x'(t)|^2dt\right)^{\frac{1}{2}}+T\|g_{M_1}\|+T\|e\|\right)\\
&\le\|f_{M_1}\|T^{\frac{1}{2}}\left(M_1'\right)^{\frac{1}{2}}+T\|g_{M_1}\|+T\|e\| := M_2',
\end{aligned}
$$

where $\|f_{M_1}\| := \max_{|x(t)|\le M_1}|f(x(t))|$.

We claim that there exists a positive constant $M_2^\star > M_2' + 1$ such that, for all $t \in \mathbb{R}$, we have

$$\|(Ax)'\| \le M_2^\star. \tag{4.12}$$

In fact, if $(Ax)'(t)$ is not bounded, then from the definition of α, there exists a positive constant M_2'' such that $\alpha(|(Ax)'|) > M_2''$ for all $(Ax)' \in \mathbb{R}$. However, from (A_2), we have

$$\alpha(|(Ax)'|)|(Ax)'| \le \phi((Ax)')(Ax)' \le |\phi(Ax)'||(Ax)'| \le M_2'|(Ax)'|.$$

Then, we can get

$$\alpha(|(Ax)'|) \le M_2', \quad \text{for all } (Ax)' \in \mathbb{R},$$

which is a contradiction. So, (4.12) holds.

By Lemma 2.2 and (4.12), we have

$$\begin{aligned} \|x'\| = \|A^{-1}Ax'\| &= \|A^{-1}(Ax)'\| \\ &\le \frac{\|(Ax)'\|}{|1-|c||} \\ &\le \frac{M_2^\star}{|1-|c||} := M_2. \end{aligned} \tag{4.13}$$

Set $M = \sqrt{M_1^2 + M_2^2} + 1$, we have

$$\Omega = \{x \in C_T^1(\mathbb{R}, \mathbb{R}) | \ \|x\| \le M + 1, \ \|x'\| \le M + 1\},$$

and we know that (4.1) has no solution on $\partial\Omega$ as $\lambda \in (0, 1)$ and when $x(t) \in \partial\Omega \cap \mathbb{R}$, $x(t) = M + 1$ or $x(t) = -M - 1$, from (4.5) we know that $M + 1 > D$. So, from (H_1), we see that

$$\frac{1}{T}\int_0^T g(t, M + 1)dt < 0,$$

$$\frac{1}{T}\int_0^T g(t, -M - 1)dt > 0,$$

since $\int_0^T e(t)dt = 0$. So condition (C_2) of Theorem 3.1 is also satisfied. Set

$$H(x, \mu) = \mu x + (1 - \mu)\frac{1}{T}\int_0^T g(t, x))dt, \quad x \in \partial\Omega \cap \mathbb{R}, \quad \mu \in [0, 1]$$

Obviously, from (H_1), we can get $xH(x, \mu) > 0$ and thus $H(x, \mu)$ is a homotopic transformation and

$$\begin{aligned} \deg\{F, \Omega \cap \mathbb{R}, 0\} &= \deg\{\frac{1}{T}\int_0^T g(t, x)dt, \Omega \cap \mathbb{R}, 0\} \\ &= \deg\{x, \Omega \cap \mathbb{R}, 0\} \ne 0. \end{aligned}$$

So condition (C_3) of Theorem 3.1 is satisfied. In view of the Theorem 3.1, there exists a solution with period T. □

Remark 4.1. When $|c| = 1$, from Theorem 4.1, we know that $\sigma_\star - (\sigma^\star + \sqrt{2}aT) > 0$ does not hold. Therefore, by applications of the above method, we do not obtain the existence of periodic solution for (4.1) in critical case ($|c| = 1$).

5 Application of Theorem 3.1: p-Laplacian operator

When $(\phi(Ax)'(t))' \equiv (\phi_p(Ax)'(t))'$, then (4.1) is rewritten

$$(\phi_p(Ax)'(t))' + f(x(t))x'(t) + g(t, x(t)) = e(t). \tag{5.1}$$

Firstly, we consider the existence of a periodic solution for (5.1) in the case that $|c| \neq 1$ by applications of Theorem 3.1.

Theorem 5.1. Suppose $|c| \neq 1$ and condition (H_1) hold. Assume that the following conditions hold:
(H_4) There exist positive constants $\alpha,\ \beta$ such that $|f(x(t))| \leq \alpha|x(t)|^{p-2} + \beta,\ \forall\ t \in \mathbb{R}$.
(H_5) There exist positive constants $\gamma,\ \eta,\ B^*$ such that

$$|g(t, x(t))| \leq \gamma|x(t)|^{p-1} + \eta, \quad \text{for } |x(t)| > B^* \text{ and } t \in \mathbb{R}.$$

Then (5.1) has at least one solution with period T if $\frac{1}{2^{p-1}}|c|\alpha + \frac{1}{2^p}(1 + |c|)\gamma T < \frac{|1-|c||^p}{T^{\frac{p}{q}}}$.

Proof. Consider the homotopic equation

$$(\phi_p(Ax)'(t))' + \lambda f(x(t)x'(t) + \lambda g(t, x(t)) = \lambda e(t). \tag{5.2}$$

We follow the same strategy and notation as in the proof of Theorem 4.1. From (H_1), we know that there exists a constant $D > 0$ such that

$$|x(t)| \leq D + \frac{1}{2}\int_0^T |x'(t)|dt. \tag{5.3}$$

Multiplying both sides of (5.2) by $(Ax)(t)$ and integrating over the interval $[0, T]$, we get

$$\int_0^T (\phi_p(Ax)'(t))'(Ax)(t)dt + \lambda \int_0^T f(x(t))x'(t)(Ax)(t)dt + \lambda \int_0^T g(t, x(t))(Ax)(t)dt = \lambda \int_0^T e(t)dt. \tag{5.4}$$

Substituting $\int_0^T (\phi_p(Ax)'(t))'(Ax)(t)dt = -\int_0^T |(Ax)'(t)|^p dt$ and $\int_0^T f(x(t))x(t)x'(t)dt = 0$ into (5.4), we have

$$\int_0^T |(Ax)'(t)|^p dt = -\lambda c \int_0^T f(x(t))x'(t)x(t-\tau)dt + \lambda \int_0^T g(t, x(t))(x(t) - cx(t-\tau))dt - \lambda \int_0^T e(t)(x(t) - x(t-\tau))dt.$$

Then, we can get

$$\int_0^T |(Ax)'(t)|^p dt \leq |c|\|x\| \int_0^T |f(x(t))||x'(t)|dt + (1 + |c|)\|x\| \int_0^T |g(t, x(t))|dt + (1 + |c|) \int_0^T |e(t)|dt.$$

Define

$$E_3 := \{t \in [0, T]|\ |x(t)| \leq B^*\}, \quad E_4 := \{t \in [0, T]|\ |x(t)| > B^*\}.$$

From (H_3) and (H_4), we have

$$
\begin{aligned}
\int_0^T |(Ax)'(t)|^p dt \le & |c|\alpha\|x\| \int_0^T |x(t)|^{p-2}|x'(t)|dt + |c|\beta\|x\| \int_0^T |x'(t)|dt \\
& + (1+|c|)\|x\| \int_{E_3+E_4} |g(t,x(t))|dt + (1+|c|)\|x\|\|e\|T \\
\le & |c|\alpha\|x\| \int_0^T |x(t)|^{p-2}|x'(t)|dt + |c|\beta\|x\| \int_0^T |x'(t)|dt \\
& + (1+|c|)\|x\|\|g_{E_3}\|T + (1+|c|)\gamma\|x\| \int_0^T |x(t)|^{p-1}dt \\
& + (1+|c|)\eta T\|x\| + (1+|c|)\|x\|\|e\|T \\
\le & |c|\alpha\|x\|^{p-1} \int_0^T |x'(t)|dt + |c|\beta\|x\| \int_0^T |x'(t)|dt \\
& + (1+|c|)\gamma T\|x\|^p + (1+|c|)T(\|g_{E_3}\| + \eta + \|e\|)\|x\|,
\end{aligned}
\tag{5.5}
$$

where $\|g_{E_3}\| := \max\limits_{|x(t)|\le B^*} |g(t,x(t))|$. Substituting (5.3) into (5.5), we have

$$
\begin{aligned}
\int_0^T |(Ax)'(t)|^p dt \le & |c|\alpha \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right)^{p-1} \int_0^T |x'(t)|dt + |c|\beta \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right) \int_0^T |x'(t)|dt \\
& + (1+|c|)\gamma T \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right)^p + (1+|c|)T(\|g_{E_3}\| + \eta + \|e\|) \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right) \\
= & |c|\alpha \left(\frac{2D}{\int_0^T |x'(t)|dt} + 1\right)^{p-1} \frac{1}{2^{p-1}} \left(\int_0^T |x'(t)|dt\right)^p + |c|\beta \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right) \int_0^T |x'(t)|dt \\
& + (1+|c|)\gamma T \left(\frac{2D}{\int_0^T |x'(t)|dt} + 1\right)^p \frac{1}{2^p} \left(\int_0^T |x'(t)|dt\right)^p \\
& + (1+|c|)T(\|g_{E_3}\| + \eta + \|e\|) \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right) \\
\le & |c|\alpha \left(1 + \frac{2Dp}{\int_0^T |x'(t)|dt}\right) \frac{1}{2^{p-1}} \left(\int_0^T |x'(t)|dt\right)^p + |c|\beta \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right) \int_0^T |x'(t)|dt \\
& + (1+|c|)\gamma T \left(1 + \frac{2D(p+1)}{\int_0^T |x'(t)|dt}\right) \frac{1}{2^p} \left(\int_0^T |x'(t)|dt\right)^p \\
& + (1+|c|)T(\|g_{E_3}\| + \eta + \|e\|) \left(D + \frac{1}{2}\int_0^T |x'(t)|dt\right),
\end{aligned}
$$

since $(1+x)^p \le 1+(1+p)x$ for $x \in [0,\delta]$, here δ is a given positive constant, which is only dependent on $k>0$. Therefore, we have

$$\begin{aligned}
\int_0^T |(Ax)'(t)|^p dt \le & \frac{1}{2^{p-1}}|c|\alpha\left(\int_0^T |x'(t)|dt\right)^p + \frac{1}{2^{p-2}}|c|\alpha Dp\left(\int_0^T |x'(t)|dt\right)^{p-1} \\
& + \frac{1}{2^p}(1+|c|)\gamma T\left(\int_0^T |x'(t)|dt\right)^p + \frac{1}{2^{p-1}}(1+|c|)\gamma TD(p+1)\left(\int_0^T |x'(t)|dt\right)^{p-1} \\
& + \frac{1}{2}|c|\beta\left(\int_0^T |x'(t)|dt\right)^2 + N_3\int_0^T |x'(t)|dt + N_4 \\
= & \left(\frac{1}{2^{p-1}}|c|\alpha + \frac{1}{2^p}(1+|c|)\gamma T\right)\left(\int_0^T |x'(t)|dt\right)^p \\
& + \left(\frac{1}{2^{p-2}}|c|\alpha Dp + \frac{1}{2^{p-1}}(1+|c|)mTD(p+1)\right)\left(\int_0^T |x'(t)|dt\right)^{p-1} \\
& + \frac{1}{2}|c|\beta\left(\int_0^T |x'(t)|dt\right)^2 + N_3\int_0^T |x'(t)|dt + N_4.
\end{aligned} \tag{5.6}$$

where $N_3 = |c|\beta D + \frac{1}{2}(1+|c|)T(\eta + \|g_{M_3}\| + \|e\|)$, $N_4 = (1+|c|)TD(\eta + \|g_{M_3}\| + \|e\|)$. By application of Lemma 2.2, we have

$$\begin{aligned}
\int_0^T |x'(t)|dt = & \int_0^T |(A^{-1}Ax')(t)|dt \\
\le & \frac{\int_0^T |(Ax)'(t)|dt}{|1-|c||} \\
\le & \frac{T^{\frac{1}{q}}\left(\int_0^T |(Ax)'(t)|^p dt\right)^{\frac{1}{p}}}{|1-|c||},
\end{aligned} \tag{5.7}$$

since $(Ax')(t) = (Ax)'(t)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Applying the inequality

$$(a+b)^k \le a^k + b^k, \quad \text{for } a,\ b > 0,\ 0 < k < 1.$$

Substituting (5.6) into (5.7), we have

$$\begin{aligned}
\int_0^T |x'(t)|dt \le & \frac{T^{\frac{1}{q}}\left(\frac{1}{2^{p-1}}|c|\alpha + \frac{1}{2^p}(1+|c|)\gamma T\right)^{\frac{1}{p}}\int_0^T |x'(t)|dt}{|1-|c||} \\
& + \frac{T^{\frac{1}{q}}\left(\frac{1}{2^{p-2}}|c|\alpha Dp + \frac{1}{2^{p-1}}(1+|c|)\gamma TD(p+1)\right)^{\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{p-1}{p}}}{|1-|c||} \\
& + \frac{T^{\frac{1}{q}}\left(\left(\frac{1}{2}|c|\beta\right)^{\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{2}{p}} + N_3^{\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{p}} + N_4^{\frac{1}{p}}\right)}{|1-|c||}.
\end{aligned}$$

Since $\frac{1}{2^{p-1}}|c|\alpha + \frac{1}{2^p}(1+|c|)\gamma T < \frac{|1-|c||^p}{T^{\frac{p}{q}}}$, it is easy to see that there exists a positive constant M_1' such that

$$\int_0^T |x'(t)|dt \le M_1'. \tag{5.8}$$

From (5.3) and (5.8), we have

$$\|x\| \le D + \frac{1}{2}\int_0^{\omega} |x'(t)|dt \le D + \frac{1}{2}M_1' := M_1. \tag{5.9}$$

As $(Ax)(0) = (Ax)(T)$, there exists $t_1 \in [0, T]$ such that $(Ax)'(t_1) = 0$, while $\phi_p(0) = 0$, we have

$$\begin{aligned} |\phi_p((Ax)'(t))| &= \left|\int_{t_1}^{t} (\phi_p((Ax)'(s)))'ds\right| \\ &\le \lambda \int_0^T |f(x(t))||x'(t)|dt + \lambda \int_0^T |g(t, x(t))|dt + \lambda \int_0^T |e(t)|dt, \end{aligned} \tag{5.10}$$

where $t \in [t_1, t_1 + T]$. In view of (H_5), (5.8), (5.9) and (5.10), we have

$$\begin{aligned} \left\|\phi_p(Ax)'\right\| &= \max_{t\in[0,T]}\{\|\phi_p((Ax)'(t))\|\} \\ &= \max_{t\in[t_1,t_1+T]}\left\{\left|\int_{t_1}^{t} (\phi_p((Ax)'(s)))'ds\right|\right\} \\ &\le \lambda \left(\int_0^T |f(x(t)||x'(t)|dt + \int_0^T |g(t, x(t))|dt + \int_0^T |e(t)|dt\right) \\ &\le \|f_{M_1}\|M_1' + \gamma M_1^{p-1} + \eta T + T\|e\| := \lambda M_2', \end{aligned} \tag{5.11}$$

where $\|f_{M_1}\| := \max_{|x|\le M_1} |f(x(t)|$.

We claim that there exists a positive constant $M_2 > M_2' + 1$ such that, for all $t \in \mathbb{R}$

$$\|x'\| \le M_2. \tag{5.12}$$

In fact, if x' is not bounded, there exists a positive constant M_2'' such that $\|x'\| > M_2''$ for some $x' \in \mathbb{R}$. Therefore, we have

$$\|\phi_p(Ax)'\| = \|\phi_p(Ax')\| = \|Ax'\|^{p-1} = (1 + |c|)^{p-1}\|x'\|^{p-1} \ge (1 + |c|)^{p-1}M_2''^{p-1} := M_2^{\star}.$$

Then, it is a contradiction. So, (5.12) holds.

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 4.1. □

Remark 5.1. Obviously, the conditions (H_4) and (H_5) are weaken than the conditions (H_2) and (H_3). Moreover, by using the method of Theorem 5.1, we can investigate (5.1) in critical case $|c| = 1$.

Next, we discuss the existence of periodic solution for (5.1) in critical case $|c| = 1$ by using Theorem 3.1.

Theorem 5.2. Suppose conditions (H_1), (H_4), (H_5) and $|c| = 1$ hold. Then (5.1) has at least one solution with period T, if one of the following conditions holds:

(i) $c = -1$ and $|\tau| = (m/n)\pi$, with m, n are coprime positive integers with m even, and $\frac{1}{2^{p-1}}(\alpha + \gamma T) < \frac{\sigma_1^p}{T^{\frac{p}{q}}}$;

(ii) $c = -1$ and $|\tau| = (m/n)\pi$, with m, n are coprime odd positive integers, and $\frac{1}{2^{p-1}}(\alpha + \gamma T) < \frac{\sigma_2^p}{T^{\frac{p}{q}}}$;

(iii) $c = -1$ and $|\tau| = (m/n)\pi$, with m, n are coprime positive integers with m odd and n even, and $\frac{1}{2^{p-1}}(\alpha + \gamma T) < \frac{\sigma_3^p}{T^{\frac{p}{q}}}$;

(iv) $c = 1$ and $|\tau| = (m/n)\pi$, with m, n are coprime positive integers with m odd, and $\frac{1}{2^{p-1}}(\alpha + \gamma T) < \frac{\sigma_4^p}{T^{\frac{p}{q}}}$;

(v) $c = 1$ and $|\tau| = \pi$, and $\frac{1}{2^{p-1}}(\alpha + \gamma T) < \frac{\sigma_5^p}{T^{\frac{p}{q}}}$.

Proof. We follow the same strategy and notation as in the proof of Theorem 5.1. Next, we consider that there exists a positive constant M_1' such that

$$\int_0^T |x'(t)|dt \le M_1'.$$

Case (i). If $c = -1$ and $|\tau| = (m/n)\pi$, with $m,\ n$ are coprime positive integers with m even. From (5.7) and Lemma 2.3, we have

$$\begin{aligned}\int_0^T |x'(t)|dt &= \int_0^T |(A^{-1}Ax')(t)|dt\\ &\le \frac{\int_0^T |(Ax)'(t)|dt}{\sigma_1}\\ &\le \frac{T^{\frac{1}{q}}\left(\int_0^T |(Ax)'(t)|^p dt\right)^{\frac{1}{p}}}{\sigma_1}.\end{aligned} \tag{5.13}$$

Substituting (5.6) into (5.13), we have

$$\begin{aligned}\int_0^T |x'(t)|dt \le{}& \frac{T^{\frac{1}{q}}\left(\frac{1}{2^{p-1}}\alpha + \frac{1}{2^{p-1}}\gamma T\right)^{\frac{1}{p}}\int_0^T |x'(t)|dt}{\sigma_1} + \frac{T^{\frac{1}{q}}\left(\frac{1}{2^{p-2}}\alpha Dp + \frac{1}{2^{p-2}}\gamma TD(p+1)\right)^{\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{p-1}{p}}}{\sigma_1}\\ &+ \frac{T^{\frac{1}{q}}\left(\left(\frac{1}{2}\beta\right)^{\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{2}{p}} + N_5^{\frac{1}{p}}\left(\int_0^T |x'(t)|dt\right)^{\frac{1}{p}} + N_6^{\frac{1}{p}}\right)}{\sigma_1},\end{aligned}$$

where $N_5 = \beta D + T(\eta + \|g_{M_3}\| + \|e\|),\ N_6 = 2TD(\eta + \|g_{M_3}\| + \|e\|)$. Since $\alpha + \gamma T < \frac{2^{p-1}\sigma_1^p}{T^{\frac{p}{q}}}$, it is easy to see that there exists a positive constant M_1' such that

$$\int_0^T |x'(t)|dt \le M_1'.$$

Similarly, we can get Case (ii)-Case (v). This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 5.1. □

6 Examples

Example 6.1. Consider the following ϕ-Laplacian Liénard equation:

$$(\phi(x(t) - \frac{1}{10}x(t-\tau))')' + (\cos x + 3)x'(t) + \frac{1}{10}(\cos 2t + 1)x(t-\sigma) = \sin 2t, \tag{6.1}$$

where relativistic operator $\phi(u) = \frac{u}{\sqrt{1-\left(\frac{|u|}{c^*}\right)^2}}$, here c^* is the speed of light in the vacuum and $c^* > 0,\ \tau,\ \sigma$ are constants and $0 \le \tau,\ \sigma < T$.

Comparing (6.1) to (4.1), it is easy to see that $f(x) = \cos x + 3$, $g(t,x) = \frac{1}{10}(\cos 2t + 2)x$, $e(t) = \sin 2t,\ T = \pi,\ c = \frac{1}{10}$. Obviously, we get

$$\left(\frac{u}{\sqrt{1-\left(\frac{|u|}{c^*}\right)^2}} - \frac{v}{\sqrt{1-\left(\frac{|v|}{c^*}\right)^2}}\right)(u-v) \ge 0,$$

and

$$\phi(u) \cdot u = \frac{|u|^2}{\sqrt{1 - \left(\frac{|u|}{c^*}\right)^2}}.$$

So, the conditions (A_1) and (A_2) hold. Moreover, it is easy to see that there exists a constant $D = 1$ such that (H_1) holds. $2 \leq |f(x)| = |\cos x + 3| \leq 4$, here $\sigma_\star = 2,\ \sigma^\star = 4$, condition (H_2) holds. Consider $|g(t,x)| = |\frac{1}{10}(\cos 2t+2)x| \leq \frac{3}{10}|x|+1$, here $a = \frac{3}{10},\ b = 1$. So, condition (H_3) is satisfied. Next, we consider the condition

$$\begin{aligned} \sigma_\star - (|c|\sigma^\star + \frac{\sqrt{2}}{2}(1+|c|)aT) &= 2 - \left(\frac{1}{10} \times 4 + \frac{\sqrt{2}}{2}\left(1 + \frac{1}{10}\right) \times \frac{3}{10} \times \pi\right) \\ &= 2 - \left(\frac{2}{5} + \frac{33\sqrt{2}\pi}{200}\right) > 0. \end{aligned}$$

Therefore, by Theorem 4.1, we know that (6.1) has at least one positive π-periodic solution.

Example 6.2. Consider the p-Laplacian neutral Liénard equation:

$$(\phi_p(x(t) - 11x(t-\tau))')' + (x^4+3)x'(t) + (5+\sin t)x^5(t-\sigma) = \cos t, \tag{6.2}$$

where $p = 6,\ \tau,\ \sigma$ are constants and $0 \leq \tau,\ \sigma < T$.

It is clear that $T = 2\pi$, $g(t,x) = (5+\sin t)x^5(t-\sigma)$, $f(x) = x^4+3$, $e(t) = \cos t$. It is obvious that there exists a constant $D = 1$ such that (H_1) holds. $|f(x)| = |x^4+3| \leq |x|^4 + 5$, here $\alpha = 1,\ \beta = 5$, condition (H_4) holds. Consider $|g(t,x)| = |(5+\sin t)x^5| \leq 6|x|^5 + 1$, here $\gamma = 6,\ \eta = 1$. So, condition (H_3) is satisfied. Next, we consider the condition

$$\begin{aligned} \frac{T^{\frac{p}{q}}\left(\frac{1}{2^{p-1}}|c|\alpha + \frac{1}{2^p}(1+|c|)\gamma T\right)}{|1-|c||^p} &= \frac{(2\pi)^5\left(\frac{1}{2^5} \times 11 + \frac{1}{2^6} \times (1+11) \times 6 \times 2\pi\right)}{10^6} \\ &\approx \frac{72343}{1000000} < 1. \end{aligned}$$

Therefore, by applications of Theorem 5.1, we know that (6.2) has at least one positive periodic solution.

Competing interests

The authors declare that they have no competing interests concerning the publication of this manuscript.

Author's contributions

The authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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