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Research Article

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Completeness theorem for probability models with finitely many valued measure

Abstract: The aim of the paper is to prove the completeness theorem for probability models with finitely many valued measure.

1 Introduction

Probability logics introduced by H. J. Keisler are logics appropriate for the study of structures of the form $(\mathfrak{A}, \mu)$ arising in Probability Theory, where $\mathfrak{A}$ is a first order structure and μ is a probability measure on A. The reader can find detailed presentations on the host of probability logics in [9] and the monograph [11]. The basic probability logic $L_{\mathcal{A}P}$ is similar to the infinitary logic $L_{\mathcal{A}}$[1] except that instead of the ordinary quantifiers $\forall x$ and $\exists x$, the logic $L_{\mathcal{A}P}$ possesses the probability quantifiers $(Px > r)$.

In this paper using the ideas from [2, 5, 10] we introduce logic $L_{\mathcal{A}P}^{\mathrm{fin}}$ which is complete for Σ_1 definable theories with respect to the class of probability models with finitely many valued measure. Let us note that our work could be seen as the first step towards the widening of application frame for probability logics since in applied mathematics one often deals with (very large but) finite phenomena.

2 $L_{\mathcal{A}P}^{\mathrm{fin}}$ logic

The main result which enable us to prove the corresponding Completeness Theorem is the following theorem (see [2]).

Theorem 2.1. *Let $\mathcal{F}$ be a field of subsets of a set Ω. Then μ is a finitely many valued probability measure on $\mathcal{F}$ if and only if there is a real number $c > 0$ such that $\mu(A) > c$ whenever $A \in \mathcal{F}$ and $\mu(A) > 0$.*

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[1] Here and subsequently, $\mathcal{A}$ is an admissible subset of $\mathbb{HC}$ containing ω. We refer the reader to [1, 7] for detailed treatments of admissible sets and the infinitary logic $L_{\mathcal{A}}$. Briefly, we note that the set of formulas of $L_{\mathcal{A}}$ is the set of all expressions in $\mathcal{A}$ that are built from atomic formulas using negation, finite or infinite conjunction, and the quantifier $\forall x$.

The logic $L_{\mathcal{A}P}^{\text{fin}}$ has all the axiom schemas and rules of inference of $L_{\mathcal{A}P}$ (listed in [9, 11]) as well as the following *axiom of finitely many valued measure*:

$$\bigvee_{c \in Q^+} \bigwedge_{\varphi \in \Phi_n} \left((P\vec{x} > 0)\varphi(\vec{x}) \rightarrow (P\vec{x} > c)\varphi(\vec{x}) \right),$$

where $\Phi_n \in \mathcal{A}$ and $\Phi_n = \{\varphi : \varphi$ has n free variables$\}$.

Completeness theorem will be proven by combining a consistency property argument, such as that of [9] or [6], and a weak-middle-strong construction, such as that of [10]. We need two sorts of auxiliary structures.

Definition 2.2. (i) A weak structure for $L_{\mathcal{A}P}^{\text{fin}}$ is a structure $(\mathfrak{A}, \mu_n)_{n \geq 1}$ such that each μ_n is a finitely additive probability measure on A^n with each singleton measurable and the set $\varphi_{\vec{a}} = \{\vec{b} : (\mathfrak{A}, \mu_n) \models \varphi[\vec{a}, \vec{b}]\}$ is μ_n-measurable for each $\varphi(\vec{x}, \vec{y}) \in L_{\mathcal{A}P}^{\text{fin}}$ and $\vec{a} \in A^m$.

(ii) A middle structure for $L_{\mathcal{A}P}^{\text{fin}}$ is a weak structure $(\mathfrak{A}, \mu_n)$ such that the following is true: there is a $c > 0$ such that for each formula $\varphi(\vec{x}, \vec{y}) \in L_{\mathcal{A}P}^{\text{fin}}$ and each $\vec{a} \in A^m$, if $\mu_n(\varphi_{\vec{a}}) > 0$, then $\mu_n(\varphi_{\vec{a}}) > c$.

Using a consistency property similarly as in [6] or [9] we prove that Σ_1 definable theory of $L_{\mathcal{A}P}^{\text{fin}}$ is consistent if and only if it has a weak model in which each theorem of $L_{\mathcal{A}P}^{\text{fin}}$ is true. Let $C \in \mathcal{A}$ be a set of new constant symbols introduced in this Henkin construction and let $K = L \cup C$.

Theorem 2.3. *(Middle Completeness Theorem) A Σ_1 definable theory T of $K_{\mathcal{A}P}^{\text{fin}}$ is consistent if and only if it has a middle model in which each theorem of $K_{\mathcal{A}P}^{\text{fin}}$ is true.*

Proof.

In order to prove that consistent Σ_1 definable theory T of $K_{\mathcal{A}P}^{\text{fin}}$ has a middle model, we introduce language M with three sorts of variables, such as that of [10]: $X, Y, Z, \ldots$ variable for sets, $x, y, z, \ldots$ variable for urelements and $r, s, t, \ldots$ variables for reals from $[0, 1]$. The predicates of M are $\leq$ for reals, $E_n(\vec{x}, X)$ for $n \geq 1$ and $\vec{x} = x_1, \ldots, x_n$ (with the canonical meaning $\vec{x} \in X$) and $\mu(X, r)$ (with the meaning $\mu(X) = r$). The constant symbols are set constant symbols X_φ, for each $\varphi \in K_{\mathcal{A}P}^{\text{fin}}$ and $\bar{r}$ for each $r \in [0, 1] \cap \mathcal{A}$. The functional symbols are $+$ and $\cdot$ for reals.

Let S be the first-order theory of $M_{\mathcal{A}}$ which has the following list of formulas:

1. *Axiom of well-definedness*
   $(\forall X) \bigwedge_{n < m} \neg(\exists \vec{x}, \vec{y})\left(E_m(\vec{x}, \vec{y}, X) \wedge E_n(\vec{x}, X)\right)$, where $\{\vec{x}\} \cap \{\vec{y}\} = \emptyset$;

2. *Axiom of extensionality*
   $(\forall \vec{x})\left(E_n(\vec{x}, X) \leftrightarrow E_n(\vec{x}, Y)\right) \leftrightarrow X = Y$;

3. *Axioms of satisfaction*
   (a) $(\forall \vec{x})\left(E_n(\vec{x}, X_{\wedge \Phi}) \leftrightarrow \bigwedge_{\varphi \in \Phi} E_n(\vec{x}, X_\varphi)\right)$
   (b) $(\forall \vec{x})\left(E_n(\vec{x}, X_{\neg\varphi}) \leftrightarrow \neg E_n(\vec{x}, X_\varphi)\right)$
   (c) $(\forall \vec{x})\left(E_n(\vec{x}, X_{(P\vec{x} \geq r)\varphi}) \leftrightarrow\right.$
       $\leftrightarrow \left((\exists_1 X)(\mu(X) \geq \bar{r} \wedge (\forall \vec{y})(E_{n+m}(\vec{x}, \vec{y}, X_\varphi) \leftrightarrow E_m(\vec{y}, X)))\right)\big)$,
       where $\mu(X) \geq \bar{r}$ is the formula $(\exists s)(s \geq \bar{r} \wedge \mu(X, s))$;
   (d) $(\forall \vec{x})\left((\forall \vec{y})E_{n+m}(\vec{y}, \vec{x}, X_R) \leftrightarrow E_{n+m}(\vec{c}, \vec{x}, X_R)\right)$,
       for each predicate $R(\vec{c}, \vec{x}) \in \mathcal{A}$;

4. *Axioms of measure*
   (a) $(\forall X)(\exists_1 r)\mu(X, r)$;
   (b) $(\forall X)(\forall Y)\left((\mu(X, r) \wedge \mu(Y, s) \wedge \neg \bigvee_{n \geq 1}(\exists \vec{x})(E_n(\vec{x}, X) \wedge E_n(\vec{x}, Y))) \rightarrow\right.$
       $\left.\rightarrow (\exists Z)\left(\bigwedge_{n \geq 1}(\forall \vec{x})(E_n(\vec{x}, Z) \leftrightarrow (E_n(\vec{x}, X) \vee E_n(\vec{x}, Y))) \wedge \mu(Z, r + s)\right)\right)$;

5. *Axiom of finitely many valued measure*

   $(\star)$ $(\exists c > 0)(\forall X)(\mu(X) > \bar{0} \rightarrow \mu(X) > c)$,

   where $\mu(X) > \bar{r}$ is the formula $(\exists s)(s > \bar{r} \wedge \mu(X, s))$;

6. *Axioms for an Archimedean field* (for real numbers);
7. *Axioms which are transformations of axioms of $K_{\mathcal{A}P}^{\text{fin}}$*
   $(\forall \vec{x}) E_n(\vec{x}, X_\varphi)$, where φ is an axiom of $K_{\mathcal{A}P}^{\text{fin}}$;
8. *Axiom of realizability of T*
   $(\forall x) E_1(x, X_\varphi)$, for each sentence φ in T.

A *standard structure* for $M_{\mathcal{A}}$ is the structure

$$\mathfrak{B} = \left(B, P, E_n^{\mathfrak{B}}, \mu^{\mathfrak{B}}, +, \cdot, \leq, X_\varphi^{\mathfrak{B}}, r \right)_{n \geq 1, \varphi \in K', r \in F},$$

where $P \subseteq \bigcup_{n \geq 1} \mathcal{P}(B)$, $E_n^{\mathfrak{B}} \subseteq B^n \times P$, $F = F' \cap [0, 1]$, $F' \subseteq \mathbb{R}$ is a field, $\mu^{\mathfrak{B}} : P \to F$, $+, \cdot : F^2 \to F$, $\leq \subseteq F^2$, $X_\varphi^{\mathfrak{B}} \in P$ and $K' \subseteq K_{\mathcal{A}P}^{\text{fin}}$.

The theory S is Σ_1 definable over $\mathcal{A}$. To prove that S is consistent it is enough, by Barwise Compactness Theorem (see [1]), to show that $S_0 \subseteq S$, $S_0 \in \mathcal{A}$ has a standard model. First, note that a weak structure $(\mathfrak{A}, \mu_n)$ for $K_{\mathcal{A}P}^{\text{fin}}$ can be transformed into a standard structure by taking: $X_\varphi^{\mathfrak{B}} = \{\vec{a} : (\mathfrak{A}, \mu_n) \models \varphi[\vec{a}]\}$ and $P = \{X_\varphi^{\mathfrak{B}} : \varphi \in K_{\mathcal{A}P}^{\text{fin}}\}$. Since the axiom

$$\bigvee_{c \in Q^+} \bigwedge_{\varphi \in (S_0')_n} \left((P\vec{x} > 0)\varphi(\vec{x}) \to (P\vec{x} > c)\varphi(\vec{x}) \right)$$

holds in the weak model $(\mathfrak{A}, \mu_n)$, where $S_0 \subseteq S_0'$, $S_0' \in \mathcal{A}$ is the closure for the substitution of constant symbols from C and disjunction and $(S_0')_n = \{\varphi \in S_0' : \varphi \text{ has } n \text{ free variables}\}$, it follows that

$$\left(A, P, E_n^{\mathfrak{A}}, \mu^{\mathfrak{A}}, +, \cdot, \leq, \{\vec{a} \in A^n ; (\mathfrak{A}, \mu_n) \models \varphi[\vec{a}, \vec{c}]\}, r \right)_{n \geq 1, \varphi \in S_0, r \in [0,1] \cap \mathcal{A}}$$

where $P = \{\{\vec{a} \in A^n ; (\mathfrak{A}, \mu_n) \models \varphi[\vec{a}, \vec{c}]\} : \varphi \in S_0\}$, is the standard model for S_0' and S_0, too.

Lastly, note that a standard model $\mathfrak{B}$ of S can be transformed into a middle model $\overline{\mathfrak{B}}$ of T by taking:

– $\vec{x} \in R^{\overline{\mathfrak{B}}}$ iff $E_n^{\mathfrak{B}}(\vec{x}, X_R)$ for an n-ary relational symbol $R \in L$,
– $\mu_n^{\overline{\mathfrak{B}}}(X) = r$ iff $\mu^{\mathfrak{B}}(X, r)$ for $X \in \mathcal{P}(B)$.

$\square$

It follows from the Loeb-Hoover-Keisler construction (see [6, 9, 11]) that the axiom of finitely many valued measures implies that (⋆) holds for all internal sets in the nonstandard superstructure. The property (⋆) also holds for all Loeb measurable sets because these can be approximated by internal ones. Thus, it follows from Theorem 2.1. that each middle model in which all theorems of $L_{\mathcal{A}P}^{\text{fin}}$ hold is elementary equivalent to a probability model for $L_{\mathcal{A}P}^{\text{fin}}$. As a consequence of the preceding we obtain the following theorem.

Theorem 2.4. *(Completeness Theorem for $L_{\mathcal{A}P}^{\text{fin}}$) A Σ_1 definable theory T of $L_{\mathcal{A}P}^{\text{fin}}$ is consistent if and only if T has a probability model with finitely many valued measure.*

Finally, let us note that structure $(\mathfrak{A}, \mu)$, where μ is a finitely many valued probability measure, cannot be axiomatized so that extended completeness theorem holds. The following example of a countable consistent theory T in $L_{\mathcal{A}P}^{\text{fin}}$ does not have a probability model with finitely many valued measure.

Example. Let $L = \{R_1(x), R_2(x), \ldots\}$ be a Δ_1 definable set which is not a subset of an element of $\mathcal{A}$, and let $\varphi_1, \varphi_2, \ldots$ be an enumeration of all formulas from $L_{\mathcal{A}P}^{\text{fin}}$. Then there exists the first predicate, denoted by $R_{\varphi_n}(x)$, not occurring in $\varphi_1, \ldots, \varphi_n$; otherwise $L \subseteq TC(\varphi_1) \cup \ldots \cup TC(\varphi_n) \in \mathcal{A}$, which would imply that $L \in \mathcal{A}$ as a Δ_1 definable set.

It is obvious that the countable theory

$$T = \{(Pxy \geq 1)x \neq y\} \cup \{(Px > 0)(R_{\varphi_1}(x) \wedge \ldots \wedge R_{\varphi_n}(x)) : n \in \omega\}$$
$$\cup \{(Px < 1/2^n)R_{\varphi_n}(x) : n \in \omega\}$$

does not have any probability model with finitely many valued measure. We prove that T is consistent in $L_{\mathcal{A}P}^{\text{fin}}$.

Let I be a unit interval $[0, 1]$ and let μ be a Lebegue measure on I. For $B_n = [0, 1/2^{n+1})$ we have $0 < \mu(B_n) < 1/2^n$. Let $A_{n_1 \ldots n_k}^{i_1 \ldots i_k} = B_{n_1}^{i_1} \cap \ldots \cap B_{n_k}^{i_k}$ be a Boolean atom, where $B_n^i = B_n$ for $i = 1$ and $B_n^i = I \setminus B_n$ for $i = -1$.

We interpret the predicates by taking $R_n^{(\mathfrak{A},\mu)} = \begin{cases} B_n, & \text{if } R_n = R_{\varphi_m} \text{ for some } m \\ B_1, & \text{otherwise.} \end{cases}$

Since only finitely many predicates R_{φ_n} can occur in an element of $\mathcal{A}$, it follows that the set

$$\left\{ A_{n_1\ldots n_k}^{i_1\ldots i_k} \; : \; \mu\left(A_{n_1\ldots n_k}^{i_1\ldots i_k}\right) > 0 \right\}$$

is finite. The theory T and all axioms of $L_{\mathcal{AP}}^{\text{fin}}$ are satisfied except perhaps the axiom of finitely many valued measure. But, for

$$c = \min\left\{ \mu\left(A_{n_1\ldots n_k}^{i_1\ldots i_k}\right) \; : \; \mu\left(A_{n_1\ldots n_k}^{i_1\ldots i_k}\right) > 0 \right\},$$

T is consistent in $L_{\mathcal{AP}}^{\text{fin}}$.

3 Conclusion

In this paper we have used very fruitful technique introduced by Raskovic in [10]. The technique is developed for solving Keisler's problem about probalilistic logics with two measures. More precisely, Keisler proposed research problem of developing a model theory and particulary proving the completeness theorem for the class of biprobability models, and specially for structures with two measures μ_1, μ_2 such that μ_1 is absolutely continuous with respect to μ_2. After paper [10], the method which was used in the proof becomes a very powerful technique for producing new results. In the first instance, the method has given completeness for many logics that appear in Probability Theory. As an example (for more see [11]) we point out a corresponding result for the extension $L(\int_1, \int_2)_{\mathcal{A}}$ of logic with integrals related to analytic models, [3]. Furthermore, using the same ideas, the completeness theorem is proven for the logic appropriate for the study of topologies on proper classes [4]. The idea for future research is to systematize many applications of this method. Future research should be dedicated to finding appropriate logics for classes of structures equipped with two monotone collections connected in the different ways.

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