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Solutions with concentration and cavitation to the Riemann problem for the isentropic relativistic Euler system for the extended Chaplygin gas

Abstract: The solutions to the Riemann problem for the isentropic relativistic Euler system for the extended Chaplygin gas are constructed for all kinds of situations by using the method of phase plane analysis. The asymptotic limits of solutions to the Riemann problem for the relativistic extended Chaplygin Euler system are investigated in detail when the pressure given by the equation of state of extended Chaplygin gas becomes that of the pressureless gas. During the process of vanishing pressure, the phenomenon of concentration can be identified and analyzed when the two-shock Riemann solution tends to a delta shock wave solution as well as the phenomenon of cavitation also being captured and observed when the two-rarefaction-wave Riemann solution tends to a two-contact-discontinuity solution with a vacuum state between them.

1 Introduction

It is very important to understand the relativistic fluid dynamics in the study of various astrophysical phenomena [1], such as the gravitational collapse, the supernova explosion and the formation and acceleration of the universe. Nowadays, there exists a vast amount of literature in various models of relativistic fluid dynamics since the fundamental work of Taub [2]. However, only a few analytical theories have been developed such as in [3–6] due to the complicated structures of various relativistic fluid dynamics models. In this present work, we draw our attention on the isentropic Euler system of two conservation laws consisting of energy and momentum in special relativity in the following form [3, 5–7]

$$\begin{cases} \left(\dfrac{p(\rho) + \rho c^2}{c^2} \dfrac{v^2}{c^2 - v^2} + \rho \right)_t + \left((p(\rho) + \rho c^2) \dfrac{v}{c^2 - v^2} \right)_x = 0, \\ \left((p(\rho) + \rho c^2) \dfrac{v}{c^2 - v^2} \right)_t + \left((p(\rho) + \rho c^2) \dfrac{v^2}{c^2 - v^2} + p(\rho) \right)_x = 0. \end{cases} \tag{1.1}$$

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Here the unknown state variables $\rho(x, t)$ and $v(x, t)$ stand for the proper-energy density and the particle speed respectively and the unknown function $p(\rho)$ is used to denote scalar pressure which is a function of ρ for the isentropic situation. In addition, the constant c is the speed of light. The system (1.1) was often used to describe the dynamics of plane waves in special relativistic fluids in the two-dimensional Minkowski space-time [3].

In our present study, the equation of state $p(\rho)$ is chosen as the third-order form of the extended Chaplygin gas [8, 9] as follows:

$$p(\rho) = A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha}, \qquad 0 < \alpha < 1, \tag{1.2}$$

in which $A_1, A_2, A_3 \geq 0$ and $B > 0$. It requires that the speed of sound $\sqrt{p'(\rho)}$ is less than the speed of light c, such that the condition $A_1 + 2A_2\rho + 3A_3\rho^2 + B\alpha\rho^{-\alpha-1} < c^2$ is satisfied. The Chaplygin gas with the equation of state given by $p(\rho) = -\frac{B}{\rho}$ with the constant $B > 0$ was first introduced by Chaplygin [10] as an effective mathematical approximation to compute the lifting force on a wing of an airplane. The equation of state for the Chaplygin gas is also very suitable to describe the dark energy and the dark matter in the universe within the framework of string theory [11]. In order to be consistent with the observed data, the equation of state was generalized to the form $p(\rho) = -\frac{B}{\rho^\alpha}$ for the generalized Chaplygin gas [12] and subsequently was further modified to the form $p(\rho) = A\rho - \frac{B}{\rho^\alpha}$ for the modified Chaplygin gas [13], in which $A, B > 0$ and $0 < \alpha \leq 1$. It is essential to deal with a two fluid model about the equation of state for the modified Chaplygin gas for the reason that the first term $A\rho$ gives an ordinary fluid obeying a linear barotropic equation of state while the second one $-\frac{B}{\rho^\alpha}$ is the pressure to some power of the inverse of energy density. However, it is possible to consider the barotropic fluid, whose equation of state is quadratic and to even higher orders. In view of the aforementioned facts, the extended Chaplygin gas with the equation of state $p(\rho) = \sum\limits_{k=1}^{n} A_k\rho^k - \frac{B}{\rho^\alpha}$ has been proposed by Pourhassan and Kahya [8]. It is easy to know that the extended Chaplygin gas recovers all the above Chaplygin gases by selecting $A_k(k = 1, \ldots, n)$ and α suitably. It is worthwhile to notice that the third-order form of the extended Chaplygin gas with the equation of state (1.2) has a good agreement with the cosmological parameters such as dark energy density, scale factor and Hubble expansion parameter [9, 14–16]. Of course we can also carry out the study for higher n terms of the extended Chaplygin gas, but the effects of more corrected terms are infinitesimal and are therefore of less importance [9]. Due to the above results, we shall draw our attention on the third-order form of the extended Chaplygin gas with the equation of state (1.2).

It is well known that the explicit solution can help us to understand the formation mechanism of singularities. For this purpose, we restrict ourselves to consider the system (1.1)-(1.2) with the Riemann-type initial data which is taken to be

$$(\rho, v)(0, x) = \begin{cases} (\rho_-, v_-), & x < 0, \\ (\rho_+, v_+), & x > 0. \end{cases} \tag{1.3}$$

Formally, if we adopt the Newtonian limit (namely the limit $\frac{v}{c} \to 0$ is taken), then the system (1.1)-(1.2) becomes the classical isentropic Euler system for the compressible fluid in the form

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p(\rho))_x = 0, \end{cases} \tag{1.4}$$

which has been widely studied as in [17, 18]. On the other hand, if the limit $A_1, A_2, A_3, B \to 0$ is taken, then the system (1.1)-(1.2) turns out to be the following zero-pressure relativistic Euler system

$$\begin{cases} \left(\dfrac{\rho}{c^2 - v^2}\right)_t + \left(\dfrac{\rho v}{c^2 - v^2}\right)_x = 0, \\ \left(\dfrac{\rho v}{c^2 - v^2}\right)_t + \left(\dfrac{\rho v^2}{c^2 - v^2}\right)_x = 0. \end{cases} \tag{1.5}$$

The system (1.5) is a non-strictly hyperbolic and completely linearly degenerate system, whose elementary wave only involves the contact discontinuity. More specifically, the solution to the Riemann problem (1.3) and

(1.5) is either a delta shock wave solution when $v_- > v_+$ or a two-contact-discontinuity solution with a vacuum state between them when $v_- < v_+$. It is worth mentioning that the evolution of universe is in agreement with the pressureless fluid of the dark matter era at the early stage and subsequently is also consistent with the cosmic fluid to mimic the cosmological constant of the dark energy era at the later stage [14]. Motivated by the above observation, it is of great interest to investigate the transition between the two different stages of the universe by studying the vanishing pressure limits of solutions to the Riemann problem (1.1)-(1.3) where $A_1, A_2, A_3, B \to 0$ is taken.

The first task of this paper is to solve the Riemann problem for the isentropic relativistic Euler system (1.1) associated with the equation of state (1.2). It is easy to get that the system (1.1) associated with (1.2) is strictly hyperbolic and each of the two characteristic fields is genuinely nonlinear. As a consequence, the solutions to the Riemann problem (1.1)-(1.3) are four kinds of different combinations between 1-shock (or 1-rarefaction) wave and 2-shock (or 2-rarefaction) wave, which depends on the choice of initial Riemann data (1.3). The second task of this paper is to consider the limits $A_1, A_2, A_3, B \to 0$ of solutions to the Riemann problem (1.1)-(1.3) as the pressure tends to zero. Our discussion should be divided into two parts: (1) $c > v_- > v_+ > -c$ and (2) $-c < v_- < v_+ < c$ according to the two different structures of the solutions to the Riemann problem (1.3) and (1.5). To be more precise, the phenomenon of concentration can be identified and analyzed for the case $c > v_- > v_+ > -c$, where the limit of solution consisting of two shock waves to the Riemann problem (1.1)-(1.3) tends to a δ-shock wave solution as $A_1, A_2, A_3, B \to 0$ while the intermediate density between the two shock waves tends to be a weight Dirac δ-measure. In contrast, the phenomenon of cavitation can also be captured and observed for the case $-c < v_- < v_+ < c$, where the limit of the solution consisting of two rarefaction waves to the Riemann problem (1.1)-(1.3) tends to a two-contact-discontinuity solution with a vacuum state between them as $A_1, A_2, A_3, B \to 0$ while the intermediate density between the two rarefaction waves tends to be zero (namely a vacuum state).

For the related work about the isentropic relativistic Euler system (1.1), the equation of state $p(\rho) = k^2\rho$ was first investigated by Smoller and Temple [3] where the global existence of BV weak solutions to the Cauchy problem for the system (1.1) was proved analytically by employing Glimm's scheme. Furthermore, the Riemann problem for the system (1.1) with the equation of state given by a smooth function $p(\rho)$ and then the Cauchy problem for the system (1.1) with the equation of state obeying the γ law were also considered in [5]. When the perturbation is arbitrarily large, the uniqueness of Riemann solution to the system (1.1) was established by Chen and Li [6] in the class of entropy solutions in $L^\infty \cap BV_{loc}$ by making use of the detailed analysis of the global behavior of shock wave curves in the half-upper (ρ, v) phase space. Li, Feng and Wang [7] made a step further to construct the global entropy solutions to the Cauchy problem for the system (1.1) with a class of large initial data including the interaction between shock waves and rarefaction waves.

The formation of vacuum state and delta shock wave to the Riemann problem for the zero-pressure gas dynamics system [19, 20] was considered initially for the isothermal case [21] with the equation of state given by $p(\rho) = c\rho$ and the isentropic case [22] with the equation of state given by $p(\rho) = c\rho^\gamma, 1 < \gamma < 3$ by making use of the vanishing pressure limit approach. The result was further extended to the generalized zero-pressure gas dynamics system in [23]. Also see for the other related works [24–27] and the references cited therein. It is worthwhile to notice that the limits of solutions to the Riemann problems from the various isentropic Chaplygin gas dynamic systems [28–32] to the zero-pressure gas dynamic system have also been widely investigated in a variety of contents [33–38], which are not described in detail any more in the paper.

As for the formation of vacuum state and delta shock wave to the Riemann problem for the zero-pressure relativistic Euler system (1.5), Yin and Sheng first investigated the vanishing pressure limits of solutions to the Riemann problems about the Euler system of conservation laws consisting of energy and momentum in special relativity for the isothermal [39] and isentropic [40] situations, in which the phenomena of concentration and cavitation can be observed and analyzed in detail. Subsequently, Li and Shao [41] considered the vanishing pressure limits of solutions to the Riemann problem (1.1)-(1.3) for the isentropic relativistic Euler system for the generalized Chaplygin gas where $A_i = 0$ ($i = 1, 2, 3$) was taken in (1.2), in which the delta shock wave was also involved in the solution to the Riemann problem (1.1)-(1.3) for the generalized Chaplygin gas when $A_i = 0$ ($i = 1, 2, 3$) in (1.2). Yin and Sheng [42] made a step further to generalize the above results to the Euler system consisting of three conservation laws to describe baryon numbers, energy and

momentum in special relativity. Furthermore, Yin and Song [43] considered the vanishing pressure limits of solutions to the Riemann problems about the Euler system of conservation laws consisting of baryon numbers and momentum in special relativity for the Chaplygin gas. In addition, Yang and Zhang [44] introduced the flux approximation approach to study the formation of vacuum state and delta shock wave to the Riemann problem for the zero-pressure relativistic Euler system (1.5).

The paper is arranged as follows. In Section 2, we are mainly concerned with the construction of solutions to the Riemann problems for the isentropic relativistic Euler system (1.1) associated with the equation of state (1.2) in detail. In addition, we give a brief description of the solutions to the Riemann problem for the zero-pressure relativistic Euler system (1.5). In Section 3, we shall focus on the vanishing pressure limits of solutions to the Riemann problems from the system (1.1)-(1.2) to the zero-pressure relativistic Euler system (1.5) for the case $c > v_- > v_+ > -c$ when the limit $A_1, A_2, A_3, B \to 0$ is taken, in which the formation of δ-shock wave can be observed and analyzed. In Section 4, we turn back to investigate the formation of vacuum state for the case $-c < v_- < v_+ < c$ when the limit $A_1, A_2, A_3, B \to 0$ is taken.

2 The Riemann problems for the isentropic and zero-pressure relativistic Euler systems

In this section, we first illustrate the solutions to the Riemann problem for the isentropic relativistic Euler system (1.1) associated with the equation of state (1.2). Then, we recollect the related results for the zero-pressure relativistic Euler system (1.5), whose Riemann solution is a delta shock wave solution when $c > v_- > v_+ > -c$ or a two-contact-discontinuity solution with a vacuum state between them when $-c < v_- < v_+ < c$.

2.1 The Riemann problem for the system (1.1) with the equation of state (1.2)

In this subsection, we shall first analyze the properties of elementary waves and then construct the solutions to the Riemann problem (1.1)-(1.3) for all kinds of situations. Since the speed of sound $\sqrt{p'(\rho)}$ is less than the speed of light c, the condition $A_1 + 2A_2\rho + 3A_3\rho^2 + B\alpha\rho^{-\alpha-1} < c^2$ has to hold. There exist ρ_1 and ρ_2 satisfying $0 < \rho_1 < \rho < \rho_2 < +\infty$ for the fixed A_1, A_2, A_3, B, in which ρ_1 and ρ_2 are determined by

$$A_1 + 2A_2\rho + 3A_3\rho^2 + B\alpha\rho^{-\alpha-1} = c^2. \tag{2.1}$$

In fact, the above ρ_1 and ρ_2 can be calculated numerically when all the coefficients A_1, A_2, A_3, B and α are given. More precisely, we can estimate ρ_1 and ρ_2 simply for sufficiently small A_1, A_2, A_3, B. It can be concluded that the following two inequalities

$$B\alpha\rho_1^{-\alpha-1} < c^2 \quad \text{and} \quad A_1 + 2A_2\rho_2 + 3A_3\rho_2^2 < c^2 \tag{2.2}$$

hold simultaneously, which enables us to have at least

$$\rho_1 > \left(\frac{B\alpha}{c^2}\right)^{\frac{1}{\alpha+1}} \quad \text{and} \quad \rho_2 < \frac{A_2 + \sqrt{3A_3(c^2 - A_1) + A_1^2}}{3A_3}. \tag{2.3}$$

Thus, the physically relevant region of solutions for the fixed A_1, A_2, A_3, B is restricted to

$$V = \left\{ (\rho, v) : \rho_1 < \rho < \rho_2, \ |v| < c \right\}. \tag{2.4}$$

In addition, it is easy to know from (2.1) that

$$\lim_{A_1, A_2, A_3, B \to 0} \rho_1 = 0 \quad \text{and} \quad \lim_{A_1, A_2, A_3, B \to 0} \rho_2 = +\infty. \tag{2.5}$$

The system (1.1)-(1.2) can be rewritten in the following quasi-linear form

$$C \begin{pmatrix} \rho \\ v \end{pmatrix}_t + D \begin{pmatrix} \rho \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.6}$$

where the matrixes C and D are given respectively by

$$C = \begin{pmatrix} \dfrac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1})v^2 + c^4}{c^2(c^2 - v^2)} & \dfrac{2v(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)}{(c^2 - v^2)^2} \\ \dfrac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1} + c^2)v}{c^2 - v^2} & \dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)(c^2 + v^2)}{(c^2 - v^2)^2} \end{pmatrix}$$

and

$$D = \begin{pmatrix} \dfrac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1} + c^2)v}{c^2 - v^2} & \dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)(c^2 + v^2)}{(c^2 - v^2)^2} \\ \dfrac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1} + v^2)c^2}{c^2 - v^2} & \dfrac{2vc^2(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)}{(c^2 - v^2)^2} \end{pmatrix}.$$

By means of a direct calculation, we can achieve two real and distinct eigenvalues

$$\lambda_1(\rho, v) = \frac{c^2(v - \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 - v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}} < \frac{c^2(v + \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 + v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}} = \lambda_2(\rho, v). \tag{2.7}$$

Corresponding to each $\lambda_i (i = 1, 2)$, the right eigenvectors are calculated respectively by

$$\vec{r_1} = \left(-\frac{1}{c^2 - v^2}, \frac{\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}{A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2} \right)^T, \qquad \vec{r_2} = \left(\frac{1}{c^2 - v^2}, \frac{\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}{A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2} \right)^T. \tag{2.8}$$

Thus the system (1.1)-(1.2) is strictly hyperbolic [3, 6, 46]. Let us introduce the notion $\nabla = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial v})$, by a direct calculation, then we have

$$\frac{\partial \lambda_1(\rho, v)}{\partial \rho} = \frac{c^2[2A_2 + 6A_3\rho - \alpha(\alpha + 1)B\rho^{-\alpha-2}](v^2 - c^2)}{2\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}(c^2 - v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})^2},$$

$$\frac{\partial \lambda_1(\rho, v)}{\partial v} = \frac{c^2[c^2 - (A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1})]}{(c^2 - v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})^2},$$

and

$$\frac{\partial \lambda_2(\rho, v)}{\partial \rho} = \frac{c^2(2A_2 + 6A_3\rho - \alpha(\alpha + 1)B\rho^{-\alpha-2})(c^2 - v^2)}{2\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}(c^2 + v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})^2},$$

$$\frac{\partial \lambda_2(\rho, v)}{\partial v} = \frac{c^2(c^2 - (A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}))}{(c^2 + v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})^2}.$$

As a consequence, the following can be obtained

$$\begin{cases} \nabla \lambda_1 \cdot \vec{r_1} = \dfrac{c^2 p''(\rho)(p(\rho) + \rho c^2) + 2c^2 p'(\rho) - 2p'(\rho)^2}{2\sqrt{p'(\rho)}(p(\rho) + \rho c^2)(c^2 - v\sqrt{p'(\rho)})^2} \neq 0, \\ \\ \nabla \lambda_2 \cdot \vec{r_2} = \dfrac{c^2 p''(\rho)(p(\rho) + \rho c^2) + 2c^2 p'(\rho) - 2p'(\rho)^2}{2\sqrt{p'(\rho)}(p(\rho) + \rho c^2)(c^2 + v\sqrt{p'(\rho)})^2} \neq 0, \end{cases} \tag{2.9}$$

in which

$$p(\rho) = A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha},$$

$$p'(\rho) = A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1},$$

$$p''(\rho) = 2A_2 + 6A_3\rho - \alpha(\alpha + 1)B\rho^{-\alpha-2}.$$

Both the characteristic fields of λ_1 and λ_2 are genuinely nonlinear. That being said, we shall show that the elementary waves for each of the two characteristic fields are either rarefaction waves or shock waves [3, 6, 46].

Let us first consider the rarefaction wave curves. Both the system (1.1)-(1.2) and the Riemann initial data (1.3) are unchanged under the scalable coordinates: $(x, t) \to (kx, kt)$ ($k > 0$ is a constant). Therefore, we want to solve the self-similar solutions of the form

$$(\rho, v)(x, t) = (\rho, v)(\xi), \qquad \xi = \frac{x}{t}. \tag{2.10}$$

Now, we can use the following boundary value problems of ordinary differential equations to take the place of the Riemann problem (1.1)-(1.3) as follows:

$$\begin{cases} -\xi\left(\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v^2}{c^2(c^2 - v^2)} + \rho\right)_\xi + \left(\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2}\right)_\xi = 0, \\[4mm] -\xi\left(\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2}\right)_\xi \\[4mm] + \left(\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v^2}{c^2 - v^2} + A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha}\right)_\xi = 0, \\[4mm] (\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \end{cases} \tag{2.11}$$

For smooth solutions, (2.11) is reduced to

$$\begin{pmatrix} E & F \\ G & H \end{pmatrix} \begin{pmatrix} d\rho \\ dv \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{2.12}$$

where

$$E = \frac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1} + c^2)v}{c^2 - v^2} - \xi\frac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1})v^2 + c^4}{c^2(c^2 - v^2)},$$

$$F = \frac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)(c^2 + v^2)}{(c^2 - v^2)^2} - \xi\frac{2v(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)}{(c^2 - v^2)^2},$$

$$G = \frac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1} + v^2)c^2}{c^2 - v^2} - \xi\frac{(A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1} + c^2)v}{c^2 - v^2},$$

$$H = \frac{2vc^2(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)}{(c^2 - v^2)^2} - \xi\frac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)(c^2 + v^2)}{(c^2 - v^2)^2}.$$

If $(d\rho, dv) = (0, 0)$, then it is easy to get the trivial solution that (ρ, v) is a constant state. Otherwise, if $(d\rho, dv) \neq (0, 0)$, by a trivial and tedious calculation, then we can obtain the singular solutions

$$\begin{cases} \xi = \lambda_1 = \dfrac{c^2(v - \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 - v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}, \\[4mm] \dfrac{\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}{A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2}\,d\rho = \dfrac{1}{v^2 - c^2}\,dv, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \xi = \lambda_2 = \dfrac{c^2(v + \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 + v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}, \\[4mm] \dfrac{\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}{A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2}\,d\rho = \dfrac{1}{c^2 - v^2}\,dv. \end{cases} \tag{2.14}$$

One can obtain $\rho_1 < \rho < \rho_- < \rho_2$ directly from the requirement $\lambda_1(\rho) > \lambda_1(\rho_-)$. Let the left state (ρ_-, v_-) be fixed, then integrating the second equation in (2.13) from ρ_- to ρ enables us to obtain the 1-rarefaction wave

curve

$$R_1(\rho_-, v_-) : \begin{cases} \xi = \lambda_1(\rho, v) = \dfrac{c^2(v - \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 - v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}, \\[4mm] \ln\dfrac{c-v}{c+v} - \ln\dfrac{c-v_-}{c+v_-} = 2c\displaystyle\int_{\rho_-}^{\rho} \dfrac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\,ds, \\[4mm] v > v_-, \qquad \rho < \rho_-. \end{cases} \tag{2.15}$$

By virtue of a straight-forward computation, it is easy to find $v_\rho = \dfrac{(v^2 - c^2)\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}{A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2} <$
0. That is to say, v decreases as ρ increases for the curve $R_1(\rho_-, v_-)$. Analogously, due to $\rho_\xi = 1 > 0$, we can derive the 2-rarefaction wave curve as follows:

$$R_2(\rho_-, v_-) : \begin{cases} \xi = \lambda_2(\rho, v) = \dfrac{c^2(v + \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 + v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}, \\[4mm] \ln\dfrac{c-v}{c+v} - \ln\dfrac{c-v_-}{c+v_-} = -2c\displaystyle\int_{\rho_-}^{\rho} \dfrac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\,ds, \\[4mm] v > v_-, \qquad \rho > \rho_-. \end{cases} \tag{2.16}$$

By a direct calculation, we find $v_\rho = \dfrac{(c^2 - v^2)\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}{A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2} > 0$. It means that v increases as ρ increases for the curve $R_2(\rho_-, v_-)$.

For the 1-rarefaction wave, owing to a tedious but straightforward calculation for the second equation of (2.13), we have $v_{\rho\rho} > 0$ for all the $\rho_1 < \rho < \rho_2$. In other words, the 1-rarefaction wave curve R_1 is convex in the half-upper (ρ, v) phase plane. Analogously, we can also have $v_{\rho\rho} < 0$ for all the $\rho_1 < \rho < \rho_2$ from the second equation in (2.14). That is to say, the 2-rarefaction wave curve R_2 is concave in the half-upper (ρ, v) phase plane.

From now on, we focus our attention on the shock wave curves. The Rankine-Hugoniot conditions are as follows

$$\begin{cases} \sigma\left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v^2}{c^2(c^2 - v^2)} + \rho\right] = \left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2}\right], \\[4mm] \sigma\left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2}\right] \\[4mm] \quad = \left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v^2}{c^2 - v^2} + A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha}\right], \end{cases} \tag{2.17}$$

where $[\rho] = \rho - \rho_-$ denotes the jump across the discontinuity. We call σ the speed of the discontinuity, where $\sigma = \frac{dx}{dt}$. On the one hand, if $\sigma = 0$, then it can be obtained that $(\rho, v) = (\rho_-, v_-)$. On the other hand, if $\sigma \neq 0$, by removing σ from (2.17), we can obtain

$$\left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v^2}{c^2(c^2 - v^2)} + \rho\right]\left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v^2}{c^2 - v^2} + A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha}\right]$$

$$= \left[\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2}\right]^2. \tag{2.18}$$
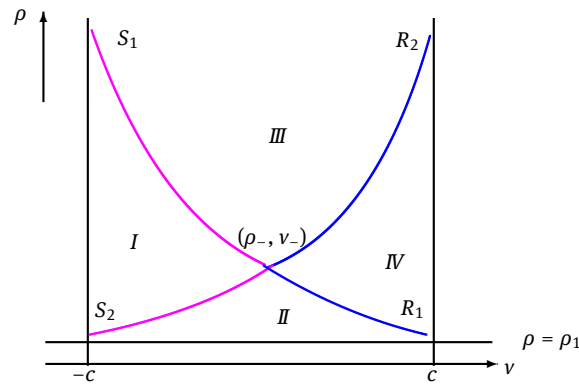
Fig. 1 For the given left state (ρ_-, v_-), the elementary wave curves emanating from the fixed left state (ρ_-, v_-) are shown in the half-upper (ρ, v) phase plane for the Riemann problem (1.1)-(1.3).

From direct calculation and simplification, (2.18) turns out to be

$$\frac{(v - v_-)^2}{(c^2 - vv_-)^2} = \frac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} - A_1\rho_- - A_2\rho_-^2 - A_3\rho_-^3 + B\rho_-^{-\alpha})(\rho - \rho_-)}{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho_-c^2)(A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha} + \rho c^2)}. \tag{2.19}$$

For the sake of simplicity, we set

$$\Psi(\rho, \rho_-) = \frac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} - A_1\rho_- - A_2\rho_-^2 - A_3\rho_-^3 + B\rho_-^{-\alpha})(\rho - \rho_-)}{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho_-c^2)(A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha} + \rho c^2)}. \tag{2.20}$$

As a consequence, (2.19) further reduces to

$$\frac{v - v_-}{c^2 - v_-^2} = -\frac{\sqrt{\Psi(\rho, \rho_-)}}{1 - v_-\sqrt{\Psi(\rho, \rho_-)}}. \tag{2.21}$$

To sum up for the given left state (ρ_-, v_-), the two shock waves are shown respectively as

$$S_1(\rho_-, v_-): \begin{cases} \sigma = \dfrac{\dfrac{c^2(\rho v^2 + (A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha}))}{c^2 - v^2} - \dfrac{c^2(\rho_- v_-^2 + (A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha}))}{c^2 - v_-^2}}{\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2} - \dfrac{(A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha} + \rho_- c^2)v_-}{c^2 - v_-^2}}, \\[4mm] \dfrac{v - v_-}{c^2 - v_-^2} = -\dfrac{\sqrt{\Psi(\rho, \rho_-)}}{1 - v_-\sqrt{\Psi(\rho, \rho_-)}}, \qquad v < v_-, \qquad \rho > \rho_-, \end{cases} \tag{2.22}$$

and

$$S_2(\rho_-, v_-): \begin{cases} \sigma = \dfrac{\dfrac{c^2(\rho v^2 + (A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha}))}{c^2 - v^2} - \dfrac{c^2(\rho_- v_-^2 + (A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha}))}{c^2 - v_-^2}}{\dfrac{(A_1\rho + A_2\rho^2 + A_3\rho^3 - B\rho^{-\alpha} + \rho c^2)v}{c^2 - v^2} - \dfrac{(A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha} + \rho_- c^2)v_-}{c^2 - v_-^2}}, \\[4mm] \dfrac{v - v_-}{c^2 - v_-^2} = -\dfrac{\sqrt{\Psi(\rho, \rho_-)}}{1 - v_-\sqrt{\Psi(\rho, \rho_-)}}, \qquad v < v_-, \qquad \rho < \rho_-. \end{cases} \tag{2.23}$$

From either (2.22) or (2.23), a tedious but straightforward computation shows that

$$v_\rho = -\frac{\Psi'(c^2 - v_-^2)}{2\sqrt{\Psi}(1 - v_-\sqrt{\Psi})^2}, \tag{2.24}$$

where

$$\Psi' = \frac{(p'(\rho)(p(\rho_-) + \rho c^2)(\rho - \rho_-) + (p(\rho) + \rho_- c^2)(p(\rho) - p(\rho_-)))(p(\rho_-) + \rho_- c^2)}{((p(\rho_-) + \rho c^2) + (p(\rho) + \rho_- c^2))^2}. \tag{2.25}$$

It is easy to see that $v_\rho < 0$ from $\rho_2 > \rho > \rho_- > \rho_1$ for the 1-shock curve and $v_\rho > 0$ from $\rho_1 < \rho < \rho_- < \rho_2$ for the 2-shock curve. It follows that v decreases as ρ increases for the curve $S_1(\rho_-, v_-)$ while v increases as ρ

increases for the curve $S_2(\rho_-, v_-)$. Comparing with the 1-rarefaction (or 2-rarefaction) curve, similar convexity (or concavity) are to be found in the 1-shock (or 2-shock) curve. The computation is tedious and trivial and thus the details are omitted here.

By combining (2.15), (2.16), (2.22) and (2.23), it is clear that the elementary wave curves $R_1(\rho_-, v_-)$, $R_2(\rho_-, v_-)$, $S_1(\rho_-, v_-)$ and $S_2(\rho_-, v_-)$ emanating from the fixed left state (ρ_-, v_-) divide the half-upper (ρ, v) phase plane into four regions I, II, III and IV (see Fig.1). Let (ρ_-, v_-) be fixed, then the solution to the Riemann problem (1.1)-(1.3) is determined uniquely by the above four regions. More precisely, the solution can be expressed as $S_1 + S_2$ when $(\rho_+, v_+) \in I$, $R_1 + S_2$ when $(\rho_+, v_+) \in II$, $S_1 + R_2$ when $(\rho_+, v_+) \in III$ or $R_1 + R_2$ when $(\rho_+, v_+) \in IV$ respectively. Here and in what follows, the symbol $S_1 + S_2$ is adopted to represent a 1-shock wave S_1 followed by a 2-shock wave S_2, etc.

2.2 The Riemann problem for the zero-pressure relativistic Euler system (1.5)

In this subsection, we shall briefly summarize the solutions to the Riemann problem for the zero-pressure relativistic Euler system (1.5), which have been well described such as in [39, 40]. The system (1.5) has the two coincident eigenvalues $\lambda_1 = \lambda_2 = v$, which means that the system (1.5) is non-strictly hyperbolic. The corresponding right eigenvector is $\overrightarrow{r_i} = (1, 0)$, $(i = 1, 2)$. Thus, the characteristic field of each $\lambda_i(i = 1, 2)$ is linear degenerate as a result of $\nabla \lambda_i \cdot \overrightarrow{r_i} = 0$, $(i = 1, 2)$, where $\nabla = (\frac{\partial}{\partial \rho}, \frac{\partial}{\partial v})$.

As before, if we look for the self-similar solution $(\rho, v)(x, t) = (\rho, v)(\xi)$, $\xi = \frac{x}{t}$, then the Riemann problem (1.3) and (1.5) is reduced to the boundary value problem of the following system of ordinary differential equations

$$\begin{cases} -\xi\left(\dfrac{\rho}{c^2 - v^2}\right)_\xi + \left(\dfrac{\rho v}{c^2 - v^2}\right)_\xi = 0, \\[2mm] -\xi\left(\dfrac{\rho v}{c^2 - v^2}\right)_\xi + \left(\dfrac{\rho v^2}{c^2 - v^2}\right)_\xi = 0, \\[2mm] (\rho, v)(\pm\infty) = (\rho_\pm, v_\pm). \end{cases} \tag{2.26}$$

In the case $v_- < v_+$, the solutions $(\rho, v)(\xi)$ including two contact discontinuities with a vacuum state between them can be written as

$$(\rho, v)(\xi) = \begin{cases} (\rho_-, v_-), & -\infty < \xi < v_-, \\ (0, v(\xi)), & v_- < \xi < v_+, \\ (\rho_+, v_+), & v_+ < \xi < +\infty. \end{cases} \tag{2.27}$$

Otherwise, in the case $v_- > v_+$, a delta shock wave solution is generated due to the overlapping characteristics for the Riemann problem (1.3) and (1.5). It is necessary to introduce the definition of δ−measure [19, 22, 45] in order to depict the delta shock wave solution to the Riemann problem (1.3) and (1.5).

Definition 2.1. *Let $\Gamma = \{(x(s), t(s)) : a < s < b\}$ be a parameterized smooth curve, then a two-dimensional weighted Dirac delta function $\omega(t)\delta_\Gamma$ with the support on Γ being defined as*

$$\langle \omega(s)\delta_s, \varphi(x(s), t(s)) \rangle = \int_a^b \omega(s)\varphi(x(s), t(s))ds, \tag{2.28}$$

for any test function $\varphi(x, t) \in C_0^\infty(R \times R_+)$.

For the purpose of completeness, it is necessary to offer the following generalized definition of delta shock wave solution introduced by Danilov et al. [47−50]. Let I be a finite index set, then we make the assumption that $\Gamma = \{\gamma_i | i \in I\}$ is a graph in the upper-half plane $(x, t) \in R \times R_+$ involving Lipschitz continuous curves γ_i for $i \in I$. Later, let I_0 be a subset of I, then the curves γ_i with $i \in I_0$ originate from the x−axis. In the end, let $\Gamma_0 = \{x_k^0 | k \in I_0\}$ be the set of initial points of γ_k with $k \in I_0$.

Definition 2.2. *Consider the δ-measure type initial data*

$$(\rho, v)(x, 0) = \left(\hat{\rho}_0(x) + \sum_{k \in I_0} w_k(x_k^0, 0)\delta(x - x_k^0), v_0(x)\right), \tag{2.29}$$

where $\hat{\rho}_0, v_0 \in L^\infty(R)$. Then, a pair of distributions (ρ, v) is called a δ–shock wave type solution for the system (1.5) with the initial data (2.29) if and only if the following integral identities

$$\iint_{R_+ R} \left(\frac{\hat{\rho}\varphi_t}{c^2 - v^2} + \frac{\hat{\rho}v\varphi_x}{c^2 - v^2}\right) dxdt + \sum_{i \in I} \int_{\gamma_i} \frac{w_i(x, t)}{c^2 - v_\delta^2} \frac{\partial\varphi(x, t)}{\partial l} dl + \int_R \frac{\hat{\rho}_0(x)\varphi(x, 0)}{c^2 - v_0(x)^2} dx + \sum_{k \in I_0} \frac{w_k(x_k^0, 0)\varphi(x_k^0, 0)}{c^2 - v_0(x_k^0)^2} = 0, \tag{2.30}$$

$$\iint_{R_+ R} \left(\frac{\hat{\rho}v\varphi_t}{c^2 - v^2} + \frac{\hat{\rho}v^2\varphi_x}{c^2 - v^2}\right) dxdt + \sum_{i \in I} \int_{\gamma_i} \frac{w_i(x, t)v_\delta}{c^2 - v_\delta^2} \frac{\partial\varphi(x, t)}{\partial l} dl + \int_R \frac{\hat{\rho}_0(x)v_0(x)\varphi(x, 0)}{c^2 - v_0(x)^2} dx$$

$$+ \sum_{k \in I_0} \frac{w_k(x_k^0, 0)v_0(x_k^0)\varphi(x_k^0, 0)}{c^2 - v_0(x_k^0)^2} = 0, \tag{2.31}$$

hold for any test function $\varphi \in C_c^\infty(R \times R_+)$.

It is clear to see that the Riemann initial data (1.3) is the simplest example of the δ-measure type initial data (2.29) that the graph contains only one arc and the initial strength of δ-measure is zero. In consideration of Definitions 2.1 and 2.2, if $v_- > v_+$, then a delta shock wave solution to the Riemann problem (1.3) and (1.5) can be provided in the following form [39, 40]

$$(\rho, v)(x, t) = \begin{cases} (\rho_-, v_-), & x < \sigma t, \\ (\omega(t)\delta(x - x(t)), v_\delta(t)), & x = \sigma t, \\ (\rho_+, v_+), & x > \sigma t, \end{cases} \tag{2.32}$$

where

$$\sigma = v_\delta(t) = \frac{v_+\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + v_-\sqrt{\frac{\rho_-}{c^2 - v_-^2}}}{\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + \sqrt{\frac{\rho_-}{c^2 - v_-^2}}}, \qquad \omega(t) = \sqrt{\frac{\rho_-\rho_+}{(c^2 - v_-^2)(c^2 - v_+^2)}} \cdot (c^2 - \sigma^2)(v_- - v_+)t. \tag{2.33}$$

Moreover, the delta shock wave solution (2.32) in comparison with (2.33) obeys the generalized Rankine-Hugoniot conditions listed below

$$\begin{cases} \dfrac{dx}{dt} = v_\delta(t), \\[2mm] \dfrac{d\left(\dfrac{\omega(t)}{c^2 - v_\delta(t)^2}\right)}{dt} = v_\delta(t)\left[\dfrac{\rho}{c^2 - v^2}\right] - \left[\dfrac{\rho v}{c^2 - v^2}\right], \\[2mm] \dfrac{d\left(\dfrac{\omega(t)v_\delta(t)}{c^2 - v_\delta(t)^2}\right)}{dt} = v_\delta(t)\left[\dfrac{\rho v}{c^2 - v^2}\right] - \left[\dfrac{\rho v^2}{c^2 - v^2}\right]. \end{cases} \tag{2.34}$$

In order to guarantee the uniqueness of solution, it should also obey the over-compressive δ-entropy condition

$$v_+ < v_\delta(t) < v_-. \tag{2.35}$$

In addition, the above-constructed delta shock wave solution (2.32) in comparison with (2.33) is satisfied with the system (1.5) in the sense of distributions. In other words, the weak form of the system (1.5) as below

$$\begin{cases} \left\langle\dfrac{\rho}{c^2 - v^2}, \phi_t\right\rangle + \left\langle\dfrac{\rho v}{c^2 - v^2}, \phi_x\right\rangle = 0, \\[2mm] \left\langle\dfrac{\rho v}{c^2 - v^2}, \phi_t\right\rangle + \left\langle\dfrac{\rho v^2}{c^2 - v^2}, \phi_x\right\rangle = 0, \end{cases} \tag{2.36}$$

holds for any test function $\phi(x, t) \in C_0^\infty((-\infty, +\infty) \times (0, +\infty))$, in which

$$\left\langle \frac{\rho}{c^2 - v^2}, \phi \right\rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0}{c^2 - v^2} \phi \, dx dt + \langle \omega \delta_s, \phi \rangle, \tag{2.37}$$

$$\left\langle \frac{\rho v}{c^2 - v^2}, \phi \right\rangle = \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0}{c^2 - v^2} \phi \, dx dt + \langle \sigma \omega \delta_s, \phi \rangle. \tag{2.38}$$

Here, we have used $\rho_0 = \rho_- + (\rho_+ - \rho_-)H(x - \sigma t)$ and $v_0 = v_- + (v_+ - v_-)H(x - \sigma t)$, in which H is the Heaviside function. In fact, the generalized Rankine-Hugoniot conditions (2.34) can be derived directly from (2.36) together with (2.37) and (2.38). The process of derivation is completely similar to that for the zero-pressure Euler system in [19], thus the details are omitted here. As a consequence, the existence and uniqueness of delta shock wave solution in the form (2.32) can be checked as in [44] by using the generalized Rankine-Hugoniot conditions (2.34) together with the over-compressive δ-entropy condition (2.35).

It is remarkable that the delta shock wave solution (2.32) together with (2.33) are no longer in the space of BV or L^∞ functions. However, the divergences of certain entropy and entropy flux fields are still in the space of Radon measures [22]. It is natural to discuss this problem in the theory of divergence-measure fields and thus the delta shock wave solution (2.32) together with (2.33) can be understood in the form of Tartar-Murat measure solution [51–53], in which the velocity must take a value at the point of the jump.

3 The formation of delta shock wave as $A_1, A_2, A_3, B \to 0$ when $v_- > v_+$

It is easy to know from [22, 26] that if $v_- > v_+$, then the solution to the Riemann problem (1.1)-(1.3) consists of two shock waves for sufficiently small positive numbers A_1, A_2, A_3 and B. In this section, we are mainly concerned with the formation of delta shock wave solution from the two-shock Riemann solution to the isentropic relativistic Euler system (1.1) associated with the equation of state (1.2) as A_1, A_2, A_3 and B tend to zero when the requirements $c > v_- > v_+ > -c$ and $\rho_1 < \rho_\pm < \rho_2$ are satisfied in the Riemann initial data (1.3).

Let $(\rho_\star, v_\star)$ be the intermediate state between two shock waves, we obtain the solution which joins (ρ_-, v_-) and $(\rho_\star, v_\star)$ by means of the 1-shock wave S_1 with the speed σ_1 and then joins $(\rho_\star, v_\star)$ and (ρ_+, v_+) by means of the 2-shock wave S_2 with the speed σ_2. To be more specific, we have

$$S_1 : \begin{cases} \sigma_1 = \dfrac{\dfrac{c^2(\rho_\star v_\star^2 + (A_1\rho_\star + A_2\rho_\star^2 + A_3\rho_\star^3 - B\rho_\star^{-\alpha}))}{c^2 - v_\star^2} - \dfrac{c^2(\rho_- v_-^2 + (A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha}))}{c^2 - v_-^2}}{\dfrac{(A_1\rho_\star + A_2\rho_\star^2 + A_3\rho_\star^3 - B\rho_\star^{-\alpha} + \rho_\star c^2)v_\star}{c^2 - v_\star^2} - \dfrac{(A_1\rho_- + A_2\rho_-^2 + A_3\rho_-^3 - B\rho_-^{-\alpha} + \rho_- c^2)v_-}{c^2 - v_-^2}}, \\[4mm] \dfrac{v_\star - v_-}{c^2 - v_-^2} = -\dfrac{\sqrt{\Psi(\rho_\star, \rho_-)}}{1 - v_-\sqrt{\Psi(\rho_\star, \rho_-)}}, \quad v_\star < v_-, \quad \rho_\star > \rho_-, \end{cases} \tag{3.1}$$

and

$$S_2 : \begin{cases} \sigma_2 = \dfrac{\dfrac{c^2(\rho_+ v_+^2 + (A_1\rho_+ + A_2\rho_+^2 + A_3\rho_+^3 - B\rho_+^{-\alpha}))}{c^2 - v_+^2} - \dfrac{c^2(\rho_\star v_\star^2 + (A_1\rho_\star + A_2\rho_\star^2 + A_3\rho_\star^3 - B\rho_\star^{-\alpha}))}{c^2 - v_\star^2}}{\dfrac{(A_1\rho_+ + A_2\rho_+^2 + A_3\rho_+^3 - B\rho_+^{-\alpha} + \rho_+ c^2)v_+}{c^2 - v_+^2} - \dfrac{(A_1\rho_\star + A_2\rho_\star^2 + A_3\rho_\star^3 - B\rho_\star^{-\alpha} + \rho_\star c^2)v_\star}{c^2 - v_\star^2}}, \\[4mm] \dfrac{v_+ - v_\star}{c^2 - v_\star^2} = -\dfrac{\sqrt{\Psi(\rho_+, \rho_\star)}}{1 - v_\star\sqrt{\Psi(\rho_+, \rho_\star)}}, \quad v_+ < v_\star, \quad \rho_+ < \rho_\star. \end{cases} \tag{3.2}$$

Then, the two second equations in (3.1) and (3.2) can be combined into

$$\frac{v_- - v_+}{c^2 - v_- v_+} = \frac{\sqrt{\Psi(\rho_\star, \rho_-)} + \sqrt{\Psi(\rho_+, \rho_\star)}}{1 + c^2\sqrt{\Psi(\rho_\star, \rho_-) \cdot \Psi(\rho_+, \rho_\star)}}. \tag{3.3}$$

In what follows, we shall give some lemmas which are related to the limiting behavior of the solution to the Riemann problem (1.1)-(1.3) as A_1, A_2, A_3 and $B \to 0$ when $c > v_- > v_+ > -c$.

Lemma 3.1. *We can establish the limiting relations* $\lim\limits_{A_1,A_2,A_3,B\to 0} \rho_\star = +\infty$ *and*

$$\lim_{A_1,A_2,A_3,B\to 0} \left(A_1\rho_\star + A_2\rho_\star^2 + A_3\rho_\star^3 - B\rho_\star^{-\alpha} \right) = \frac{c^4\rho_-\rho_+(v_- - v_+)^2}{(\rho_- + \rho_+)(c^2 - v_-^2)(c^2 - v_+^2) + 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2)}}.$$
(3.4)

Furthermore, we also have

$$\lim_{A_1,A_2,A_3,B\to 0} \sigma_1 = \lim_{A_1,A_2,A_3,B\to 0} \sigma_2 = \lim_{A_1,A_2,A_3,B\to 0} v_\star = \frac{v_+\sqrt{\dfrac{\rho_+}{c^2 - v_+^2}} + v_-\sqrt{\dfrac{\rho_-}{c^2 - v_-^2}}}{\sqrt{\dfrac{\rho_+}{c^2 - v_+^2}} + \sqrt{\dfrac{\rho_-}{c^2 - v_-^2}}} = \sigma.$$
(3.5)

Proof. Without loss of generality, we suppose $\lim\limits_{A_1,A_2,A_3,B\to 0} \rho_\star = m \in (max(\rho_-,\rho_+), +\infty)$. By a direct calculation, we have $\lim\limits_{A_1,A_2,A_3,B\to 0} \sqrt{\Psi(\rho_\star,\rho_-)} = \lim\limits_{A_1,A_2,A_3,B\to 0} \sqrt{\Psi(\rho_+,\rho_\star)} = 0$ under the above assumption that $\lim\limits_{A_1,A_2,A_3,B\to 0} \rho_\star$ is bounded. Taking the limit of (3.3) as $A_1, A_2, A_3, B \to 0$ leads to $v_- - v_+ = 0$, which contradicts with the fact $c > v_- > v_+ > -c$. Hence, it can be verified that $\lim\limits_{A_1,A_2,A_3,B\to 0} \rho_\star = +\infty$.

Let $\lim\limits_{A_1,A_2,A_3,B\to 0} (A_1\rho_\star + A_2\rho_\star^2 + A_3\rho_\star^3 - B\rho_\star^{-\alpha}) = M$, then we have

$$\lim_{A_1,A_2,A_3,B\to 0} \Psi(\rho_\star,\rho_-) = \frac{M}{(M + \rho_-c^2)c^2}, \qquad \lim_{A_1,A_2,A_3,B\to 0} \Psi(\rho_+,\rho_\star) = \frac{M}{(M + \rho_+c^2)c^2}.$$
(3.6)

As a result, (3.3) takes the following form

$$\frac{c(v_- - v_+)}{c^2 - v_-v_+} = \frac{\sqrt{\dfrac{M}{(M + \rho_-c^2)}} + \sqrt{\dfrac{M}{(M + \rho_+c^2)}}}{1 + \sqrt{\dfrac{M^2}{(M + \rho_-c^2)(M + \rho_+c^2)}}},$$
(3.7)

which enables us to have

$$\frac{c^2(v_-^2 - 2v_-v_+ + v_+^2)}{c^4 - 2v_-v_+c^2 + v_-^2v_+^2} = \frac{\dfrac{M}{M + \rho_-c^2} + \dfrac{M}{M + \rho_+c^2} + 2\sqrt{\dfrac{M^2}{(M + \rho_-c^2)(M + \rho_+c^2)}}}{1 + \dfrac{M^2}{(M + \rho_-c^2)(M + \rho_+c^2)} + 2\sqrt{\dfrac{M^2}{(M + \rho_-c^2)(M + \rho_+c^2)}}}.$$
(3.8)

Then (3.8) reduces to

$$\frac{c^2(v_-^2 - 2v_-v_+ + v_+^2)}{c^4 - (v_-^2 + v_+^2)c^2 + v_-^2v_+^2} = \frac{\dfrac{M}{M + \rho_-c^2} + \dfrac{M}{M + \rho_+c^2} + 2\sqrt{\dfrac{M^2}{(M + \rho_-c^2)(M + \rho_+c^2)}}}{\dfrac{\rho_-\rho_+c^4}{(M + \rho_-c^2)(M + \rho_+c^2)}}.$$
(3.9)

Furthermore, we have

$$\frac{\rho_-\rho_+c^6(v_- - v_+)^2}{(c^2 - v_-^2)(c^2 - v_+^2)} - M(2M + (\rho_- + \rho_+)c^2) = 2M\sqrt{(M + \rho_-c^2)(M + \rho_+c^2)}.$$
(3.10)

Squaring both sides of (3.10) and then simplifying, yields

$$\left((\rho_- - \rho_+)^2c^4 - 4\frac{\rho_-\rho_+c^6(v_- - v_+)^2}{(c^2 - v_-^2)(c^2 - v_+^2)}\right)M^2 - 2(\rho_- + \rho_+)c^2\frac{\rho_-\rho_+c^6(v_- - v_+)^2}{(c^2 - v_-^2)(c^2 - v_+^2)}M + \left(\frac{\rho_-\rho_+c^6(v_- - v_+)^2}{(c^2 - v_-^2)(c^2 - v_+^2)}\right)^2 = 0, \quad (3.11)$$

namely

$$\left((\rho_- + \rho_+)^2(c^2 - v_-^2)(c^2 - v_+^2) - 4\rho_-\rho_+(c^2 - v_-v_+)^2\right)M^2 - 2c^4(\rho_- + \rho_+)\rho_-\rho_+(v_- - v_+)^2 M + \frac{c^8\rho_-^2\rho_+^2(v_- - v_+)^4}{(c^2 - v_-^2)(c^2 - v_+^2)} = 0. \quad (3.12)$$

This is a quadratic form of M and can be solved as

$$M = \frac{(\rho_- + \rho_+)c^4\rho_-\rho_+(v_- - v_+)^2 \pm 2c^4\rho_-\rho_+(v_- - v_+)^2(c^2 - v_-v_+)\sqrt{\dfrac{\rho_-\rho_+}{(c^2 - v_-^2)(c^2 - v_+^2)}}}{(\rho_- + \rho_+)^2(c^2 - v_-^2)(c^2 - v_+^2) - 4\rho_-\rho_+(c^2 - v_-v_+)^2}$$

$$= \frac{c^4\rho_-\rho_+(v_- - v_+)^2\left((\rho_- + \rho_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)} \pm 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+}\right) \cdot \dfrac{1}{\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}}{\left((\rho_- + \rho_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)} + 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+}\right)\left((\rho_- + \rho_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)} - 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+}\right)}.$$

Thus, one has

$$M = \frac{c^4\rho_-\rho_+(v_- - v_+)^2}{(\rho_- + \rho_+)(c^2 - v_-^2)(c^2 - v_+^2) \pm 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2)}}. \quad (3.13)$$

If the negative sign in (3.13) is chosen, then one has

$$\frac{M}{(M + \rho_-c^2)c^2} = \frac{\dfrac{c^4\rho_-\rho_+(v_- - v_+)^2}{(\rho_- + \rho_+)(c^2 - v_-^2)(c^2 - v_+^2) - 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2)}}}{\left(\dfrac{c^4\rho_-\rho_+(v_- - v_+)^2}{(\rho_- + \rho_+)(c^2 - v_-^2)(c^2 - v_+^2) - 2(c^2 - v_-v_+)\sqrt{\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2)}} + \rho_-c^2\right)c^2}$$

$$= \frac{\rho_-\rho_+(v_- - v_+)^2}{\left(\sqrt{\rho_-\rho_+}(c^2 - v_-v_+) - \rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)^2}. \quad (3.14)$$

Analogously, one also has

$$\frac{M}{(M + \rho_+c^2)c^2} = \frac{\rho_-\rho_+(v_- - v_+)^2}{\left(\sqrt{\rho_-\rho_+}(c^2 - v_-v_+) - \rho_+\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)^2}. \quad (3.15)$$

Let v_- and v_+ be fixed to satisfy $c > v_- > v_+ > -c$, then it is easy to know that we can choose ρ_- and ρ_+ suitably to satisfy either

$$\rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)} < \sqrt{\rho_-\rho_+}(c^2 - v_-v_+) < \rho_+\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)} \quad (3.16)$$

or

$$\rho_+\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)} < \sqrt{\rho_-\rho_+}(c^2 - v_-v_+) < \rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}. \quad (3.17)$$

If the Riemann initial data (1.3) satisfy $c > v_- > v_+ > -c$ and (3.16) at the same time, then we have

$$\frac{\sqrt{\dfrac{M}{(M + \rho_-c^2)c^2}} + \sqrt{\dfrac{M}{(M + \rho_+c^2)c^2}}}{1 + c^2\sqrt{\dfrac{M}{(M + \rho_-c^2)c^2} \cdot \dfrac{M}{(M + \rho_+c^2)c^2}}}$$

$$= \frac{\dfrac{\sqrt{\rho_-\rho_+}(v_- - v_+)}{\sqrt{\rho_-\rho_+}(c^2 - v_-v_+) - \rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}} - \dfrac{\sqrt{\rho_-\rho_+}(v_- - v_+)}{\sqrt{\rho_-\rho_+}(c^2 - v_-v_+) - \rho_+\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}}{1 - \dfrac{c^2\rho_-\rho_+(v_- - v_+)^2}{\left(\sqrt{\rho_-\rho_+}(c^2 - v_-v_+) - \rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)\left(\sqrt{\rho_-\rho_+}(c^2 - v_-v_+) - \rho_+\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)}}$$

$$= \frac{\sqrt{\rho_-\rho_+}(v_- - v_+)(\rho_- - \rho_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}{2\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2) - \sqrt{\rho_-\rho_+}(\rho_- + \rho_+)(c^2 - v_-v_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}$$

$$= \frac{(v_- - v_+)(\rho_- - \rho_+)}{2\sqrt{\rho_-\rho_+(c^2 - v_-^2)(c^2 - v_+^2)} - (\rho_- + \rho_+)(c^2 - v_-v_+)}$$

which means that (3.7) is not satisfied. Similarly, if the Riemann initial data (1.3) satisfy $c > v_- > v_+ > -c$ and (3.17) at the same time, then we can also see that (3.7) is still not satisfied. As a consequence, it can be concluded from the above discussion that the negative sign cannot be chosen in (3.13) for the reason that (3.7) does not always hold for any given Riemann initial data (1.3) satisfying $c > v_- > v_+ > -c$.

On the other hand, if we choose the positive sign in (3.13), then it yields

$$\frac{M}{(M + \rho_- c^2)c^2} = \frac{\rho_- \rho_+ (v_- - v_+)^2}{\left(\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_- \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)^2}, \tag{3.18}$$

and

$$\frac{M}{(M + \rho_+ c^2)c^2} = \frac{\rho_- \rho_+ (v_- - v_+)^2}{\left(\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_+ \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)^2}. \tag{3.19}$$

Thus, it is easy to get

$$\frac{\sqrt{\dfrac{M}{(M + \rho_- c^2)c^2}} + \sqrt{\dfrac{M}{(M + \rho_+ c^2)c^2}}}{1 + c^2 \sqrt{\dfrac{M}{(M + \rho_- c^2)c^2} \cdot \dfrac{M}{(M + \rho_+ c^2)c^2}}}$$

$$= \frac{\dfrac{\sqrt{\rho_- \rho_+}(v_- - v_+)}{\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_- \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}} + \dfrac{\sqrt{\rho_- \rho_+}(v_- - v_+)}{\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_+ \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}}{1 + \dfrac{c^2 \rho_- \rho_+ (v_- - v_+)^2}{\left(\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_- \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)\left(\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_+ \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}\right)}}$$

$$= \frac{\sqrt{\rho_- \rho_+}(v_- - v_+)(2\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + (\rho_- + \rho_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)})}{2\rho_- \rho_+(c^2 - v_- v_+)^2 + \sqrt{\rho_- \rho_+}(\rho_- + \rho_+)(c^2 - v_- v_+)\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}$$

$$= \frac{v_- - v_+}{c^2 - v_- v_+},$$

which enables us to see that (3.7) is indeed satisfied. In a word, it can be concluded from the above discussion that

$$M = \frac{c^4 \rho_- \rho_+ (v_- - v_+)^2}{(\rho_- + \rho_+)(c^2 - v_-^2)(c^2 - v_+^2) + 2(c^2 - v_- v_+)\sqrt{\rho_- \rho_+(c^2 - v_-^2)(c^2 - v_+^2)}}. \tag{3.20}$$

As a consequence, the limiting relation (3.4) can be established.

It is deduced from (3.1) that

$$v_\star = v_- - \frac{(c^2 - v_-^2)\sqrt{\Psi(\rho_\star, \rho_-)}}{1 - v_-\sqrt{\Psi(\rho_\star, \rho_-)}} = \frac{v_- - c^2\sqrt{\Psi(\rho_\star, \rho_-)}}{1 - v_-\sqrt{\Psi(\rho_\star, \rho_-)}}, \tag{3.21}$$

such that we have

$$\lim_{A_1, A_2, A_3, B \to 0} v_\star = \frac{v_- - c^2 \lim\limits_{A_1, A_2, A_3, B \to 0} \sqrt{\Psi(\rho_\star, \rho_-)}}{1 - v_- \lim\limits_{A_1, A_2, A_3, B \to 0} \sqrt{\Psi(\rho_\star, \rho_-)}}. \tag{3.22}$$

Substituting (3.6) and (3.20) into (3.22) leads to

$$\lim_{A_1, A_2, A_3, B \to 0} v_\star = \frac{v_- - \dfrac{\sqrt{\rho_- \rho_+}(v_- - v_+)c^2}{\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_- \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}}{1 - \dfrac{v_- \sqrt{\rho_- \rho_+}(v_- - v_+)}{\sqrt{\rho_- \rho_+}(c^2 - v_- v_+) + \rho_- \sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}}$$

$$= \frac{v_+\sqrt{\rho_-\rho_+}(c^2 - v_-^2) + v_-\rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}}{\sqrt{\rho_-\rho_+}(c^2 - v_-^2) + \rho_-\sqrt{(c^2 - v_-^2)(c^2 - v_+^2)}} = \frac{v_+\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + v_-\sqrt{\frac{\rho_-}{c^2 - v_-^2}}}{\sqrt{\frac{\rho_+}{c^2 - v_+^2}} + \sqrt{\frac{\rho_-}{c^2 - v_-^2}}} = \sigma. \quad (3.23)$$

On the one hand, it can be clearly seen that

$$\lim_{A_1,A_2,A_3,B \to 0} \sigma_1 = \lim_{A_1,A_2,A_3,B \to 0} \frac{\frac{c^2(\rho_\star v_\star^2 + p(\rho_\star))}{c^2 - v_\star^2} - \frac{c^2(\rho_- v_-^2 + p(\rho_-))}{c^2 - v_-^2}}{\frac{(p(\rho_\star) + \rho_\star c^2)v_\star}{c^2 - v_\star^2} - \frac{(p(\rho_-) + \rho_- c^2)v_-}{c^2 - v_-^2}}$$

$$= \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2[(\rho_\star v_\star^2 + p(\rho_\star))(c^2 - v_-^2) - (\rho_- v_-^2 + p(\rho_-))(c^2 - v_\star^2)]}{v_\star(p(\rho_\star) + \rho_\star c^2)(c^2 - v_-^2) - v_-(p(\rho_-) + \rho_- c^2)(c^2 - v_\star^2)}. \quad (3.24)$$

Taking into account $\lim_{A_1,A_2,A_3,B \to 0} \rho_\star = +\infty$, (3.4) and (3.23), we then have

$$\lim_{A_1,A_2,A_3,B \to 0} \sigma_1 = \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2(\rho_\star v_\star^2 + p(\rho_\star))(c^2 - v_-^2)}{v_\star(p(\rho_\star) + \rho_\star c^2)(c^2 - v_-^2)} = \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2(\rho_\star v_\star^2 + p(\rho_\star))}{v_\star(p(\rho_\star) + \rho_\star c^2)}$$

$$= \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2(v_\star^2 + \frac{p(\rho_\star)}{\rho_\star})}{v_\star(\frac{p(\rho_\star)}{\rho_\star} + c^2)} = \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2 v_\star^2}{v_\star c^2} = \lim_{A_1,A_2,A_3,B \to 0} v_\star = \sigma. \quad (3.25)$$

On the other hand, it is easy to see that

$$\lim_{A_1,A_2,A_3,B \to 0} \sigma_2 = \lim_{A_1,A_2,A_3,B \to 0} \frac{\frac{c^2(\rho_+ v_+^2 + p(\rho_+))}{c^2 - v_+^2} - \frac{c^2(\rho_\star v_\star^2 + p(\rho_\star))}{c^2 - v_\star^2}}{\frac{(p(\rho_+) + \rho_+ c^2)v_+}{c^2 - v_+^2} - \frac{(p(\rho_\star) + \rho_\star c^2)v_\star}{c^2 - v_\star^2}}$$

$$= \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2[(\rho_+ v_+^2 + p(\rho_+))(c^2 - v_\star^2) - (\rho_\star v_\star^2 + p(\rho_\star))(c^2 - v_+^2)]}{v_+(p(\rho_+) + \rho_+ c^2)(c^2 - v_\star^2) - v_\star(p(\rho_\star) + \rho_\star c^2)(c^2 - v_+^2)}. \quad (3.26)$$

Analogously, we have

$$\lim_{A_1,A_2,A_3,B \to 0} \sigma_2 = \lim_{A_1,A_2,A_3,B \to 0} \frac{-c^2(\rho_\star v_\star^2 + p(\rho_\star))(c^2 - v_+^2)}{-v_\star(p(\rho_\star) + \rho_\star c^2)(c^2 - v_+^2)}$$

$$= \lim_{A_1,A_2,A_3,B \to 0} \frac{c^2(v_\star^2 + \frac{p(\rho_\star)}{\rho_\star})}{v_\star(\frac{p(\rho_\star)}{\rho_\star} + c^2)} = \sigma. \quad (3.27)$$

The proof is completed. □

Lemma 3.2. *The limiting relations of mass and momentum between the two shock waves as $A_1, A_2, A_3, B \to 0$ can be established as follows:*

$$\lim_{A_1,A_2,A_3,B \to 0} \int_{\sigma_1}^{\sigma_2} \rho_\star d\xi = \left(\sigma\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right]\right)\left(c^2 - \sigma^2\right), \quad (3.28)$$

$$\lim_{A_1,A_2,A_3,B \to 0} \int_{\sigma_1}^{\sigma_2} \rho_\star v_\star d\xi = \left(\sigma\left[\frac{\rho v}{c^2 - v^2}\right] - \left[\frac{\rho v^2}{c^2 - v^2}\right]\right)\left(c^2 - \sigma^2\right), \quad (3.29)$$

in which the jump of ρ is given by $[\rho] = \rho_+ - \rho_-$, etc.

Proof. We turn back to consider the Rankine-Hugoniot conditions (2.17). Carrying out the two shock waves S_1 and S_2 in the first equation of (2.17), one has

$$
\begin{cases}
\sigma_1\left(\dfrac{p(\rho_\star)v_\star^2}{c^2(c^2-v_\star^2)}+\dfrac{\rho_\star v_\star^2}{c^2-v_\star^2}+\rho_\star-\dfrac{p(\rho_-)v_-^2}{c^2(c^2-v_-^2)}-\dfrac{\rho_- v_-^2}{c^2-v_-^2}-\rho_-\right)=\dfrac{p(\rho_\star)v_\star}{c^2-v_\star^2}+\dfrac{\rho_\star v_\star c^2}{c^2-v_\star^2}-\dfrac{p(\rho_-)v_-}{c^2-v_-^2}-\dfrac{\rho_- v_- c^2}{c^2-v_-^2}, \\[3mm]
\sigma_2\left(\dfrac{p(\rho_+)v_+^2}{c^2(c^2-v_+^2)}+\dfrac{\rho_+ v_+^2}{c^2-v_+^2}+\rho_+-\dfrac{p(\rho_\star)v_\star^2}{c^2(c^2-v_\star^2)}-\dfrac{\rho_\star v_\star^2}{c^2-v_\star^2}-\rho_\star\right)=\dfrac{p(\rho_+)v_+}{c^2-v_+^2}+\dfrac{\rho_+ v_+ c^2}{c^2-v_+^2}-\dfrac{p(\rho_\star)v_\star}{c^2-v_\star^2}-\dfrac{\rho_\star v_\star c^2}{c^2-v_\star^2},
\end{cases}
$$
$$(3.30)$$

which brings about

$$
\lim_{A_1,A_2,A_3,B\to 0}(\sigma_2-\sigma_1)\rho_\star=\lim_{A_1,A_2,A_3,B\to 0}\left((\sigma_1-\sigma_2)\dfrac{p(\rho_\star)v_\star^2}{c^2(c^2-v_\star^2)}-\sigma_1\dfrac{p(\rho_-)v_-^2}{c^2(c^2-v_-^2)}-\sigma_1\dfrac{\rho_- c^2}{c^2-v_-^2}+\sigma_2\dfrac{p(\rho_+)v_+^2}{c^2(c^2-v_+^2)}\right.
$$
$$
\left.+\sigma_2\dfrac{\rho_+ c^2}{c^2-v_+^2}+\dfrac{p(\rho_-)v_-}{c^2-v_-^2}+\dfrac{\rho_- v_- c^2}{c^2-v_-^2}-\dfrac{p(\rho_+)v_+}{c^2-v_+^2}-\dfrac{\rho_+ v_+ c^2}{c^2-v_+^2}\right)\dfrac{c^2-v_\star^2}{c^2}
$$
$$
=\left(-\sigma\dfrac{\rho_-}{c^2-v_-^2}+\sigma\dfrac{\rho_+}{c^2-v_+^2}+\dfrac{\rho_- v_-}{c^2-v_-^2}-\dfrac{\rho_+ v_+}{c^2-v_+^2}\right)\left(c^2-\sigma^2\right)
$$
$$
=\left(\sigma\left[\dfrac{\rho}{c^2-v^2}\right]-\left[\dfrac{\rho v}{c^2-v^2}\right]\right)\left(c^2-\sigma^2\right). \tag{3.31}
$$

Then, using the similar way to deal with the second equation of (2.17), one also has

$$
\begin{cases}
\sigma_1\left(\dfrac{p(\rho_\star)v_\star}{c^2-v_\star^2}+\dfrac{\rho_\star v_\star c^2}{c^2-v_\star^2}-\dfrac{p(\rho_-)v_-}{c^2-v_-^2}-\dfrac{\rho_- v_- c^2}{c^2-v_-^2}\right)=\dfrac{p(\rho_\star)v_\star^2}{c^2-v_\star^2}+\dfrac{\rho_\star v_\star^2 c^2}{c^2-v_\star^2}+p(\rho_\star)-\dfrac{p(\rho_-)v_-^2}{c^2-v_-^2}-\dfrac{\rho_- v_-^2 c^2}{c^2-v_-^2}-p(\rho_-), \\[3mm]
\sigma_2\left(\dfrac{p(\rho_+)v_+}{c^2-v_+^2}+\dfrac{\rho_+ v_+ c^2}{c^2-v_+^2}-\dfrac{p(\rho_\star)v_\star}{c^2-v_\star^2}-\dfrac{\rho_\star v_\star c^2}{c^2-v_\star^2}\right)=\dfrac{p(\rho_+)v_+^2}{c^2-v_+^2}+\dfrac{\rho_+ v_+^2 c^2}{c^2-v_+^2}+p(\rho_+)-\dfrac{p(\rho_\star)v_\star^2}{c^2-v_\star^2}-\dfrac{\rho_\star v_\star^2 c^2}{c^2-v_\star^2}-p(\rho_\star),
\end{cases}
$$
$$(3.32)$$

which gives rise to

$$
\lim_{A_1,A_2,A_3,B\to 0}(\sigma_2-\sigma_1)\rho_\star v_\star=\lim_{A_1,A_2,A_3,B\to 0}\left((\sigma_1-\sigma_2)\dfrac{p(\rho_\star)v_\star}{c^2-v_\star^2}-\sigma_1\dfrac{p(\rho_-)v_-}{c^2-v_-^2}-\sigma_1\dfrac{\rho_- v_- c^2}{c^2-v_-^2}+\sigma_2\dfrac{p(\rho_+)v_+}{c^2-v_+^2}+\sigma_2\dfrac{\rho_+ v_+ c^2}{c^2-v_+^2}\right.
$$
$$
\left.+\dfrac{p(\rho_-)v_-^2}{c^2-v_-^2}+\dfrac{\rho_- v_-^2 c^2}{c^2-v_-^2}+p(\rho_-)-\dfrac{p(\rho_+)v_+^2}{c^2-v_+^2}-\dfrac{\rho_+ v_+^2 c^2}{c^2-v_+^2}-p(\rho_+)\right)\dfrac{c^2-v_\star^2}{c^2}
$$
$$
=\left(-\sigma\dfrac{\rho_- v_-}{c^2-v_-^2}+\sigma\dfrac{\rho_+ v_+}{c^2-v_+^2}+\dfrac{\rho_- v_-^2}{c^2-v_-^2}-\dfrac{\rho_+ v_+^2}{c^2-v_+^2}\right)\left(c^2-\sigma^2\right)
$$
$$
=\left(\sigma\left[\dfrac{\rho v}{c^2-v^2}\right]-\left[\dfrac{\rho v^2}{c^2-v^2}\right]\right)\left(c^2-\sigma^2\right). \tag{3.33}
$$

As a consequence, the limiting relations (3.28) and (3.29) can be established directly from (3.31) and (3.33). ☐

Theorem 3.3. *When $-c<v_+<v_-<c$, let us suppose that $(\rho(\xi),v(\xi))$ is a two-shock-wave solution to the Riemann problem (1.1)-(1.3) for sufficiently small A_1,A_2,A_3,B, then the limit of solution as $A_1,A_2,A_3,B\to 0$ converges to the δ-shock wave solution (2.32) linked with (2.33) in the sense of distributions, which is identical with that for the pressureless relativistic Euler system (1.5). Besides, the limits of momentums $\dfrac{\rho}{c^2-v^2}$ and $\dfrac{\rho v}{c^2-v^2}$ as $A_1,A_2,A_3,B\to 0$ are the sum of a step function and a Dirac δ-function with weights in the form*

$$
\left(\sigma\left[\dfrac{\rho}{c^2-v^2}\right]-\left[\dfrac{\rho v}{c^2-v^2}\right]\right)t \quad\text{and}\quad \left(\sigma\left[\dfrac{\rho v}{c^2-v^2}\right]-\left[\dfrac{\rho v^2}{c^2-v^2}\right]\right)t
$$

respectively.

Proof. Given $\xi=\dfrac{x}{t}$, for each fixed $A_1,A_2,A_3,B>0$, the solutions comprising of two-shock waves to the Riemann problem (1.1)-(1.3) may be indicated as

$$
(\rho,v)(\xi)=\begin{cases}
(\rho_-,v_-), & -\infty<\xi<\sigma_1, \\
(\rho_\star,v_\star), & \sigma_1<\xi<\sigma_2, \\
(\rho_+,v_+), & \sigma_2<\xi<+\infty,
\end{cases} \tag{3.34}
$$

which should satisfy the following weak forms

$$-\int_{-\infty}^{+\infty}(v(\xi)-\xi)\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi'(\xi)d\xi + \int_{-\infty}^{+\infty}\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi(\xi)d\xi$$

$$-\int_{-\infty}^{+\infty}(1-\frac{\xi v(\xi)}{c^2})\frac{p(\rho(\xi))v(\xi)}{c^2-v(\xi)^2}\phi'(\xi)d\xi + \int_{-\infty}^{+\infty}\frac{p(\rho(\xi))v(\xi)^2}{c^2(c^2-v(\xi)^2)}\phi(\xi)d\xi = 0, \tag{3.35}$$

and

$$-\int_{-\infty}^{+\infty}(v(\xi)-\xi)\frac{\rho(\xi)c^2v(\xi)}{c^2-v(\xi)^2}\phi'(\xi)d\xi + \int_{-\infty}^{+\infty}\frac{\rho(\xi)c^2v(\xi)}{c^2-v(\xi)^2}\phi(\xi)d\xi - \int_{-\infty}^{+\infty}(v(\xi)-\xi)\frac{p(\rho(\xi))v(\xi)}{c^2-v(\xi)^2}\phi'(\xi)d\xi$$

$$+\int_{-\infty}^{+\infty}\frac{p(\rho(\xi))v(\xi)}{c^2(c^2-v(\xi)^2)}\phi(\xi)d\xi - \int_{-\infty}^{+\infty}p(\rho(\xi))\phi'(\xi)d\xi = 0, \tag{3.36}$$

for any $\phi(\xi) \in C_0^\infty(-\infty, +\infty)$.

Splitting the first integral in (3.35), yields

$$\int_{-\infty}^{+\infty}(v(\xi)-\xi)\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi'(\xi)d\xi = \left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\right)(v(\xi)-\xi)\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi'(\xi)d\xi. \tag{3.37}$$

To sum up the first and last terms of (3.37), we obtain

$$\lim_{A_1,A_2,A_3,B\to0}\int_{-\infty}^{\sigma_1}(v(\xi)-\xi)\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi'(\xi)d\xi + \lim_{A_1,A_2,A_3,B\to0}\int_{\sigma_2}^{+\infty}(v(\xi)-\xi)\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi'(\xi)d\xi$$

$$=\lim_{A_1,A_2,A_3,B\to0}\int_{-\infty}^{\sigma_1}(v_--\xi)\frac{\rho_-c^2}{c^2-v_-^2}\phi'(\xi)d\xi + \lim_{A_1,A_2,A_3,B\to0}\int_{\sigma_2}^{+\infty}(v_+-\xi)\frac{\rho_+c^2}{c^2-v_+^2}\phi'(\xi)d\xi$$

$$=\lim_{A_1,A_2,A_3,B\to0}\left(\frac{\rho_-v_-c^2}{c^2-v_-^2}\phi(\sigma_1)-\frac{\rho_-c^2}{c^2-v_-^2}\sigma_1\phi(\sigma_1)-\frac{\rho_+v_+c^2}{c^2-v_+^2}\phi(\sigma_2)+\frac{\rho_+c^2}{c^2-v_+^2}\sigma_2\phi(\sigma_2)\right. \tag{3.38}$$

$$\left.+\frac{\rho_-c^2}{c^2-v_-^2}\int_{-\infty}^{\sigma_1}\phi(\xi)d\xi+\frac{\rho_+c^2}{c^2-v_+^2}\int_{\sigma_2}^{+\infty}\phi(\xi)d\xi\right)$$

$$=c^2\left(\sigma\left[\frac{\rho}{c^2-v^2}\right]-\left[\frac{\rho v}{c^2-v^2}\right]\right)\phi(\sigma)+c^2\int_{-\infty}^{+\infty}\frac{\rho_0}{c^2-v_0^2}H(\xi-\sigma))\phi(\xi)d\xi$$

with

$$\frac{\rho_0}{c^2-v_0^2}(\xi)=\frac{\rho_-}{c^2-v_-^2}+\left[\frac{\rho}{c^2-v^2}\right]H(\xi),$$

where $H(\xi)$ is the normal Heaviside function. For the second part of (3.37), based on Lemma 3.1, we obtain

$$\lim_{A_1,A_2,A_3,B\to 0}\int_{\sigma_1}^{\sigma_2}(v(\xi)-\xi)\frac{\rho(\xi)c^2}{c^2-v(\xi)^2}\phi'(\xi)d\xi$$

$$=\lim_{A_1,A_2,A_3,B\to 0}\int_{\sigma_1}^{\sigma_2}(v_\star-\xi)\frac{\rho_\star c^2}{c^2-v_\star^2}\phi'(\xi)d\xi$$

$$=\lim_{A_1,A_2,A_3,B\to 0}\left(\rho_\star(\sigma_2-\sigma_1)\frac{c^2 v_\star}{c^2-v_\star^2}\left(\frac{\phi(\sigma_2)-\phi(\sigma_1)}{\sigma_2-\sigma_1}\right)-\rho_\star(\sigma_2-\sigma_1)\frac{c^2}{c^2-v_\star^2}\left(\frac{\sigma_2\phi(\sigma_2)-\sigma_1\phi(\sigma_1)}{\sigma_2-\sigma_1}\right)\right.\tag{3.39}$$

$$\left.+\rho_\star(\sigma_2-\sigma_1)\frac{c^2}{c^2-v_\star^2}\frac{\int_{\sigma_1}^{\sigma_2}\phi(\xi)d\xi}{\sigma_2-\sigma_1}\right)$$

$$=c^2\left(\sigma\left[\frac{\rho}{c^2-v^2}\right]-\left[\frac{\rho v}{c^2-v^2}\right]\right)(\sigma\phi'(\sigma)-\sigma\phi'(\sigma)-\phi(\sigma)+\phi(\sigma))=0,$$

in which we have used $\phi(\xi)\in C_0^\infty(-\infty,+\infty)$ and $\lim\limits_{A_1,A_2,A_3,B\to 0}\sigma_1=\lim\limits_{A_1,A_2,A_3,B\to 0}\sigma_2=\lim\limits_{A_1,A_2,A_3,B\to 0}v_\star=\sigma$.

Moreover, the third and fourth integrals in (3.35) can be split into

$$\lim_{A_1,A_2,A_3,B\to 0}\left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\right)(1-\frac{\xi v(\xi)}{c^2})\frac{p(\rho(\xi))v(\xi)}{c^2-v(\xi)^2}\phi'(\xi)d\xi$$

$$=\lim_{A_1,A_2,A_3,B\to 0}\left(\frac{p(\rho_-)v_-}{c^2-v_-^2}\phi(\sigma_1)-\frac{p(\rho_-)v_-^2}{c^2(c^2-v_-^2)}\sigma_1\phi(\sigma_1)-\frac{p(\rho_+)v_+}{c^2-v_+^2}\phi(\sigma_2)+\frac{p(\rho_+)v_+^2}{c^2(c^2-v_+^2)}\sigma_2\phi(\sigma_2)\right.$$

$$+\frac{p(\rho_-)v_-^2}{c^2(c^2-v_-^2)}\int_{-\infty}^{\sigma_1}\phi(\xi)d\xi+\frac{p(\rho_+)v_+^2}{c^2(c^2-v_+^2)}\int_{\sigma_2}^{+\infty}\phi(\xi)d\xi+\frac{p(\rho_\star)v_\star}{c^2-v_\star^2}(\phi(\sigma_2)-\phi(\sigma_1))-\frac{p(\rho_\star)v_\star^2}{c^2(c^2-v_\star^2)}(\sigma_2\phi(\sigma_2)-\sigma_1\phi(\sigma_1))$$

$$\left.+\frac{p(\rho_\star)v_\star^2}{c^2(c^2-v_\star^2)}\int_{\sigma_1}^{\sigma_2}\phi(\xi)d\xi\right)=0,$$

$$\tag{3.40}$$

and

$$\lim_{A_1,A_2,A_3,B\to 0}\left(\int_{-\infty}^{\sigma_1}+\int_{\sigma_1}^{\sigma_2}+\int_{\sigma_2}^{+\infty}\right)\frac{p(\rho(\xi))v(\xi)^2}{c^2(c^2-v(\xi)^2)}\phi(\xi)d\xi$$

$$=\lim_{A_1,A_2,A_3,B\to 0}\frac{p(\rho_-)v_-^2}{c^2(c^2-v_-^2)}\int_{-\infty}^{\sigma_1}\phi(\xi)d\xi+\frac{p(\rho_+)v_+^2}{c^2(c^2-v_+^2)}\int_{\sigma_2}^{+\infty}\phi(\xi)d\xi+\frac{p(\rho_\star)v_\star^2}{c^2(c^2-v_\star^2)}\int_{\sigma_1}^{\sigma_2}\phi(\xi)d\xi\tag{3.41}$$

$$=0,$$

respectively. By substituting (3.38)-(3.41) into (3.35), it follows that

$$\lim_{A_1,A_2,A_3,B\to 0}\int_{-\infty}^{+\infty}\left(\frac{\rho(\xi)}{c^2-v(\xi)^2}-\frac{\rho_0}{c^2-v_0^2}(\xi-\sigma)\right)\phi(\xi)d\xi=\left(\sigma\left[\frac{\rho}{c^2-v^2}\right]-\left[\frac{\rho v}{c^2-v^2}\right]\right)\phi(\sigma).\tag{3.42}$$

On the other hand, if the integral equation (3.36) is carried out in the same way as before, then

$$\lim_{A_1,A_2,A_3,B\to 0}\int_{-\infty}^{+\infty}\left(\frac{\rho(\xi)v(\xi)}{c^2-v(\xi)^2}-\frac{\rho_0 v_0}{c^2-v_0^2}(\xi-\sigma)\right)\phi(\xi)d\xi=\left(\sigma\left[\frac{\rho v}{c^2-v^2}\right]-\left[\frac{\rho v^2}{c^2-v^2}\right]\right)\phi(\sigma).\tag{3.43}$$

In the end, we are concerned with the limits of $\frac{\rho}{c^2 - v^2}$ and $\frac{\rho v}{c^2 - v^2}$ as $A_1, A_2, A_3, B \to 0$. Let $\psi(x, t) \in C_0^\infty(R \times R_+)$, then it can be concluded from (3.42) that

$$\lim_{A_1,A_2,A_3,B\to0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho}{c^2 - v^2}\left(\frac{x}{t}\right)\psi(x, t)dxdt$$

$$= \lim_{A_1,A_2,A_3,B\to0} \int_0^{+\infty} t\left(\int_{-\infty}^{+\infty} \frac{\rho(\xi)}{c^2 - v(\xi)^2}\psi(\xi t, t)d\xi\right)dt = \int_0^{+\infty} t\left(\lim_{A_1,A_2,A_3,B\to0} \int_{-\infty}^{+\infty} \frac{\rho(\xi)}{c^2 - v(\xi)^2}\psi(\xi t, t)d\xi\right)dt$$

$$= \int_0^{+\infty} t\left(\int_{-\infty}^{+\infty} \frac{\rho_0}{c^2 - v_0^2}(\xi - \sigma)\psi(\xi t, t)d\xi + \left(\sigma\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right]\right)\psi(\sigma t, t)\right)dt \tag{3.44}$$

$$= \int_0^{+\infty} t\left(t^{-1}\int_{-\infty}^{+\infty} \frac{\rho_0}{c^2 - v_0^2}(x - \sigma t)\psi(x, t)dx + \left(\sigma\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right]\right)\psi(\sigma t, t)\right)dt$$

$$= \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0}{c^2 - v_0^2}(x - \sigma t)\psi(x, t)dxdt + \int_0^{+\infty} t\left(\sigma\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right]\right)\psi(\sigma t, t)dt.$$

According to Definition 2.1, the last part of (3.44) is equivalent to $\langle\omega_1(t)\delta_s, \psi(\cdot, \cdot)\rangle$, in which

$$\omega_1(t) = \left(\sigma\left[\frac{\rho}{c^2 - v^2}\right] - \left[\frac{\rho v}{c^2 - v^2}\right]\right)t. \tag{3.45}$$

In the same way as before, from (3.43), we also obtain

$$\lim_{A_1,A_2,A_3,B\to0} \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho v}{c^2 - v^2}\left(\frac{x}{t}\right)\psi(x, t)dxdt = \int_0^{+\infty} \int_{-\infty}^{+\infty} \frac{\rho_0 v_0}{c^2 - v_0^2}(x - \sigma t)\psi(x, t)dxdt + \langle\omega_2(t)\delta_s, \psi(\cdot, \cdot)\rangle, \tag{3.46}$$

in which

$$\omega_2(t) = \left(\sigma\left[\frac{\rho v}{c^2 - v^2}\right] - \left[\frac{\rho v^2}{c^2 - v^2}\right]\right)t. \tag{3.47}$$

Thus, the conclusion of Theorem 3.3 can be drawn. □

4 The formation of vacuum state as $A_1, A_2, A_3, B \to 0$ when $v_- < v_+$

In this section, let us consider the situation $-c < v_- < v_+ < c$ where the formation of vacuum state from the two-rarefaction Riemann solution to the isentropic relativistic Euler system (1.1) associated with the equation of state (1.2) as A_1, A_2, A_3 and B tend to zero when $-c < v_- < v_+ < c$ is satisfied in the Riemann initial data (1.3). It is easy to obtain from [22, 26] that the solution of the Riemann problem (1.1)-(1.3) includes two rarefaction waves for sufficiently small positive numbers A_1, A_2, A_3 and B when $-c < v_- < v_+ < c$. Let $(\rho_\star, v_\star)$ be the intermediate state between the two rarefaction waves, then we can get the 1-rarefaction wave R_1 connecting (ρ_-, v_-) and $(\rho_\star, v_\star)$ as well as the 2-rarefaction wave R_2 connecting $(\rho_\star, v_\star)$ and (ρ_+, v_+). To be more specific, it can be deduced from (2.15) and (2.16) that

$$R_1 : \begin{cases} \xi = \lambda_1(\rho, v) = \dfrac{c^2(v - \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 - v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}, \\[4mm] \ln\dfrac{c - v_\star}{c + v_\star} - \ln\dfrac{c - v_-}{c + v_-} = 2c\displaystyle\int_{\rho_-}^{\rho_\star} \dfrac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}ds, \\[4mm] v_\star > v_-, \qquad \rho_\star < \rho_-, \end{cases} \tag{4.1}$$

and

$$
R_2 : \begin{cases}
\xi = \lambda_2(\rho, v) = \dfrac{c^2(v + \sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}})}{c^2 + v\sqrt{A_1 + 2A_2\rho + 3A_3\rho^2 + \alpha B\rho^{-\alpha-1}}}, \\[4mm]
\ln \dfrac{c - v_+}{c + v_+} - \ln \dfrac{c - v_\star}{c + v_\star} = -2c \displaystyle\int_{\rho_\star}^{\rho_+} \dfrac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\, ds, \\[4mm]
v_+ > v_\star, \qquad \rho_+ > \rho_\star.
\end{cases}
\tag{4.2}
$$

Theorem 4.1. *When $-c < v_- < v_+ < c$, let us suppose that $(\rho(\xi), v(\xi))$ is a two-rarefaction-wave solution to the Riemann problem (1.1)-(1.3) for sufficiently small A_1, A_2, A_3, B, then the limit of solution to the Riemann problem (1.1)-(1.3) as $A_1, A_2, A_3, B \to 0$ is a two-contact-discontinuity solution with a vacuum state between them in the form (2.27), which is identical with that for the pressureless relativistic Euler system (1.5).*

Proof. Taking into account (4.1) and (4.2), we obtain

$$
\begin{cases}
\ln \dfrac{c - v_\star}{c + v_\star} - \ln \dfrac{c - v_-}{c + v_-} = 2c \displaystyle\int_{\rho_-}^{\rho_\star} \dfrac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\, ds, \\[4mm]
\ln \dfrac{c - v_+}{c + v_+} - \ln \dfrac{c - v_\star}{c + v_\star} = -2c \displaystyle\int_{\rho_\star}^{\rho_+} \dfrac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\, ds,
\end{cases}
\tag{4.3}
$$

where $\rho_\star \le \min(\rho_-, \rho_+)$. Thus, it is easy to get that

$$
\ln \frac{c - v_+}{c + v_+} - \ln \frac{c - v_-}{c + v_-} = 2c \int_{\rho_-}^{\rho_\star} \frac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\, ds + 2c \int_{\rho_+}^{\rho_\star} \frac{\sqrt{A_1 + 2A_2 s + 3A_3 s^2 + \alpha B s^{-\alpha-1}}}{A_1 s + A_2 s^2 + A_3 s^3 - B s^{-\alpha} + s c^2}\, ds
$$

$$
\le 2c \left(\int_{\rho_-}^{\rho_\star} \frac{\sqrt{A_1 + 2A_2 \rho_- + 3A_3 \rho_-^2 + \alpha B \rho_-^{-\alpha-1}}}{A_1 \rho_\star + A_2 \rho_\star^2 + A_3 \rho_\star^3 - B \rho_\star^{-\alpha} + \rho_\star c^2}\, ds + \int_{\rho_+}^{\rho_\star} \frac{\sqrt{A_1 + 2A_2 \rho_+ + 3A_3 \rho_+^2 + \alpha B \rho_+^{-\alpha-1}}}{A_1 \rho_\star + A_2 \rho_\star^2 + A_3 \rho_\star^3 - B \rho_\star^{-\alpha} + \rho_\star c^2}\, ds \right)
$$

$$
\le \frac{2c(\rho_- \sqrt{A_1 + 2A_2 \rho_- + 3A_3 \rho_-^2 + \alpha B \rho_-^{-\alpha-1}} + \rho_+ \sqrt{A_1 + 2A_2 \rho_+ + 3A_3 \rho_+^2 + \alpha B \rho_+^{-\alpha-1}})}{A_1 \rho_\star + A_2 \rho_\star^2 + A_3 \rho_\star^3 - B \rho_\star^{-\alpha} + \rho_\star c^2}.
\tag{4.4}
$$

If we suppose that $\lim\limits_{A_1,A_2,A_3,B \to 0} \rho_\star > 0$, then we can arrive at $\ln \frac{c - v_+}{c + v_+} - \ln \frac{c - v_-}{c + v_-} = 0$ (namely $v_- = v_+$), which contradicts the fact $-c < v_- < v_+ < c$. Hence, it can be derived that $\lim\limits_{A_1,A_2,A_3,B \to 0} \rho_\star = 0$. The fact implies that the intermediate state turns into the vacuum state as the limit $A_1, A_2, A_3, B \to 0$ is taken. With the formation of the vacuum state, as a matter of fact, the intermediate state should not be considered as a constant state once again.

Moreover, it is worthwhile to notice that the rarefaction curves R_1 and R_2 turn out to be the lines of contact discontinuities $J_1 : v = v_-$ and $J_2 : v = v_+$ in the half-upper (ρ, v) phase plane respectively. Therefore, we take a step further to get

$$
\lim_{A_1,A_2,A_3,B \to 0} \lambda_1(\rho_-, v_-) = \lim_{A_1,A_2,A_3,B \to 0} \lambda_1(\rho_\star, v_\star) = v_-,
\tag{4.5}
$$

$$
\lim_{A_1,A_2,A_3,B \to 0} \lambda_2(\rho_\star, v_\star) = \lim_{A_1,A_2,A_3,B \to 0} \lambda_2(\rho_+, v_+) = v_+.
\tag{4.6}
$$

It is obvious to see that the rarefaction curves R_1 and R_2 tend to the contact discontinuities J_1 and J_2 with the speeds of v_- and v_+ respectively. The proof is accomplished. $\qquad\square$

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