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A note on the formulas for the Drazin inverse of the sum of two matrices

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Abstract: In this paper we derive the formula of $(P+Q)^D$ under the conditions $Q(P+Q)P(P+Q)=0$, $P(P+Q)P(P+Q)=0$ and $QPQ^2=0$. Then, a corollary is given which satisfies the conditions $(P+Q)P(P+Q)=0$ and $QPQ^2=0$. Meanwhile, we show that the additive formula provided by Bu et al. (J. Appl. Math. Comput. 38 (2012) 631-640) is not valid for all matrices which satisfies the conditions $(P+Q)P(P+Q)=0$ and $QPQ^2=0$. Also, the representation can be simplified from Višnjić (Filomat 30 (2016) 125-130) which satisfies given conditions. Furthermore, we apply our result to establish a new representation for the Drazin inverse of a complex block matrix having generalized Schur complement equal to zero under some conditions. Finally, a numerical example is given to illustrate our result.

Keywords: Drazin inverse, Block matrix, Matrix index, Generalized Schur complement

MSC: 15A09

1 Introduction

Let $C^{n\times n}$ denote the set of all $n\times n$ complex matrices. For $A\in C^{n\times n}$, we call the smallest nonnegative integer k which satisfies $rank(A^{k+1})=rank(A^k)$ the index of A, and denote k by $ind(A)$. Let $A\in C^{n\times n}$ with $ind(A)=k$, we call the matrix $X\in C^{n\times n}$ which satisfies

$$A^{k+1}X=A^k, XAX=X, XA=AX$$

the Drazin inverse of A and denote X by A^D (see [1]). The Drazin inverse of a square complex matrix always exists and is unique (see [1]). In this paper, we denote $A^\pi=I-AA^D$. A matrix $A\in C^{n\times n}$ is nilpotent if $A^k=0$ for some integer $k\geq 0$. The smallest such k is called the index of nilpotency of A.

The Drazin inverse of square complex matrices has applications in several areas, such as singular differential or difference equations, Markov chains and iterative method and so on (see [1, 2]). For applications of the Drazin inverse of a 2×2 block matrix, we refer the readers to [2–4].

Suppose $P,Q\in C^{n\times n}$. In 1958, Drazin offered the formula $(P+Q)^D=P^D+Q^D$, when $PQ=QP=0$. In the recent years many authors have considered this problem and provided the representations of $(P+Q)^D$ with some specific conditions. In [5], the authors gave the representation of $(P+Q)^D$ when $PQ=0$. In [6], the authors derived the formula for $(P+Q)^D$ under the conditions $P^2Q=0$, $Q^2=0$. The case when $P^2Q=0$, $Q^2P=0$ was studied in [7]. In [8], the authors gave the formula for $(P+Q)^D$ when $P^2QP=0$, $PQ^2P=$

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$0, P^2Q^2 = 0, PQ^3 = 0$. In [9], the formula of $(P+Q)^D$ under the conditions $(P+Q)P(P+Q) = 0$, $QPQ^2 = 0$ was given. In [10], the authors derived a result under the conditions $P(P+Q)Q = 0$.

In this short paper, we derive the formula of $(P+Q)^D$ under the conditions $Q(P+Q)P(P+Q) = 0$, $P(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$ in Section 3. We also get that a formula for $(P+Q)^D$ from [9] is not valid for all matrices which satisfy conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$. Meanwhile, a corollary is given which is valid for all matrices under the mentioned conditions. Furthermore, we offer an example which shows that the formula from [9] is not valid for all matrices which satisfy conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$.

Another aim of this paper is to derive a representation of the Drazin inverse of 2×2 complex block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

where A and D are square matrices. This problem was firstly posed in 1979 by Compbell and Meyer [11]. No formula for M^D has yet been offered without any restrictions upon the blocks. Special cases of this problem have been studied. In some papers the expression of M^D is given under conditions which concern the generalized Schur complement of matrix M defined by $S = D - CA^DB$. Here we list some of them:

(i) $CA^\pi = 0$, $A^\pi B = 0$, and $S = 0$ [12],
(ii) $CA^\pi B = 0$, $AA^\pi B = 0$, and $S = 0$ [3],
(iii) $CA^\pi B = 0$, $CAA^\pi = 0$, and $S = 0$ [3],
(iv) $ABCA^\pi = 0$, BCA^π is nilpotent, and $S = 0$ [6],
(v) $A^\pi BCA = 0$, $A^\pi BC$ is nilpotent, and $S = 0$ [6],
(vi) $ABCA^\pi = 0$, $A^\pi ABC = 0$, and $S = 0$ [7],
(vii) $ABCA^\pi = 0$, $CBCA^\pi = 0$, and $S = 0$ [7],
(viii) $ABCA^\pi A = 0$, $ABCA^\pi B = 0$, and $S = 0$ [8],
(ix) $AA^\pi BCA = 0$, $CA^\pi BCA = 0$, and $S = 0$ [8].

In Section 4, we derive a new representation for M^D under the conditions $ABCA^\pi A = 0$, $A^\pi BCA = 0$, $A^\pi BCB = 0$ and $S = 0$.

2 Some lemmas

In order to give the main results, we first give some lemmas as follows.

Lemma 2.1. [11] *Let $A \in C^{m\times n}$, $B \in C^{n\times m}$, then*

$$(AB)^D = A\left((BA)^D\right)^2 B.$$

Lemma 2.2. [4] *Let $M = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}$, where A and B are square matrices with $ind(A) = r$ and $ind(B) = s$. Then $max\{r,s\} \le ind(M) \le r+s$, and*

$$M^D = \begin{pmatrix} A^D & 0 \\ X & B^D \end{pmatrix},$$

where $X = \sum_{k=0}^{s-1} B^\pi B^k C(A^D)^{k+2} + \sum_{k=0}^{r-1}(B^D)^{k+2}CA^kA^\pi - B^DCA^D$.

Lemma 2.3. [5] *Let $P, Q \in C^{n\times n}$ be such that $ind(P) = r$ and $ind(Q) = s$. If $PQ = 0$, then*

$$(P+Q)^D = \sum_{i=0}^{s-1} Q^\pi Q^i (P^D)^{i+1} + \sum_{i=0}^{r-1}(Q^D)^{i+1}P^iP^\pi.$$

Lemma 2.4. [13] *Let $P, Q \in C^{n\times n}$, such that $ind(P) = r$, $ind(Q) = s$. If $PQP = 0$ and $PQ^2 = 0$, then*

$$(P+Q)^D = Y_1 + Y_2 + (Y_1(P^D)^2 + (Q^D)^2Y_2 - Q^D(P^D)^2 - (Q^D)^2P^D)PQ,$$

where $Y_1 = \sum_{i=0}^{s-1} Q^\pi Q^i (P^D)^{i+1}$, $Y_2 = \sum_{i=0}^{r-1} (Q^D)^{i+1} P^i P^\pi$.

Lemma 2.5. [6] *Let* $P, Q \in C^{n\times n}$.

1. *If* $PQ = QP = 0$, *then* $(P+Q)^D = P^D + Q^D$.
2. *If* $PQ = 0$ *and* P *is* $r-$*nilpotent, then* $((P+Q)^D)^j = \sum_{i=0}^{r-1} (Q^D)^{i+j} P^i$, $\forall j \geq 1$.
3. *If* $PQ = 0$ *and* Q *is* $s-$*nilpotent, then* $((P+Q)^D)^j = \sum_{i=0}^{s-1} Q^i (P^D)^{i+j}$, $\forall j \geq 1$.

Lemma 2.6. [12] *Let* M *be a matrix of the form* (1.1), *such that* $S = 0$. *If* $A^\pi B = 0$, $CA^\pi = 0$, *then*

$$M^D = \begin{bmatrix} I \\ CA^D \end{bmatrix} ((AW)^D)^2 A \begin{bmatrix} I & A^D B \end{bmatrix},$$

where $W = AA^D + A^D BCA^D$.

3 Additive results

In [9], Bu et al. gave the representation of $(P+Q)^D$ when $(P+Q)P(P+Q) = 0$, $QPQ^2 = 0$ which is not correct for all matrices. In this paper we give the representation of $(P+Q)^D$ when $Q(P+Q)P(P+Q) = 0$, $P(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$ are satisfied, from which we will get the correct formula under the conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$.

Theorem 3.1. *Let* $P, Q \in C^{n\times n}$, *if* $Q(P+Q)P(P+Q) = 0$, $P(P+Q)P(P+Q) = 0$ *and* $QPQ^2 = 0$, *then*

$$(P+Q)^D = (P+Q)^3((Q^2+QP)^D)^2 = (P+Q)^2((Q^2+QP)^D)^2(P+Q),$$

where

$$((Q^2+QP)^D)^2 = \sum_{i=0}^{t-1} Q^\pi Q^{2i}((QP)^D)^{i+2} + \sum_{i=0}^{t-1} (Q^D)^{2(i+2)}(QP)^i(QP)^\pi - (Q^D)^2(QP)^D,$$

for $t = max\{ind(QP), ind(Q^2)\}$.

Proof. Using definition of the Drazin inverse, we have that

$$(P+Q)^D = (P+Q)((P+Q)^2)^D = (P+Q)\left(\begin{pmatrix} I & PQ+P^2 \end{pmatrix}\begin{pmatrix} Q^2+QP \\ I \end{pmatrix}\right)^D. \tag{3.1}$$

Denote by $M = \begin{pmatrix} I & PQ+P^2 \end{pmatrix}\begin{pmatrix} Q^2+QP \\ I \end{pmatrix}$.

By Lemma 2.1 and $Q(P+Q)P(P+Q) = 0$, we have

$$M^D = \begin{pmatrix} I & PQ+P^2 \end{pmatrix}\left(\begin{pmatrix} Q^2+QP & 0 \\ I & PQ+P^2 \end{pmatrix}^D\right)^2 \cdot \begin{pmatrix} Q^2+QP \\ I \end{pmatrix}. \tag{3.2}$$

Denote by

$$N = \begin{pmatrix} Q^2+QP & 0 \\ I & PQ+P^2 \end{pmatrix}.$$

From $P(P+Q)P(P+Q) = 0$, we get $(PQ+P^2)^D$=0. After applying Lemma 2.2 we get

$$N^D = \begin{pmatrix} (Q^2+QP)^D & 0 \\ X & 0 \end{pmatrix}, \tag{3.3}$$

where $X = \left((Q^2+QP)^D\right)^2 + (PQ+P^2)\left((Q^2+QP)^D\right)^3$.
Since $QPQ^2 = 0$ and Lemma 2.3, we obtain

$$(Q^2+QP)^D = \sum_{i=0}^{t-1} Q^\pi Q^{2i}((QP)^D)^{i+1} + \sum_{i=0}^{t-1}(Q^D)^{2(i+1)}(QP)^i(QP)^\pi,$$

where $t = max\{ind(QP), ind(Q^2)\}$.

Substituting (3.3) into (3.2), we get

$$M^D = (Q^2+QP)^D + (PQ+P^2)(Q^2+QP)^D(Q^2+QP).$$

By $P(P+Q)P(P+Q) = 0$, we have

$$M^D = (Q^2+QP)^D + (PQ+P^2)((Q^2+QP)^D)^2. \tag{3.4}$$

Substituting (3.4) into (3.1), we get

$$\begin{aligned}(P+Q)^D &= (P+Q)(Q^2+QP)^D + (P+Q)(PQ+P^2)((Q^2+QP)^D)^2\\ &= (P+Q)(Q^2+QP)((Q^2+QP)^D)^2 + (P+Q)(PQ+P^2)((Q^2+QP)^D)^2\\ &= (P+Q)(P+Q)^2((Q^2+QP)^D)^2\\ &= (P+Q)^3((Q^2+QP)^D)^2.\end{aligned}$$

Similarly, using definition of Drazin inverse, $(P+Q)^D = ((P+Q)^2)^D(P+Q)$.
So,

$$(P+Q)^D = (P+Q)^2((Q^2+QP)^D)^2(P+Q).$$

The proof is completed. □

The next theorem is a symmetrical formulation of Theorem 3.1.

Theorem 3.2. *Let $P, Q \in C^{n\times n}$. If $(P+Q)Q(P+Q)P = 0$, $(P+Q)P(P+Q)P = 0$, $Q^2PQ = 0$, then*

$$(P+Q)^D = (P+Q)^3((Q^2+PQ)^D)^2 = (P+Q)^2((Q^2+PQ)^D)^2(P+Q),$$

where

$$((Q^2+PQ)^D)^2 = \sum_{i=0}^{t-1}(PQ)^\pi(PQ)^i((Q)^D)^{2(i+2)} + \sum_{i=0}^{t-1}((PQ)^D)^{i+2}Q^{2i}Q^\pi - (PQ)^D(Q^D)^2,$$

for $t = max\{ind(PQ), ind(Q^2)\}$.

Notice that one special case of Theorem 3.1 is when matrices P, Q satisfy the conditions $(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$, we get following additive formula.

Corollary 3.3. *Let $P, Q \in C^{n\times n}$, such that $max\{ind(QP), ind(Q^2)\} = t$, if $(P+Q)P(P+Q) = 0$, $QPQ^2 = 0$, then*

$$(P+Q)^D = (P+Q)^2((Q^2+QP)^D)^2(P+Q) = (P+Q)(Q^2+QP)^D,$$

where

$$(Q^2+QP)^D = \sum_{i=0}^{t-1} Q^\pi Q^{2i}((QP)^D)^{i+1} + \sum_{i=0}^{t-1}(Q^D)^{2(i+1)}(QP)^i(QP)^\pi,$$
$$((Q^2+QP)^D)^2 = \sum_{i=0}^{t-1} Q^\pi Q^{2i}((QP)^D)^{i+2} + \sum_{i=0}^{t-1}(Q^D)^{2(i+2)}(QP)^i(QP)^\pi - (Q^D)^2(QP)^D.$$

Remark 3.4. *In [9, Theorem 3.1], the authors studied conditions from Corollary 3.3 and obtained the formula*

$$(P+Q)^D = (P+Q)^2(\sum_{i=0}^{t-1}(Q^D)^{2i+4}(QP)^i(QP)^\pi - (Q^D)^2(QP)^D)(P+Q). \tag{3.5}$$

Notice that in Corollary 3.3 we have additional element

$$(P+Q)^2\Big(\sum_{i=0}^{t-1} Q^{\pi} Q^{2i}((QP)^D)^{i+2}\Big)(P+Q).$$

In fact, this element doesn't have to be equal to zero. So, the formula from [9] is not correct for all matrices which satisfy conditions $(P+Q)P(P+Q)=0$ *and* $QPQ^2=0$.

Remark 3.5. *In [10, Corollary 3.2], the author obtained the formula satisfy the same conditions from Corollary 3.3*

$$(P+Q)^D = (P+Q)^2((Q^2+QP)^D)^2(P+Q). \tag{3.6}$$

Since $(P+Q)P(P+Q)=0$, *by Theorem* 3.1, (3.6) *can be checked easily that*

$$(P+Q)^D = (P+Q)^3((Q^2+QP)^D)^2 = (P+Q)Q(P+Q)((Q^2+QP)^D)^2 + (P+Q)P(P+Q)((Q^2+QP)^D)^2$$
$$= (P+Q)(Q^2+QP)^D.$$

Next we consider matrices which satisfy conditions $(P+Q)P(P+Q)=0$ and $QPQ^2=0$ and for which is $(P+Q)^2(\sum_{i=0}^{t-1} Q^{\pi} Q^{2i}((QP)^D)^{i+2})(P+Q) \neq 0$. Furthermore, we demonstrate the application of Corollary 3.3.

Example 3.6. *Let P and Q be matrices*: $P = \begin{pmatrix} 0 & -2 \\ 8 & 4 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

It can be checked easily that $(P+Q)P(P+Q)=0$, $QPQ^2=0$. *Hence the conditions of Corollary 3.3 are satisfied. Also, we have that* $(P+Q)^2(\sum_{i=0}^{t-1} Q^{\pi} Q^{2i}((QP)^D)^{i+2})(P+Q) = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \neq 0$. *Moreover, by the formula from Corollary* 3.3, $(P+Q)^D = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}$. *But, if we apply the formula* (3.5) *from* [9], *we get that* $(P+Q)^D = 0$.

Example 3.7. [9] *Let P and Q be matrices*: $P = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$, $Q = \begin{pmatrix} 0&0&0&0 \\ 1&1&-1&0 \\ 0&1&0&1 \\ 1&1&-1&0 \end{pmatrix}$.

It is obvious that $(P+Q)P(P+Q)=0$, $QPQ^2=0$. *Hence the conditions of Corollary* 3.3 *are satisfied. After computation, we get*

$$(QP)^D = 0,\ Q^D = \begin{pmatrix} 0&0&0&0 \\ -\frac{1}{2}&-\frac{1}{4}&\frac{1}{2}&\frac{1}{4} \\ -\frac{1}{2}&-\frac{3}{4}&\frac{1}{2}&-\frac{1}{4} \\ -\frac{1}{2}&-\frac{1}{4}&\frac{1}{2}&\frac{1}{4} \end{pmatrix},\ (Q^2+QP)^D = \begin{pmatrix} 0&0&0&0 \\ -\frac{1}{4}&-\frac{1}{4}&\frac{1}{4}&-\frac{1}{4} \\ \frac{1}{4}&\frac{1}{4}&-\frac{1}{4}&-\frac{3}{4} \\ -\frac{1}{4}&-\frac{1}{4}&\frac{1}{4}&-\frac{1}{4} \end{pmatrix}.$$

Moreover, by the formula from Corollary 3.3,

$$(P+Q)^D = \begin{pmatrix} -\frac{1}{4}&-\frac{1}{4}&\frac{1}{4}&-\frac{1}{4} \\ -\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2} \\ -\frac{3}{4}&-\frac{3}{4}&\frac{3}{4}&-\frac{3}{4} \\ -\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2} \end{pmatrix}.$$

But, in [9], *the authors got that*

$$(P+Q)^D = \begin{pmatrix} -\frac{1}{4}&0&\frac{1}{4}&0 \\ -\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2} \\ -\frac{3}{4}&0&\frac{3}{4}&0 \\ -\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2} \end{pmatrix}.$$

This is illustrated easily by the definition of Drazin inverse, the result of $(P+Q)^D$ *is not correct from* [9].

4 A formula of the Drazin inverse for block matrix

In this section, we use the additive formula in section 3 to give representation for the Drazin inverse of a complex block matrix.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square matrices with generalized Schur complement $S = D - CA^D B$ of matrix M equal to zero. In [6], Martinez-Serrano and Castro-Gonzalez gave the representation for the Drazin inverse of M under the condition $ABC = 0$. In [7], Bu et al. gave the formula for M^D under conditions $ABCA^\pi = 0$ and $A^\pi ABC = 0$. In the following Theorem 4.1, we give the representation for the Drazin inverse of M under the conditions $ABCA^\pi A = 0$, $A^\pi BCA = 0$, $A^\pi BCB = 0$ and $S = 0$.

Theorem 4.1. *Let M be a matrix of the form* (1.1) *such that $S = 0$. If $ABCA^\pi A = 0$, $A^\pi BCA = 0$ and $A^\pi BCB = 0$, then*

$$M^D = L^D + (L^D)^2 \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix} + (L^D)^3 \begin{pmatrix} A^\pi BC & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$L^D = \begin{pmatrix} A & AA^D B \\ C & CA^D B \end{pmatrix} (Q^D)^2 + \begin{pmatrix} A & AA^D B \\ C & CA^D B \end{pmatrix} (Q^D)^4 \begin{pmatrix} AA^D BCA^\pi & 0 \\ CA^D BCA^\pi & 0 \end{pmatrix},$$

$$(Q^D)^n = \begin{pmatrix} I \\ CA^D \end{pmatrix} ((AW)^D)^{n+1} A \begin{pmatrix} I & A^D B \end{pmatrix} + \begin{pmatrix} ((AA^\pi)^D)^n & 0 \\ 0 & 0 \end{pmatrix},$$

$W = AA^D + A^D BCA^D$, *for $n \geq 1$.*

Proof. Consider the splitting of matrix M,

$$M = \begin{pmatrix} A & B \\ C & CA^D B \end{pmatrix} = \begin{pmatrix} A & AA^D B \\ C & CA^D B \end{pmatrix} + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}.$$

If we denote by $L = \begin{pmatrix} A & AA^D B \\ C & CA^D B \end{pmatrix}$ and $H = \begin{pmatrix} 0 & A^\pi B \\ 0 & 0 \end{pmatrix}$, we get $H^2 = 0$, $H^D = 0$. From $A^\pi BCA = 0$, $A^\pi BCB = 0$, it is obvious that $HLH = 0$, $HL^2 = 0$. Hence, the conditions of Lemma 2.4 are satisfied, and

$$M^D = (H + L)^D = L^D + (L^D)^2 H + (L^D)^3 HL. \tag{4.1}$$

Let

$$Q = \begin{pmatrix} A & AA^D B \\ CAA^D & CA^D B \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 \\ CA^\pi & 0 \end{pmatrix},$$

then $L = P + Q$, $P^2 = 0$. From $ABCA^\pi A = 0$, we have $Q(P+Q)P(P+Q) = 0$, $P(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$. Hence, the conditions of Theorem 3.1 are satisfied and

$$L^D = (P + Q)^D = (P + Q) \left((Q^2 + QP)^D + (PQ + P^2)((Q^2 + QP)^D)^2 \right), \tag{4.2}$$

where $(Q^2 + QP)^D$ is defined as in Theorem 3.1.

Notice $(QP)^2 = 0$, so $(QP)^D = 0$. Thus $((Q^2 + QP)^D)^n = (Q^D)^{2n} + (Q^D)^{2n+2} QP$, $n \geq 1$.

If we split matrix Q as

$$Q = \begin{pmatrix} A^2 A^D & AA^D B \\ CAA^D & CA^D B \end{pmatrix} + \begin{pmatrix} AA^\pi & 0 \\ 0 & 0 \end{pmatrix},$$

and denote by

$$Q_1 = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & CA^DB \end{pmatrix}, Q_2 = \begin{pmatrix} AA^\pi & 0 \\ 0 & 0 \end{pmatrix},$$

we get $Q_2Q_1 = Q_1Q_2 = 0$. After applying Lemma 2.5,

$$Q^D = (Q_1)^D + (Q_2)^D.$$

Let S_1 be the generalized Schur complement of Q_1, then we have

$$S_1 = CA^DB - CAA^D(A^2A^D)^D AA^DB = 0,$$

and

$$(A^2A^D)^\pi AA^DB = 0,\ CAA^D(A^2A^D)^\pi = 0,$$

so from Lemma 2.6, we get

$$((Q_1)^D)^n = \begin{pmatrix} I \\ CA^D \end{pmatrix} ((AW)^D)^{n+1} A \begin{pmatrix} I & A^DB \end{pmatrix},$$

where $W = AA^D + A^DBCA^D$, *for* $n \geq 1$.

After computation we get

$$(Q^D)^n = \begin{pmatrix} I \\ CA^D \end{pmatrix} ((AW)^D)^{n+1} A \begin{pmatrix} I & A^DB \end{pmatrix} + \begin{pmatrix} ((AA^\pi)^D)^n & 0 \\ 0 & 0 \end{pmatrix}.$$

After substituting this expressions and (4.2) into (4.1), we complete the proof. □

5 Numerical example

The following numerical example illustrates the application of Theorem 4.1.

Example 5.1. *Consider the matrix* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where*

$$A = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix}, B = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0&1&1&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \end{pmatrix}, D = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}.$$

By computing we know the generalized Schur complement $S = D - CA^DB$ is zero and M satisfies the condition $ABCA^\pi A = 0$, $A^\pi BCA = 0$, $A^\pi BCB = 0$, in Theorem 4.1 in the paper. We have $ind(A) = 2$, and

$$A^D = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, A^\pi = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},$$

then according to the formula in Theorem 4.1, we have

$$M^D = \begin{pmatrix} 1&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}.$$

6 Conclusions

In this paper we derive the formula of $(P+Q)^D$ under the conditions $Q(P+Q)P(P+Q) = 0$, $P(P+Q)P(P+Q) = 0$ and $QPQ^2 = 0$. Then, a corollary is given which satisfies the conditions $(P + Q)P(P + Q) = 0$ and $QPQ^2 = 0$. Furthermore, we apply the new result of $(P + Q)^D$ to establish a new representation for the Drazin inverse of complex block matrix having generalized Schur complement equal to zero under the conditions $ABCA^{\pi}A = 0$, $A^{\pi}BCA = 0$ and $A^{\pi}BCB = 0$.

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