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Stability Problems and Analytical Integration for the Clebsch's System

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Abstract: The nonlinear stability and the existence of periodic orbits of the equilibrium states of the Clebsch's system are discussed. Numerical integration using the Lie-Trotter integrator and the analytic approximate solutions using Multistage Optimal Homotopy Asymptotic Method are presented, too.

Keywords: optimal control, nonlinear ordinary differential systems, nonlinear stability, multistage optimal homotopy asymptotic method.

MSC: 34H15, 65Nxx, 65P40, 70H14, 74G10, 74H10

1 Introduction

The Clebsch's system was proposed in 1870 (see [1] for details) and it represents a specific famous case of the Kirchoff equations which describes the motion of a rigid body in an ideal fluid. The Clebsch's case was obtained from the equations:

$$\dot{x} = x \times \frac{\partial H}{\partial p}, \ \dot{p} = x \times \frac{\partial H}{\partial x} + p \times \frac{\partial H}{\partial p} \tag{1}$$

by taking the quadratic Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{3}\left(a_i p_i^2 + c_i x_i^2\right),$$

where

$$\frac{c_2 - c_3}{a_1} + \frac{c_3 - c_1}{a_2} + \frac{c_1 - c_2}{a_3} = 0.$$

The physical meaning of p is the total angular momentum, whereas x represents the total linear momentum of the system.

If we consider now the Hamiltonian

$$H_1 = \frac{1}{2}\sum_{i=1}^{3}\left(p_i^2 + a_i x_i^2\right), \tag{2}$$

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the equations (1) become:

$$\begin{cases} \dot{x_1} = x_2p_3 - x_3p_2 \\ \dot{x_2} = x_3p_1 - x_1p_3 \\ \dot{x_3} = x_1p_2 - x_2p_1 \\ \dot{p_1} = (a_3 - a_2)x_2x_3 \\ \dot{p_2} = (a_1 - a_3)x_1x_3 \\ \dot{p_3} = (a_2 - a_1)x_1x_2 \end{cases} \tag{3}$$

where a_1, a_2, a_3 are different and nonzero constants. It is well-known that its first integrals are:

$$H_2 = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2),$$

$$H_3 = x_1p_1 + x_2p_2 + x_3p_3,$$

and

$$H_4 = \frac{1}{2}(a_1p_1^2 + a_2p_2^2 + a_3p_3^2 - a_2a_3x_1^2 - a_1a_3x_2^2 - a_1a_2x_3^2).$$

During the time since its publication, a lot of problems about the Clebsch's system have been studied like its almost Lie-Poisson structure ([2]), the Lax formulation ([3]) or its Hirota-Kimura type discretization ([4]).

The paper's structure is as follows: first, the nonlinear stability of the equilibrium states of Clebsch's dynamics is discussed. About this problem, only partial results were found in [2] due to the fact that the existence of a Hamilton-Poisson structure is still an open problem, and for the almost Hamilton-Poisson structure proposed only one Casimir function was found instead of two. We use here the Arnold's method in order to obtain some new results, which does not require a Hamilton-Poisson structure. The existence of the periodic orbits around the nonlinear stable states is the subject of the second part. The last part is committed to the numerical and analytical integration. Two methods are proposed: the Lie-Trotter integrator for numerical integration and the Multistage Optimal Homotopy Asymptotic Method to find the analytic approximate solutions. Numerical simulations obtained via Mathematica 10 are presented, too.

2 Stability problems

The equilibrium states of the dynamics (3) are

$$e_1^{M,N} = (M, 0, 0, N, 0, 0), \quad e_2^{M,N} = (0, M, 0, 0, N, 0), \quad e_3^{M,N} = (0, 0, M, 0, 0, N),$$

$$e_4^{M,N,P} = (0, 0, 0, M, N, P), \qquad M, N, P \in \mathbb{R}.$$

Proposition 1. *If $a_1 < a_2$ and $a_1 < a_3$ then the equilibrium states $e_1^{M,M}$ are nonlinear stable.*

Proof. We shall make the proof using Arnold's method ([5]).
Let $F_{\alpha,\beta,\gamma} \in C^\infty\left(R^6, R\right)$ given by

$$F_{\alpha,\beta,\gamma} = H_1 + \alpha H_2 + \beta H_3 + \gamma H_4.$$

Following Arnold's method, we have successively:

$\nabla F_{\alpha,\beta,\gamma}\left(e_1^{M,M}\right) = 0$ iff $\alpha = 1 - a_1, \beta = -1, \gamma = 0$.

Considering the space

$$X = Ker(dH_2)\left(e_1^{M,M}\right) \cap Ker(dH_3)\left(e_1^{M,M}\right) \cap Ker(dH_4)\left(e_1^{M,M}\right)$$

$$= span\left(\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}\right)$$

then

$$v^t\nabla^2 F_{1-a_1,-1,0}\left(e_1^{M,M}\right)v = (a_2-a_1)a^2+(a_3-a_1)b^2+(a-c)^2+(b-d)^2,$$

for any $v \in X$, i.e. $v = \begin{pmatrix}0 & a & b & 0 & c & d\end{pmatrix}^t$, so

$$\nabla^2 F_{1-a_1,-1,0}\left(e_1^{M,M}\right) \mid X \times X$$

is positive definite if $a_1 < a_2$ and $a_1 < a_3$. □

□

Using the same arguments we obtain the following results:

Proposition 2. *The equilibrium states $e_2^{M,M}$ are nonlinearly stable if $a_2 < a_1$ and $a_2 < a_3$.*

Proposition 3. *The equilibrium states $e_3^{M,M}$ are nonlinearly stable if $a_3 < a_1$ and $a_3 < a_2$.*

Proposition 4. *The equilibrium states $e_4^{M,N,P}$ are nonlinearly stable for any $M, N, P \in \mathbf{R}$.*

Proof. For this case we consider the function $G_{\alpha,\beta} \in C^\infty\left(R^6, R\right)$,

$$G_{\alpha,\beta} = H_2 + \alpha H_1 + \beta H_4.$$

Following Arnold's method, we have successively:
$\nabla G_{\alpha,\beta}\left(e_4^{M,N,P}\right) = 0$ iff $\alpha = 0, \beta = 0$.
Let us consider the space

$$Y = Ker(dH_1)\left(e_4^{M,N,P}\right) \cap Ker(dH_4)\left(e_1^{M,N,P}\right)$$

$$= span\left(\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}\right)$$

then

$$v^t\nabla^2 G_{0,0}\left(e_4^{M,N,P}\right)v = a^2+b^2+c^2,$$

for any $v \in Y$, i.e. $v = \begin{pmatrix}0 & 0 & 0 & a & b & c\end{pmatrix}^t$, so

$$\nabla^2 G_{0,0}\left(e_4^{M,N,P}\right) \mid Y \times Y$$

is positive definite. □

□

3 Periodic orbits

Proposition 5. *If $a_1 < a_2$ and $a_1 < a_3$ then, near to $e_1^{M,M} = (M, 0, 0, M, 0, 0)$, the reduced dynamics have, for each sufficiently small value of the reduced energy, at least one periodic solution whose period is close to $\frac{2\pi}{|\lambda|}$, where*

$$\lambda = M\frac{\sqrt{2a_1 - a_2 - a_3 - 1 - \sqrt{1 - 4a_1 + 2a_2 + a_2^2 + 2a_3 - 2a_2a_3 + a_3^2}}}{\sqrt{2}}.$$

Proof. We use the Moser-Weinstein theorem with zero eigenvalue, see [6] for details:

(i) The restriction of our dynamics (3) to the coadjoint orbit:

$$x_1^2 + x_2^2 + x_3^2 = M^2,$$

$$x_1p_1 + x_2p_2 + x_3p_3 = M^2,$$

$$a_1p_1^2 + a_2p_2^2 + a_3p_3^2 - a_2a_3x_1^2 - a_1a_3x_2^2 - a_1a_2x_3^2 = (a_1 - a_2a_3)M^2$$

gives rise to a classical Hamiltonian system.

(ii) The matrix of the linear part of the reduced dynamics is:

$$A = \begin{bmatrix} 0 & p_3 & -p_2 & 0 & -x_3 & x_2 \\ -p_3 & 0 & p_1 & x_3 & 0 & -x_1 \\ p_2 & -p_1 & 0 & -x_2 & x_1 & 0 \\ 0 & (a_3 - a_2)x_3 & (a_3 - a_2)x_2 & 0 & 0 & 0 \\ (a_1 - a_3)x_3 & 0 & (a_1 - a_3)x_1 & 0 & 0 & 0 \\ (a_2 - a_1)x_2 & (a_2 - a_1)x_1 & 0 & 0 & 0 & 0 \end{bmatrix}\Bigg|_{e_1^{M,M}} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & -M \\ 0 & -M & 0 & 0 & M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (a_1 - a_3)M & 0 & 0 & 0 \\ 0 & (a_2 - a_1)M & 0 & 0 & 0 & 0 \end{bmatrix}$$

and has purely imaginary roots. More exactly:

$$\lambda_{1,2} = 0, \lambda_{3,4,5,6} = \pm iM\frac{\sqrt{\sqrt{1 - 4a_1 + 2a_2 + a_2^2 + 2a_3 - 2a_2a_3 + a_3^2} + 1 + a_2 + a_3 - 2a_1}}{\sqrt{2}}.$$

(iii) $\operatorname{span}(\nabla H_2(e_1^{M,M}), \nabla H_3(e_1^{M,M}), \nabla H_4(e_1^{M,M})) = V_0 = span\left\{\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}\right\}$, where

$$V_0 = \ker(A(e_1^{M,M})).$$

(iv) The smooth function $F_{1-a_1,-1,0} \in C^\infty(\mathbf{R}^6, \mathbf{R})$ given by

$$F_{1-a_1,-1,0} = H_1 + (1 - a_1)H_2 - H_3$$

has the following properties:
• It is a constant of motion of the dynamics (3).
• $\nabla F_{1-a_1,-1,0}\left(e_1^{M,M}\right) = 0.$
• If $a_1 < a_2$ and $a_1 < a_3$ then

$$\nabla^2 F_{1-a_1,-1,0}(e_1^{M,M})\Big|_{X\times X} > 0,$$

where

$$X := Ker(dH_2)\left(e_1^{M,M}\right) \cap Ker(dH_3)\left(e_1^{M,M}\right) \cap Ker(dH_4)\left(e_1^{M,M}\right)$$
$$= span\left(\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}\right).$$

Then our assertion follows. □

Using similar arguments, the following results hold:

Proposition 6. *If $a_2 < a_1$ and $a_2 < a_3$ then near to $e_2^{M,M} = (0, M, 0, 0, M, 0)$, the reduced dynamics has, for each sufficiently small value of the reduced energy, at least one periodic solution whose period is close to $\frac{2\pi}{|\omega|}$, where*

$$\omega = M\frac{\sqrt{2a_2 - a_1 - a_3 - 1 - \sqrt{1 - 4a_2 + 2a_3 + a_1^2 + 2a_1 - 2a_1a_3 + a_3^2}}}{\sqrt{2}}.$$

Proposition 7. *If $a_3 < a_1$ and $a_3 < a_2$ then near to $e_3^{M,M} = (0, 0, M, 0, 0, M)$, the reduced dynamics has, for each sufficiently small value of the reduced energy, at least one periodic solution whose period is close to $\frac{2\pi}{|\sigma|}$, where*

$$\sigma = M\frac{\sqrt{2a_3 - a_1 - a_2 - 1 - \sqrt{1 + 2a_2 - 4a_3 + a_2^2 + 2a_1 - 2a_1a_2 + a_3^2}}}{\sqrt{2}}.$$

Remark 1. *The existence of periodic orbits around the equilibrium states $e_4^{M,N,P}$ remains an open problem. Due to the fact that the states $e_4^{M,N,P}$ are not regular points for the Hamiltonian H_2, the method described above does not work.*

4 Numerical integration

We shall discuss now the numerical integration of the equations (3) via the Lie-Trotter formula ([7]). Figures 1 and 2 present numerical simulations obtained with MATHEMATICA 10.

Following [2] or [7], the Lie-Trotter integrator can be written as:

$$\left[x_1^{n+1}\ x_2^{n+1}\ x_3^{n+1}\ p_1^{n+1}\ p_2^{n+1}\ p_3^{n+1}\right]^t = \tag{4}$$

$$= A_1(t)A_2(t)A_3(t)B_1(t)B_2(t)B_3(t)\left[x_1^n\ x_2^n\ x_3^n\ p_1^n\ p_2^n\ p_3^n\right]^t,$$

where

$$A_1(t)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&a_1x_1(0)t&0&1&0\\0&-a_1x_1(0)t&0&0&0&1\end{bmatrix},\quad A_2(t)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&-a_2x_2(0)t&1&0&0\\0&0&0&0&1&0\\a_2x_2(0)t&0&0&0&0&1\end{bmatrix},$$

$$A_3(t)=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&a_3x_3(0)t&0&1&0&0\\-a_3x_3(0)t&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\quad B_1(t)=\begin{bmatrix}1&0&0&0&0&0\\0&cosp_1(0)t&sinp_1(0)t&0&0&0\\0&-sinp_1(0)t&cosp_1(0)t&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},$$

$$B_2(t)=\begin{bmatrix}cosp_2(0)t&0&-sinp_2(0)t&0&0&0\\0&0&0&0&0&0\\sinp_2(0)t&0&cosp_2(0)t&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\quad B_3(t)=\begin{bmatrix}cosp_3(0)t&sinp_3(0)t&0&0&0&0\\sinp_3(0)t&cosp_3(0)t&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}.$$

Some of its properties are sketched in the following proposition:

Proposition 8. *The numerical integrator (4) preserves the Hamiltonians H_1, H_2, H_3 and H_4 if*

$$x_1(0)=x_2(0)=x_3(0)=0$$

and

$$p_1(0)=p_2(0)=p_3(0)=0.$$

5 Analytic approximate solutions of the Clebsch System (3) using Multistage Optimal Homotopy Asymptotic Method

In order to find the analytical approximate solutions of the nonlinear differential system (3) with the boundary conditions

$$x_i(0)=A_i,\quad p_i(0)=A_{i+3},\ i=\overline{1,3}, \tag{5}$$

we will use the Multistage Optimal Homotopy Asymptotic Method (MOHAM) [8], as follows:

a) we divide the theoretical interval $[t_0,\ T]$ into some subintervals as $[t_0,\ t_1),\dots,[t_{j-1},\ t_j),\dots$, where $t_j=T$;

b) we apply the Optimal Homotopy Asymptotic Method (OHAM) [9, 10] to find the first-order approximate solutions using only one iteration.

The initial approximation in each interval $[t_{j-1},\ t_j)$, $j\in\mathbb{N}^\star$ is provided by the solution from the previous interval, so the analytical approximate solutions can be obtained for equations of the general form

$$\mathcal{L}\Big(F(t)\Big)+\mathcal{N}\Big(F(t)\Big)=0, \tag{6}$$

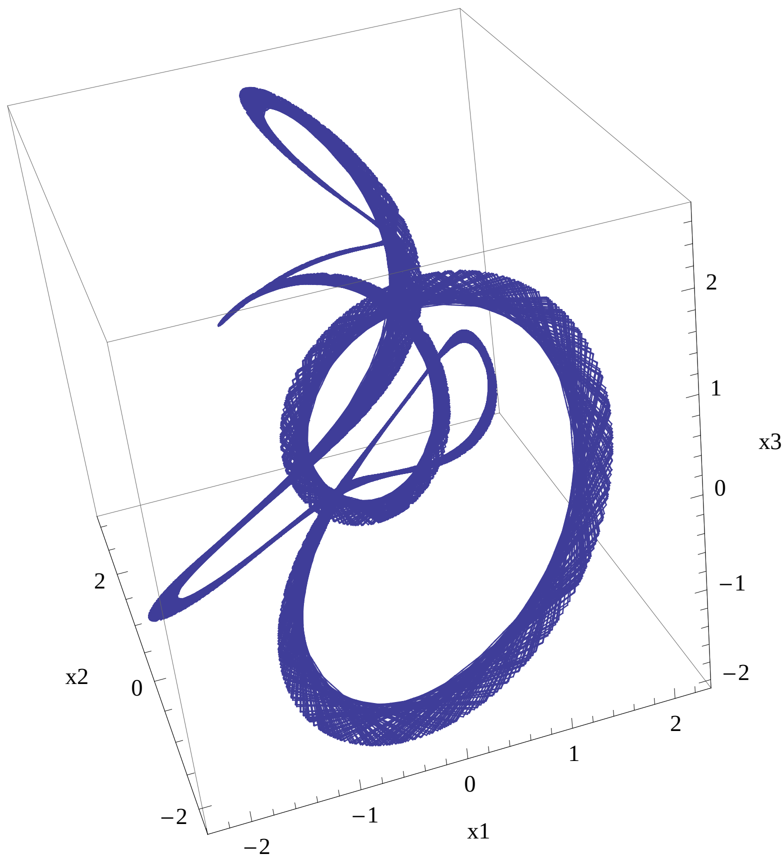

Figure 1: The Lie-Trotter integrator of the system (3), projection on $(Ox_1x_2x_3)$ plane ($a_1 = 10, a_2 = 2, a_3 = 3, x_1(0) = x_2(0) = x_3(0) = p_1(0) = p_2(0) = p_3(0) = 1$).

subject to the initial conditions (5), where $\mathcal{L}$ is a linear operator (which is not unique) and $\mathcal{N}$ is a nonlinear one.

Now, choosing the linear operators $\mathcal{L}$ as:

$$\begin{aligned}
\mathcal{L}\left[x_1(t)\right] &= \dot{x}_1 - b_3x_2(t) + b_2x_3(t),\\
\mathcal{L}\left[x_2(t)\right] &= \dot{x}_2 - b_1x_3(t) + b_3x_1(t),\\
\mathcal{L}\left[x_3(t)\right] &= \dot{x}_3 - b_2x_1(t) + b_1x_2(t),\\
\mathcal{L}\left[p_1(t)\right] &= \dot{p}_1 - (a_3 - a_2)x_2(t)x_3(t),\\
\mathcal{L}\left[p_2(t)\right] &= \dot{p}_2 - (a_1 - a_3)x_1(t)x_3(t),\\
\mathcal{L}\left[p_3(t)\right] &= \dot{p}_3 - (a_2 - a_1)x_1(t)x_2(t),
\end{aligned} \tag{7}$$

and the nonlinear operators $\mathcal{N}\left[x_i(t)\right]$ and $\mathcal{N}\left[p_i(t)\right]$, $i = \overline{1,3}$ as

$$\begin{aligned}
\mathcal{N}\left[x_1(t)\right] &= b_3x_2(t) - b_2x_3(t) + x_2(t)p_3(t) - x_3(t)p_2(t),\\
\mathcal{N}\left[x_2(t)\right] &= b_1x_3(t) - b_3x_1(t) + x_3(t)p_1(t) - x_1(t)p_3(t),\\
\mathcal{N}\left[x_3(t)\right] &= b_2x_1(t) - b_1x_2(t) + x_1(t)p_2(t) - x_2(t)p_1(t),\\
\mathcal{N}\left[p_1(t)\right] &= 0,\\
\mathcal{N}\left[p_2(t)\right] &= 0,\\
\mathcal{N}\left[p_3(t)\right] &= 0,
\end{aligned} \tag{8}$$

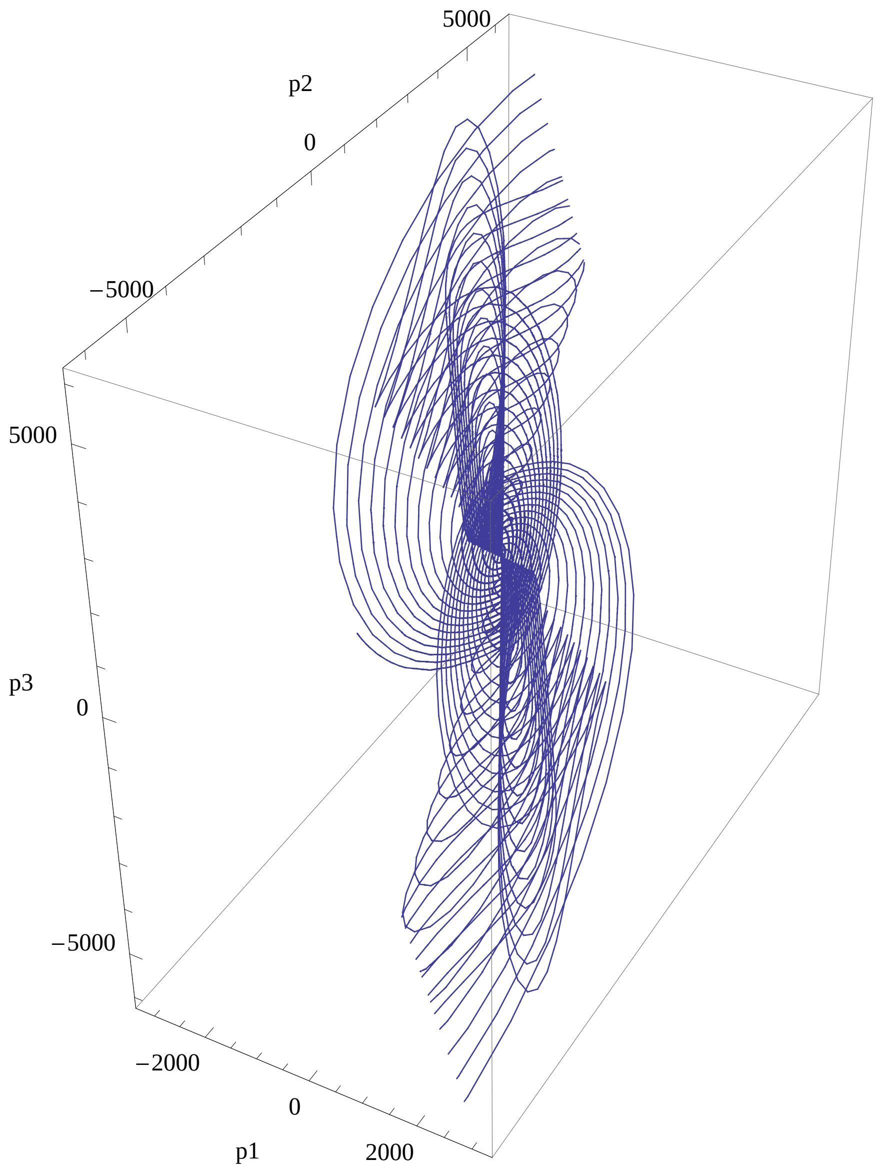

Figure 2: The Lie-Trotter integrator of the system (3), projection on $(Op_1p_2p_3)$ plane ($a_1 = 10$, $a_2 = 2$, $a_3 = 3$, $x_1(0) = x_2(0) = x_3(0) = p_1(0) = p_2(0) = p_3(0) = 1$).

$b_1,\ b_2,\ b_3 \in \mathbf{R}$, and following [9, 10], we are able to construct the homotopy given by:

$$\mathcal{H}\left[\mathcal{L}\left(F(t,p)\right),\ H(t,C_i),\ \mathcal{N}\left(F(t,p)\right)\right], \tag{9}$$

where $p \in [0,\ 1]$ is the embedding parameter, and $H(t, C_i) \neq 0$ is an auxiliary convergence-control function, depending of the variable t and the parameters $C_1, C_2, \ldots, C_s$.

The following properties hold:

$$\mathcal{H}\left[\mathcal{L}\left(F(t,0)\right),\ H(t,C_i),\ \mathcal{N}\left(F(t,0)\right)\right] = \mathcal{L}\left(F(t,0)\right) = \mathcal{L}\left(F_0(t)\right) \tag{10}$$

and

$$\mathcal{H}\left[\mathcal{L}\left(F(t,1)\right),\ H(t,C_i),\ \mathcal{N}\left(F(t,1)\right)\right] = H(t,C_i)\mathcal{N}\left(F(t,1)\right). \tag{11}$$

For the functions F of the form

$$F(t,p) = F_0(t) + pF_1(t, C_i), \tag{12}$$

the following relation is obtained:

$$\mathcal{H}\left[\mathcal{L}\Big(F(t,p)\Big),\ H(t,C_i),\ \mathcal{N}\Big(F(t,p)\Big)\right] = 0. \tag{13}$$

Considering the homotopy $\mathcal{H}$ given by:

$$\begin{aligned}\mathcal{H}\left[\mathcal{L}\Big(F(t,p)\Big),\ H(t,C_i),\ \mathcal{N}\Big(F(t,p)\Big)\right] &= \mathcal{L}\Big(F_0(t)\Big) + \\ &+p\left[\mathcal{L}\Big(F_1(t,C_i)\Big) - H(t,C_i)\mathcal{N}\Big(F_0(t)\Big)\right],\end{aligned} \tag{14}$$

and using the linear operator given by Eq. (7), the solutions of the equation

$$\mathcal{L}\Big(F_0(t)\Big) = 0, \quad \mathcal{B}\Big(F_0(t), \frac{dF_0(t)}{dt}\Big) = 0 \tag{15}$$

for the initial approximations x_{i_0} and p_{i_0}, $i = \overline{1,3}$ respectively, are

$$\begin{aligned}
&x_{1_0}(t) = M_1 + N_1\cos(\omega_0 t) + P_1\sin(\omega_0 t)\\
&x_{2_0}(t) = M_2 + N_2\cos(\omega_0 t) + P_2\sin(\omega_0 t)\\
&x_{3_0}(t) = M_3 + N_3\cos(\omega_0 t) + P_3\sin(\omega_0 t)\\
&p_{1_0}(t) = \tfrac{1}{4\omega_0}\big(4A_4\omega_0 - 4b_1\omega_0 - 4a_2M_3P_2 + 4a_3M_3P_2 - a_2N_3P_2 + a_3N_3P_2 - 4a_2M_2P_3+\\
&+4a_3M_2P_3 - a_2N_2P_3 + a_3N_2P_3 + (-4a_2M_2M_3 + 4a_3M_2M_3 - 2a_2N_2N_3 + 2a_3N_2N_3-\\
&-2a_2P_2P_3 + 2a_3P_2P_3)t\omega_0 + (4a_2M_3P_2 - 4a_3M_3P_2 + 4a_2M_2P_3 - 4a_3M_2P_3)\cos(\omega_0 t)+\\
&+(a_2N_3P_2 - a_3N_3P_2 + a_2N_2P_3 - a_3N_2P_3)\cos(2\omega_0 t) + (-4a_2M_3N_2 + 4a_3M_3N_2 - 4a_2M_2N_3+\\
&+4a_3M_2N_3)\sin(\omega_0 t) + (-a_2N_2N_3 + a_3N_2N_3 + a_2P_2P_3 - a_3P_2P_3)\sin(2\omega_0 t)\big)\\
&p_{2_0}(t) = \tfrac{1}{4\omega_0}\big(4A_5\omega_0 - 4b_2\omega_0 + 4a_1M_3P_1 - 4a_3M3P_1 + a_1N_3P_1 - a_3N_3P_1 + 4a_1M_1P_3-\\
&-4a_3M_1P_3 + a_1N_1P_3 - a3N1P3 + (+4a_1M_1M_3 - 4a_3M_1M_3 + 2a_1N_1N_3 - 2a_3N_1N_3+\\
&+2a_1P_1P_3 - 2a_3P_1P_3)t\omega_0 + (-4a_1M_3P_1 + 4a_3M_3P_1 - 4a_1M_1P_3 + 4a_3M_1P_3)\cos(\omega_0 t)+\\
&+(-a_1N_3P_1 + a_3N_3P_1 - a_1N_1P_3 + a_3N_1P_3)\cos(2\omega_0 t) + (4a_1M_3N_1 - 4a_3M_3N_1 + 4a_1M_1N_3-\\
&-4a_3M_1N_3)\sin(\omega_0 t) + (a_1N_1N_3 - a_3N_1N_3 - a_1P_1P_3 + a_3P_1P_3)\sin(2\omega_0 t)\big)\\
&p_{3_0}(t) = \tfrac{1}{4\omega_0}\big(4A_6\omega_0 - 4b_3\omega_0 - 4a_1M_2P_1 + 4a_2M_2P_1 - a_1N_2P_1 + a_2N_2P_1 - 4a_1M_1P_2+\\
&+4a_2M_1P_2 - a_1N_1P_2 + a_2N_1P_2 + (-4a_1M_1M_2 + 4a_2M_1M_2 - 2a_1N_1N_2 + 2a_2N_1N_2-\\
&-2a_1P_1P_2 + 2a_2P_1P_2)t\omega_0 + (4a_1M_2P_1 - 4a_2M_2P_1 + 4a_1M_1P_2 - 4a_2M_1P_2)\cos(\omega_0 t)+\\
&+(a_1N_2P_1 - a_2N_2P_1 + a_1N_1P_2 - a_2N_1P_2)\cos(2\omega_0 t) + (-4a_1M_2N_1 + 4a_2M_2N_1 - 4a_1M_1N_2+\\
&+4a_2M_1N_2)\sin(\omega_0 t) + (-a_1N_1N_2 + a_2N_1N_2 + a_1P_1P_2 - a_2P_1P_2)\sin(2\omega_0 t)\big)
\end{aligned} \tag{16}$$

where

$$\omega_0 = \sqrt{b_1^2 + b_2^2 + b_3^2},$$

$$M_1 = \frac{(A_1b_1 + A_2b_2 + A_3b_3)b_1}{b_1^2 + b_2^2 + b_3^2},$$

$$M_2 = \frac{(A_1b_1 + A_2b_2 + A_3b_3)b_2}{b_1^2 + b_2^2 + b_3^2},$$

$$M_3 = \frac{(A_1b_1 + A_2b_2 + A_3b_3)b_3}{b_1^2 + b_2^2 + b_3^2},$$

$$N_1 = \frac{A_1 b_2^2 + A_1 b_3^2 - A_2 b_1 b_2 - A_3 b_1 b_3}{b_1^2 + b_2^2 + b_3^2},$$
$$N_2 = \frac{A_2 b_1^2 + A_2 b_3^2 - A_1 b_1 b_2 - A_3 b_2 b_3}{b_1^2 + b_2^2 + b_3^2},$$
$$N_3 = \frac{A_3 b_1^2 + A_3 b_2^2 - A_1 b_1 b_3 - A_2 b_2 b_3}{b_1^2 + b_2^2 + b_3^2},$$
$$P_1 = \frac{(A_2 b_3 - A_3 b_2)\sqrt{b_1^2 + b_2^2 + b_3^2}}{b_1^2 + b_2^2 + b_3^2},$$
$$P_2 = \frac{(A_3 b_1 - A_1 b_3)\sqrt{b_1^2 + b_2^2 + b_3^2}}{b_1^2 + b_2^2 + b_3^2},$$
$$P_3 = \frac{(A_1 b_2 - A_2 b_1)\sqrt{b_1^2 + b_2^2 + b_3^2}}{b_1^2 + b_2^2 + b_3^2}.$$

The secular terms must be equal to zero, i.e.:

$$-4a_2 M_2 M_3 + 4a_3 M_2 M_3 - 2a_2 N_2 N_3 + 2a_3 N_2 N_3 - 2a_2 P_2 P_3 + 2a_3 P_2 P_3 = 0,$$

$$4a_1 M_1 M_3 - 4a_3 M_1 M_3 + 2a_1 N_1 N_3 - 2a_3 N_1 N_3 + 2a_1 P_1 P_3 - 2a_3 P_1 P_3 = 0$$

and

$$-4a_1 M_1 M_2 + 4a_2 M_1 M_2 - 2a_1 N_1 N_2 + 2a_2 N_1 N_2 - 2a_1 P_1 P_2 + 2a_2 P_1 P_2 = 0,$$

respectively.

Also, to compute $F_1(t, C_i)$ we solve the equation

$$\begin{aligned} &\mathcal{L}\Big(F_1(t, C_i)\Big) = H(t, C_i)\mathcal{N}\Big(F_0(t)\Big), \\ &\mathcal{B}\Big(F_1(t, C_i), \tfrac{dF_1(t,C_i)}{dt}\Big) = 0, \ i = \overline{1, s}. \end{aligned} \tag{17}$$

by taking into consideration that the nonlinear operator N presents the general form:

$$\mathcal{N}\left(F_0(t)\right) = \textstyle\sum_{i=1}^{m} h_i(t) g_i(t), \tag{18}$$

where m is a positive integer and $h_i(\eta)$ and $g_i(\eta)$ are known functions depending both on $F_0(\eta)$ and $\mathcal{N}$.

Substituting Eqs. (16) into Eqs. (8), we obtain

$$\begin{aligned} \mathcal{N}\left[x_{1_0}(t)\right] &= b_3 x_{2_0}(t) - b_2 x_{3_0}(t) + x_{2_0}(t)p_{3_0}(t) - x_{3_0}(t)p_{2_0}(t), \\ \mathcal{N}\left[x_{2_0}(t)\right] &= b_1 x_{3_0}(t) - b_3 x_{1_0}(t) + x_{3_0}(t)p_{1_0}(t) - x_{1_0}(t)p_{3_0}(t), \\ \mathcal{N}\left[x_{3_0}(t)\right] &= b_2 x_{1_0}(t) - b_1 x_{2_0}(t) + x_{1_0}(t)p_{2_0}(t) - x_{2_0}(t)p_{1_0}(t), \\ \mathcal{N}\left[p_{1_0}(t)\right] &= 0, \\ \mathcal{N}\left[p_{2_0}(t)\right] &= 0, \\ \mathcal{N}\left[p_{3_0}(t)\right] &= 0. \end{aligned} \tag{19}$$

Now, we observe that the nonlinear operators $\mathcal{N}\left[x_{i_0}(t)\right]$ and $\mathcal{N}\left[p_{i_0}(t)\right]$, $i = \overline{1, 3}$ respectively, are linear combinations of the functions

$$\cos(\omega_0 t), \sin(\omega_0 t), \cos^2(\omega_0 t), \sin^2(\omega_0 t), \cos(\omega_0 t)\sin(\omega_0 t), \cos(2\omega_0 t), \sin(2\omega_0 t), \cos^2(2\omega_0 t),$$

$$\sin^2(2\omega_0 t), \cos(2\omega_0 t)\sin(2\omega_0 t), \cos(\omega_0 t)\cos(2\omega_0 t), \sin(\omega_0 t)\sin(2\omega_0 t), \sin(\omega_0 t)\cos(2\omega_0 t),$$

$$\sin(2\omega_0 t)\cos(\omega_0 t).$$

Although the equation (17) is a nonhomogeneous linear one, in most cases its solution cannot be found.

In order to compute the function $F_1(t, C_i)$ we will use the third modified version of OHAM (see [9] for details), consisting of the following steps:

- We choose the auxiliary convergence-control functions H_i such that $H_i \cdot \mathcal{N}\left[F_0(t)\right]$ and $\mathcal{N}\left[F_0(t)\right]$ have the same form. So, the first approximation of x_{i_1} or p_{i_1}, $i = \overline{1,\ 3}$, denoted F_1, becomes:

$$\begin{aligned} F_1(t) &= B_1\cos(\omega_1 t) + B_2\cos(3\omega_1 t) + B_3\cos(\omega_2 t) + B_4\cos(3\omega_2 t) + B_5\cos(\omega_3 t) + B_6\cos(3\omega_3 t)+ \\ &+B_7\cos(\omega_4 t) + B_8\cos(3\omega_4 t) + B_9\cos(\omega_5 t) + B_{10}\cos(3\omega_5 t) + B_{11}\cos(\omega_6 t) + B_{12}\cos(3\omega_6 t)+ \\ &+C_1\sin(\omega_1 t) + C_2\sin(3\omega_1 t) + C_3\sin(\omega_2 t) + C_4\sin(3\omega_2 t) + C_5\sin(\omega_3 t) + C_6\sin(3\omega_3 t) + C_7\sin(\omega_4 t)+ \\ &+C_8\sin(3\omega_4 t) + C_9\sin(\omega_5 t) + C_{10}\sin(3\omega_5 t) + C_{11}\sin(\omega_6 t) + C_{12}\sin(3\omega_6 t), \end{aligned} \tag{20}$$

where $B_{12} = -\sum_{i=1}^{11} B_i$;

- Next, by taking into account the equation (12), the first-order analytical approximate solution of the equations (6) - (5) is:

$$\bar{F}(t, C_i) = F(t, 1) = F_0(t) + F_1(t, C_i), \tag{21}$$

where $\bar{F}$ can be $\bar{x}_i$ or $\bar{p}_i$, $i = \overline{1,\ 3}$, F_0 can be x_{i_0} or p_{i_0}, $i = \overline{1,\ 3}$, and F_1 can be x_{i_1} or p_{i_1}, $i = \overline{1,\ 3}$, respectively;

- Finally, the convergence-control parameters ω_0, ω_1 - ω_6, B_1-B_{12}, C_1-C_{12}, which determine the first-order approximate solution (21), can be optimally computed by means of various methods, such as: the least square method, the Galerkin method, the collocation method, the Kantorowich method or the weighted residual method.

6 Numerical Examples and Discussions

In this section, the accuracy and validity of the MOHAM technique is proved using a comparison of our approximate solutions with numerical results obtained via the fourth-order Runge-Kutta method in the following case: we consider the initial value problem given by (3) with initial conditions $A_i = 1$, $i = \overline{1, 6}$, $a_1 = 10$, $a_2 = 2$ and $a_3 = 3$.

The convergence-control parameters b_1, b_2, b_3, ω_0, ω_1 - ω_6, $\{B_i\}\Big|_{i=\overline{1,11}}$, $\{C_i\}\Big|_{i=\overline{1,12}}$ are optimally determined by means of the least-square method.

– *For* $\bar{x}_1$: the convergence-control parameters on the interval [0, 2] are:

$$\begin{aligned} &b_1 = 0.2283569027,\ b_2 = 0.0989000587,\ b_3 = 2.5941842774,\ B_1 = -0.6188189674, \\ &B_2 = -2.1325097805,\ B_3 = -0.3758362313,\ B_4 = 8471.7677686468,\ B_5 = 1.5400120621, \\ &B_6 = -1.0021438297,\ B_7 = 2.1916593205,\ B_8 = 0.0000325467,\ B_9 = 0.1819796718, \\ &B_{10} = 28.1134709577,\ B_{11} = -0.7724268910,\ C_1 = -1.1221754588,\ C_2 = -0.2752866428, \\ &C_3 = 1.2174215282,\ C_4 = -1.8784047879,\ C_5 = -2.2915639543,\ C_6 = -2.1942748182, \\ &C_7 = 0.0795117420,\ C_8 = 0.0002229350,\ C_9 = 0.6031276078,\ C_{10} = 6.2856803724, \\ &C_{11} = 0.8698723067,\ C_{12} = -2.2168099281,\ \omega_0 = 2.7171764693,\ \omega_1 = 5.2066860242, \\ &\omega_2 = 5.2822750608,\ \omega_3 = 5.2456387071,\ \omega_4 = 8.0830652375,\ \omega_5 = 5.2711242920, \\ &\omega_6 = 5.2822418786. \end{aligned} \tag{22}$$

and the convergence-control parameters on the interval [2, 5] are given by:

$$\begin{aligned}&b_1 = 1.4571967311,\ b_2 = 0.5249006763,\ b_3 = -0.0347559443,\ B_1 = 0.1680588883,\\ &B_2 = 0.4911503480,\ B_3 = -0.2633005330,\ B_4 = -0.0267241176,\ B_5 = 0.5389446585,\\ &B_6 = -0.0608389537,\ B_7 = -0.0690728145,\ B_8 = 0.0512340615,\ B_9 = -0.7510395754,\\ &B_{10} = 0.0009876254,\ B_{11} = -0.0967713677,\ C_1 = 0.9792979157,\ C_2 = -0.0262186055,\\ &C_3 = 15.1392159943,\ C_4 = 0.0147959078,\ C_5 = 29.0450320529,\ C_6 = 0.0407981678,\\ &C_7 = -37.3422492635,\ C_8 = -0.0693807591,\ C_9 = 0.3185006323,\ C_{10} = -0.0074308664,\\ &C_{11} = -6.8194469552,\ C_{12} = 0.0013704626,\ \omega_0 = 0.1419552102,\ \omega_1 = 5.5187682954,\\ &\omega_2 = 3.8017451606,\ \omega_3 = 3.9733238823,\ \omega_4 = 3.8809286209,\ \omega_5 = 5.6385913917,\\ &\omega_6 = 4.1185336351.\end{aligned} \tag{23}$$

– *for* $\bar{x}_2$ *: the convergence-control parameters on the interval* [0, 2] *are:*

$$\begin{aligned}&b_1 = 6.456\cdot 10^{-11}, b_2 = 0.7633375264,\ b_3 = 1.0398880285,\ B_1 = -0.5392336063,\\ &B_2 = -0.1298294288,\ B_3 = -1.5659562337,\ B_4 = -0.0011808860,\ B_5 = 0.2090800434,\\ &B_6 = 0.0372954632,\ B_7 = 2.1157347686,\ B_8 = -0.0074351146,\ B_9 = -0.6138533982,\\ &B_{10} = -0.0114690595,\ B_{11} = 0.5302233690,\ C_1 = 0.0351118148,\ C_2 = 1.3059452150,\\ &C_3 = -1.8749417378,\ C_4 = -0.0010852753,\ C_5 = -0.0615141661,\ C_6 = 0.1651855910,\\ &C_7 = -0.0687100638,\ C_8 = 0.0088237373,\ C_9 = 1.5611890958,\ C_{10} = 0.0143718926,\\ &C_{11} = -0.7124448084,\ C_{12} = -0.1610568447,\ \omega_0 = 2.6307796567,\ \omega_1 = 6.2033327841,\\ &\omega_2 = 7.2015801370,\ \omega_3 = 6.3707515131,\ \omega_4 = 5.6559017346,\ \omega_5 = 4.6657471128,\\ &\omega_6 = 6.3944311111.\end{aligned} \tag{24}$$

and the convergence-control parameters on the interval [2, 5] are given by:

$$\begin{aligned}&b_1 = 1.4173488936,\ b_2 = 0.4882085120,\ b_3 = 0.3915263714,\ B_1 = -0.2345297398,\\ &B_2 = 0.1971742651,\ B_3 = -1.0786285111,\ B_4 = -0.0486671777,\ B_5 = 0.0131930466,\\ &B_6 = -0.0133772489,\ B_7 = 2.0340028797,\ B_8 = -0.0016684368,\ B_9 = -0.9450842076,\\ &B_{10} = 0.0116617605,\ B_{11} = 0.0690318479,\ C_1 = -0.0289673387,\ C_2 = 0.8064636535,\\ &C_3 = 0.5582977833,\ C_4 = 0.0567720911,\ C_5 = 29.4749483082,\ C_6 = 0.0162850975,\\ &C_7 = 1.1060197207,\ C_8 = 0.0016094866,\ C_9 = -18.5848145096,\ C_{10} = -0.0057518090,\\ &C_{11} = -13.1084491075,\ C_{12} = -0.0078586499,\ \omega_0 = 2.0952483159,\ \omega_1 = 5.6375412723,\\ &\omega_2 = 3.7903624005,\ \omega_3 = 5.7047027326,\ \omega_4 = 4.4756746978,\ \omega_5 = 5.8040132816,\\ &\omega_6 = 5.5129048262.\end{aligned} \tag{25}$$

– *for* $\bar{x}_3$ *: the convergence-control parameters on the interval* [0, 2] *are:*

$$\begin{aligned}&b_1 = 0.2283569027,\ b_2 = 0.0989000587,\ b_3 = 2.5941842774,\ B_1 = -0.6188189674,\\ &B_2 = -2.1325097805,\ B_3 = -0.3758362313,\ B_4 = 8471.7677686468,\ B_5 = 1.5400120621,\\ &B_6 = -1.0021438297,\ B_7 = 2.1916593205,\ B_8 = 0.0000325467,\ B_9 = 0.1819796718,\\ &B_{10} = 28.1134709577,\ B_{11} = -0.7724268910,\ C_1 = -1.1221754588,\ C_2 = -0.2752866428,\\ &C_3 = 1.2174215282,\ C_4 = -1.8784047879,\ C_5 = -2.2915639543,\ C_6 = -2.1942748182,\\ &C_7 = 0.0795117420,\ C_8 = 0.0002229350,\ C_9 = 0.6031276078,\ C_{10} = 6.2856803724,\\ &C_{11} = 0.8698723067,\ C_{12} = -2.2168099281,\ \omega_0 = 2.7171764693,\ \omega_1 = 5.2066860242,\\ &\omega_2 = 5.2822750608,\ \omega_3 = 5.2456387071,\ \omega_4 = 8.0830652375,\ \omega_5 = 5.2711242920,\\ &\omega_6 = 5.2822418786.\end{aligned} \tag{26}$$

and the convergence-control parameters on interval [2, 5] are:

$$\begin{aligned}
&b_1 = 1.4571967311,\ b_2 = 0.5249006763,\ b_3 = -0.0347559443,\ B_1 = 0.1680588883,\\
&B_2 = 0.4911503480,\ B_3 = -0.2633005330,\ B_4 = -0.0267241176,\ B_5 = 0.5389446585,\\
&B_6 = -0.0608389537,\ B_7 = -0.0690728145,\ B_8 = 0.0512340615,\ B_9 = -0.7510395754,\\
&B_{10} = 0.0009876254,\ B_{11} = -0.0967713677,\ C_1 = 0.9792979157,\ C_2 = -0.0262186055,\\
&C_3 = 15.1392159943,\ C_4 = 0.0147959078,\ C_5 = 29.0450320529,\ C_6 = 0.0407981678,\\
&C_7 = -37.3422492635,\ C_8 = -0.0693807591,\ C_9 = 0.3185006323,\ C_{10} = -0.0074308664,\\
&C_{11} = -6.8194469552,\ C_{12} = 0.0013704626,\ \omega_0 = 0.1419552102,\ \omega_1 = 5.5187682954,\\
&\omega_2 = 3.8017451606,\ \omega_3 = 3.9733238823,\ \omega_4 = 3.8809286209,\ \omega_5 = 5.6385913917,\\
&\omega_6 = 4.1185336351.
\end{aligned} \tag{27}$$

– *for $\bar{p}_1$: the convergence-control parameters on the interval* [0, 2] *are:*

$$\begin{aligned}
&b_1 = 0.1036485246,\ b_2 = -0.0838988657,\ b_3 = 0.4403110806,\ B_1 = -0.0376788635,\\
&B_2 = -0.0248941204,\ B_3 = 0.0844332782,\ B_4 = 0.0310777423,\ B_5 = -0.0452108999,\\
&B_6 = 0.0282176461,\ B_7 = 0.0511653654,\ B_8 = 0.0940858033,\ B_9 = -0.0682239918,\\
&B_{10} = -0.0969008853,\ B_{11} = 0.0250573399,\ C_1 = 0.0667039160,\ C_2 = 0.3022819435,\\
&C_3 = 0.2669398708,\ C_4 = 0.0548663952,\ C_5 = -0.1717403645,\ C_6 = -0.2707589998,\\
&C_7 = -0.2306352376,\ C_8 = -0.2284873607,\ C_9 = -0.1108807314,\ C_{10} = -0.1313264955,\\
&C_{11} = 0.2092043761,\ C_{12} = 0.2694354262,\ \omega_0 = 5.0760814351,\ \omega_1 = 2.8222787903,\\
&\omega_2 = 4.9149721267,\ \omega_3 = 2.8974142315,\ \omega_4 = 5.0572416865,\ \omega_5 = 5.0454188859,\\
&\omega_6 = 2.9258916892.
\end{aligned} \tag{28}$$

and the convergence-control parameters on the interval [2, 5] are:

$$\begin{aligned}
&b_1 = 1.5724586627,\ b_2 = 0.4707847419,\ b_3 = 0.2544395948,\ B_1 = -0.0712733791,\\
&B_2 = 0.0207851682,\ B_3 = -0.2205623769,\ B_4 = -0.0033128577,\ B_5 = 0.1849911903,\\
&B_6 = -0.0413121206,\ B_7 = 0.1464344089,\ B_8 = 0.0057048496,\ B_9 = -0.4026916710,\\
&B_{10} = 0.0422517376,\ B_{11} = 0.3367134667,\ C_1 = -0.0419985180,\ C_2 = 0.0892298271,\\
&C_3 = 0.1005288960,\ C_4 = -0.0013255000,\ C_5 = -2.7869531888,\ C_6 = 1.0428982525,\\
&C_7 = -1.4509805903,\ C_8 = -0.0067469563,\ C_9 = 2.6907110004,\ C_{10} = -1.0427232874,\\
&C_{11} = 1.4538893065,\ C_{12} = 0.0197688218,\ \omega_0 = 1.9158018360,\ \omega_1 = 5.7473826070,\\
&\omega_2 = 3.0355210214,\ \omega_3 = 5.1915773192,\ \omega_4 = 4.1288373559,\ \omega_5 = 5.1917135926,\\
&\omega_6 = 3.9528743913.
\end{aligned} \tag{29}$$

– *for $\bar{p}_2$: the convergence-control parameters on the interval* [0, 2] *are:*

$$\begin{aligned}
&b_1 = 0.0001683611,\ b_2 = -2.0016481420,\ b_3 = -0.0001675070,\ B_1 = -2.0392289521,\\
&B_2 = -0.7006696104,\ B_3 = 0.7040465119,\ B_4 = -0.0569489536,\ B_5 = 0.6718048017,\\
&B_6 = 0.3983543667,\ B_7 = 0.6738145503,\ B_8 = -0.6345532338,\ B_9 = 0.1136578383,\\
&B_{10} = 0.1778469563,\ B_{11} = 0.6978480581,\ C_1 = 1.7187369022,\ C_2 = -0.2003619518,\\
&C_3 = -0.3990006850,\ C_4 = -0.5538259239,\ C_5 = -0.0404543278,\ C_6 = -0.6223400206,\\
&C_7 = 0.0674780726,\ C_8 = 0.5312964055,\ C_9 = -0.6211580141,\ C_{10} = 0.6394029170,\\
&C_{11} = 1.0642886252,\ C_{12} = -0.0755732216,\ \omega_0 = 2.1419552102,\ \omega_1 = 5.3953610209,\\
&\omega_2 = 5.5180829133,\ \omega_3 = 4.7506598358,\ \omega_4 = 4.8477963380,\ \omega_5 = 5.4670333603,\\
&\omega_6 = 3.5311907764.
\end{aligned} \tag{30}$$

and the convergence-control parameters on the interval [2, 5] are:

$$\begin{aligned}
&b_1 = 1.5140732386,\ b_2 = 0.6178462055,\ b_3 = 0.0867140300,\ B_1 = -0.2786291081,\\
&B_2 = 0.4786720197,\ B_3 = -0.5292322445,\ B_4 = 0.4322397555,\ B_5 = 0.7165692078,\\
&B_6 = 0.0812481438,\ B_7 = 0.3403175271,\ B_8 = 0.1173750787,\ B_9 = -0.2683988617,\\
&B_{10} = -1.1274223663,\ B_{11} = 0.0493435362,\ C_1 = -3.6814134305,\ C_2 = 3.8467270553,\\
&C_3 = 0.8484998229,\ C_4 = -4.9123469312,\ C_5 = -8.7928157171,\ C_6 = 1.2560165019,\\
&C_7 = 6.9962684630,\ C_8 = -1.0188150435,\ C_9 = 5.0605350584,\ C_{10} = 0.6868788798,\\
&C_{11} = -0.1766079390,\ C_{12} = 0.0029503559,\ \omega_0 = 3.1361476367,\ \omega_1 = 6.4656953483,\\
&\omega_2 = 3.1838334017,\ \omega_3 = 3.8578923929,\ \omega_4 = 3.8982724447,\ \omega_5 = 3.3948338453,\\
&\omega_6 = 5.5582071846.
\end{aligned} \tag{31}$$

– *for* $\bar{p}_3$ *: the convergence-control parameters on the interval* [0, 2] *are:*

$$\begin{aligned}
&b_1 = 0,\ b_2 = -0.0355751014,\ b_3 = -0.4758762645,\ B_1 = 3.0671655307,\\
&B_2 = -1.2783637062,\ B_3 = -0.1220942110,\ B_4 = 0.9894151494,\ B_5 = -1.6067226786,\\
&B_6 = -0.3094379114,\ B_7 = -0.4821540115,\ B_8 = -0.3708570097,\ B_9 = -1.6591913793,\\
&B_{10} = 0.0107568076,\ B_{11} = 1.7584367052,\ C_1 = 4.2347221138,\ C_2 = 1.5534081190,\\
&C_3 = -1.3745219366,\ C_4 = -0.2337265539,\ C_5 = -1.0301856996,\ C_6 = 0.2062301374,\\
&C_7 = -0.9954330338,\ C_8 = -0.8268518077,\ C_9 = 1.3017488786,\ C_{10} = 0.0901576224,\\
&C_{11} = 1.3131066060,\ C_{12} = 0.0035743896,\ \omega_0 = 0.4284845434,\ \omega_1 = 4.6137446439,\\
&\omega_2 = 2.8735244185,\ \omega_3 = 3.8183642486,\ \omega_4 = 3.3447626333,\ \omega_5 = 4.5270915130,\\
&\omega_6 = 6.2424590206.
\end{aligned} \tag{32}$$

and the convergence-control parameters on the interval [2, 5] are:

$$\begin{aligned}
&b_1 = 1.1539383919,\ b_2 = -0.0378132194,\ b_3 = 0.0764893091,\ B_1 = 3.6683385539,\\
&B_2 = 3.7078901504,\ B_3 = 0.1769224627,\ B_4 = 0.8704700821,\ B_5 = -1.8228794015,\\
&B_6 = -0.0174711644,\ B_7 = -3.1339499445,\ B_8 = -0.5873136873,\ B_9 = -1.5165399988,\\
&B_{10} = 2439.7242372375,\ B_{11} = -1.3526927320,\ C_1 = -0.6348315021,\ C_2 = -2.8138881766,\\
&C_3 = -54.5190807947,\ C_4 = 1.7557827711,\ C_5 = 7.7092155624,\ C_6 = -0.0035474772,\\
&C_7 = 50.6503693246,\ C_8 = -1.7755694908,\ C_9 = 42.7395871073,\ C_{10} = -13.2505164509,\\
&C_{11} = -43.3928223377,\ C_{12} = 13.2111837376,\ \omega_0 = 1.9658055572,\ \omega_1 = 5.7390876257,\\
&\omega_2 = 3.0355210214,\ \omega_3 = 5.5992543568,\ \omega_4 = 3.0644519992,\ \omega_5 = 4.4482240954,\\
&\omega_6 = 4.4482272978.
\end{aligned} \tag{33}$$

Finally, Tables 1 - 4 emphasizes the accuracy of the MOHAM technique by comparing the approximate analytic solutions $\bar{x}_1$ and $\bar{p}_1$ respectively presented above with the corresponding numerical integration values (via the 4th-order Runge-Kutta method), and the Lie-Trotter integrator. Finally, the results obtained using MOHAM are much closer to the original solution in comparison to the results obtained using Lie-Trotter integrator. These comparisons show the effectiveness, reliability, applicability, efficiency and accuracy of the MOHAM against to the Lie-Trotter integrator.

Remark 2. *Figures 3 and 4 present the comparisons between the analytical approximate solutions given by MOHAM and numerical results provided by Runge-Kutta 4th steps integrator. We can see that the analytical approximate solutions and Runge-Kutta 4th steps integrator's results are quite the same.*

Table 1: The comparison between the approximate solutions $\bar{x}_1$ given by Eq. (22) and the corresponding numerical solutions for $a_1 = 10$, $a_2 = 2$ and $a_3 = 3$ (relative errors: $\epsilon_{x_1} = |x_{1_{numerical}} - \bar{x}_{1_{MOHAM}}|$)

t	$x_{1_{numerical}}$	x_1 Lie-Trotter (5 iterations)	$\bar{x}_{1_{MOHAM}}$ given by Eq. (22) on $[0, 2]$	ϵ_{x_1} MOHAM
0	1	1	1	0
1/5	0.7337023135	0.3397255522	0.7337010385	$1.27 \cdot 10^{-6}$
2/5	0.1051371952	-0.1175841263	0.1051376412	$4.46 \cdot 10^{-7}$
3/5	-0.5390294341	0.4802310232	-0.5390162606	$1.31 \cdot 10^{-5}$
4/5	-0.7408940837	1.0112524913	-0.7408685929	$2.54 \cdot 10^{-5}$
1	-0.1865095491	0.6934561430	-0.1864820575	$2.74 \cdot 10^{-5}$
6/5	0.6179798375	-0.0704392741	0.6180122798	$3.24 \cdot 10^{-5}$
7/5	0.7542356906	-0.7051643433	0.7542576320	$2.19 \cdot 10^{-5}$
8/5	0.1623850803	-1.0253085094	0.1623984362	$1.33 \cdot 10^{-5}$
9/5	-0.5946553094	-0.9561760501	-0.5946418649	$1.34 \cdot 10^{-5}$
2	-0.7270211941	-0.3650442893	-0.7269848249	$3.63 \cdot 10^{-5}$

Table 2: The comparison between the approximate solutions $\bar{x}_1$ given by Eq. (23) and the corresponding numerical solutions for $a_1 = 10$, $a_2 = 2$ and $a_3 = 3$ (relative errors: $\epsilon_{x_1} = |x_{1_{numerical}} - \bar{x}_{1_{MOHAM}}|$)

t	$x_{1_{numerical}}$	x_1 Lie-Trotter (5 iterations)	$\bar{x}_{1_{MOHAM}}$ given by Eq. (23) on $[2, 5]$	ϵ_{x_1} MOHAM
2	-0.7270211941	-0.3650442893	-0.7269848249	$3.63 \cdot 10^{-5}$
23/10	0.4372022188	0.9364250346	0.4372037377	$1.51 \cdot 10^{-6}$
13/5	0.6086980787	0.3097132458	0.6088280862	$1.30 \cdot 10^{-4}$
29/10	-0.5357138405	-0.8052294850	-0.5357087054	$5.13 \cdot 10^{-6}$
16/5	-0.5145382567	1.5523267940	-0.5143525655	$1.85 \cdot 10^{-4}$
7/2	0.6856064084	1.4821107514	0.6858827501	$2.76 \cdot 10^{-4}$
19/5	0.3181857442	-1.2076590809	0.3183075232	$1.21 \cdot 10^{-4}$
41/10	-0.7368863901	-0.3502332442	-0.7367419022	$1.44 \cdot 10^{-4}$
22/5	-0.1421700459	0.5798310108	-0.1420135462	$1.56 \cdot 10^{-4}$
47/10	0.7752628531	0.9869474224	0.7751771539	$8.5 \cdot 10^{-5}$
5	-0.0560490264	0.7281825005	-0.0561680013	$1.18 \cdot 10^{-4}$

7 Conclusion

The stability problem and the existence of the periodic orbits represent important issues for any differential equations system, so a lot of methods were developed ([11]) in order to obtain better results.

The Clebsch's system arises from physics like a lot of other systems: the rigid body ([12]), the Maxwell-Bloch equations ([13]), the heavy top dynamics ([14]), the spacecraft dynamics ([15]), the Ishii's equations ([16]), and the list could continue. For all these examples, the energy-methods provided us conclusive results, being a good reason to use it again.

In this paper we analyze the nonlinear stability of the equilibrium states of Clebsch's system. Some results obtained in [2] -like the stability of the equilibrium states $e_1^{M,N}$, $e_2^{M,N}$, $e_3^{M,N}$ - are improved and some other new, like the stability of the equilibrium states $e_4^{M,N,P}$ which are developed. Then, using Moser-Weinstein theorem with zero eigenvalue, we were able to prove the existence of the periodic orbits around the nonlinear stable equilibria.

Table 3: The comparison between the approximate solutions $\bar{p}_1$ given by Eq. (28) and the corresponding numerical solutions for $a_1 = 10$, $a_2 = 2$ and $a_3 = 3$ (relative errors: $\epsilon_{p_1} = |p_{1_{numerical}} - \bar{p}_{1_{MOHAM}}|$)

t	$p_{1_{numerical}}$	p_1 Lie-Trotter (5 iterations)	$\bar{p}_{1_{MOHAM}}$ given by Eq. (28) on $[0,\ 2]$	ϵ_{p_1} MOHAM
0	1	1	1	0
1/5	1.1988857916	-5.69570694281	1.1989144264	$2.86 \cdot 10^{-5}$
2/5	1.3795883412	-12.2537269802	1.3795018229	$8.65 \cdot 10^{-5}$
3/5	1.5324439810	-15.7770580704	1.5324405923	$3.38 \cdot 10^{-6}$
4/5	1.6664441641	-18.0829524967	1.6665733306	$1.29 \cdot 10^{-4}$
1	1.7403013358	-20.7507227878	1.7401964409	$1.04 \cdot 10^{-4}$
6/5	1.7799452607	-23.4286883110	1.7798486906	$9.65 \cdot 10^{-5}$
7/5	1.8498771709	-25.3934489405	1.8500266661	$1.49 \cdot 10^{-4}$
8/5	1.8827210686	-25.5740167497	1.8827099340	$1.11 \cdot 10^{-5}$
9/5	1.8942673776	-22.8936868313	1.8941392230	$1.28 \cdot 10^{-4}$
2	1.9333046846	-16.3172763616	1.9332334105	$7.12 \cdot 10^{-5}$

Table 4: The comparison between the approximate solutions $\bar{p}_1$ given by Eq. (29) and the corresponding numerical solutions for $a_1 = 10$, $a_2 = 2$ and $a_3 = 3$ (relative errors: $\epsilon_{p_1} = |p_{1_{numerical}} - \bar{p}_{1_{MOHAM}}|$)

t	$p_{1_{numerical}}$	p_1 Lie-Trotter (5 iterations)	$\bar{p}_{1_{MOHAM}}$ given by Eq. (29) on $[2,\ 5]$	ϵ_{p_1} MOHAM
2	1.9333046846	-16.3172763616	1.9332334105	$7.12 \cdot 10^{-5}$
23/10	1.9216518990	-2.1569524154	1.9217457034	$9.38 \cdot 10^{-5}$
13/5	1.9566676109	-12.0492776630	1.9566449348	$2.26 \cdot 10^{-5}$
29/10	1.9371859786	-54.9580130501	1.9370816701	$1.04 \cdot 10^{-4}$
16/5	1.9648174687	-77.7932051440	1.9646341611	$1.83 \cdot 10^{-4}$
7/2	1.9425320654	-46.9039743670	1.9423610516	$1.71 \cdot 10^{-4}$
19/5	1.9628519104	-28.5641249959	1.9627279020	$1.24 \cdot 10^{-4}$
41/10	1.9501703138	-54.0054145407	1.9501176267	$5.26 \cdot 10^{-5}$
22/5	1.9576681782	-65.1112304012	1.9576082788	$5.98 \cdot 10^{-5}$
47/10	1.9560784138	-33.8220308602	1.9560788273	$4.13 \cdot 10^{-7}$
5	1.9519551708	7.6427858981	1.9521276801	$1.72 \cdot 10^{-4}$

In the last part, a comparison of the results obtained using numerical integrator, Lie-Trotter and Multistage Optimal Homotopy Asymptotic Method are analyzed. We summarize that the MOHAM's analytic solutions are proved to be the best.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

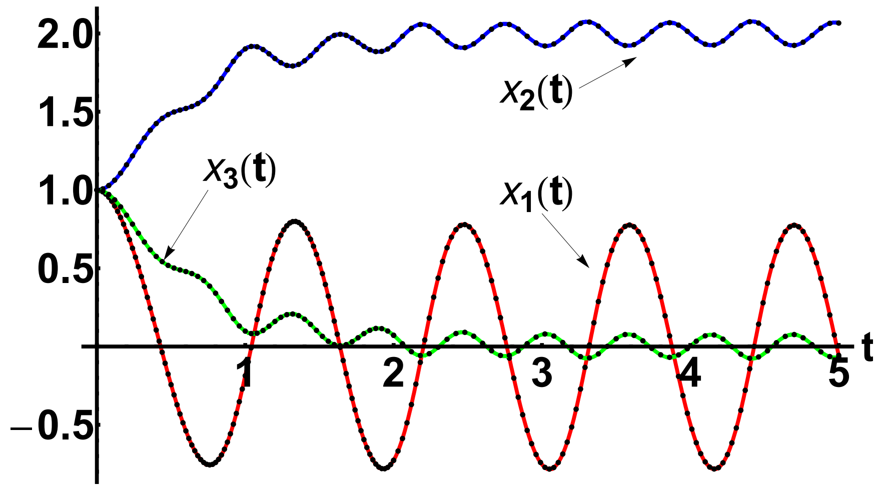

Figure 3: Profiles of the functions $\bar{x}_1$, $\bar{x}_2$, and $\bar{x}_3$ respectively, given by Eqs. (22), (24), (26) on $[0,\ 2]$ and Eqs. (23), (25), (27) on $[2,\ 5]$ respectively: – – – – – numerical solution, · · · · · · · · · · · MOHAM solution.

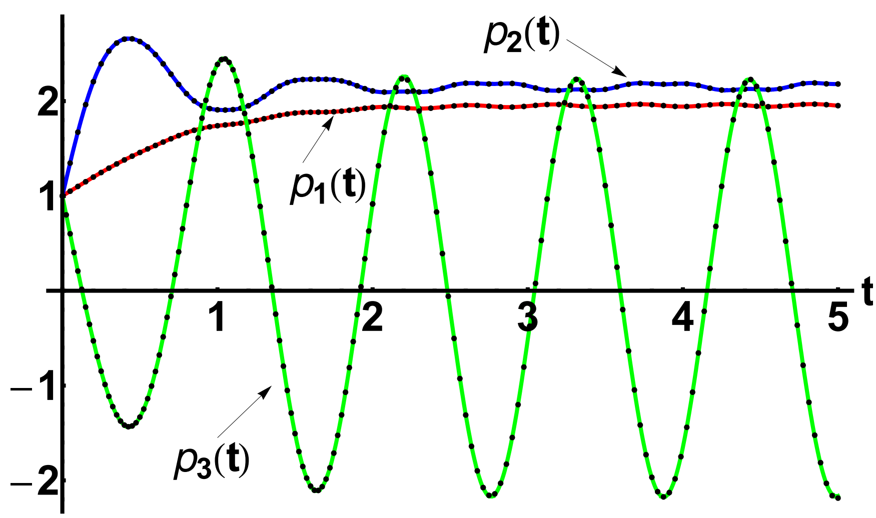

Figure 4: Profiles of the functions $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ respectively, given by Eq. (28), (30), (32) on $[0,\ 2]$ and Eqs. (29), (31), (33) on $[2,\ 5]$ respectively: – – – – – numerical solution, · · · · · · · · · · · MOHAM solution.

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