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Hopf bifurcation and stability in a Beddington-DeAngelis predator-prey model with stage structure for predator and time delay incorporating prey refuge

Abstract: In this paper, we consider a Beddington-DeAngelis predator-prey system with stage structure for predator and time delay incorporating prey refuge. By analyzing the characteristic equations, we study the local stability of the equilibrium of the system. Using the delay as a bifurcation parameter, the model undergoes a Hopf bifurcation at the coexistence equilibrium when the delay crosses some critical values. After that, by constructing a suitable Lyapunov functional, sufficient conditions are derived for the global stability of the system. Finally, the influence of prey refuge on densities of prey species and predator species is discussed.

1 Introduction

Predator-prey model is one of the basic models between different species in real world. As we all know, there are always many species going through two stages, immature and mature, which reflect the different characteristics of species at each stage. Therefore, to exhibit the real world phenomenon, stage structure population models are more reasonable than other models. In recent years, numerous papers have considered the predator-prey system with stage structure (see [1-5]).

On the other hand, in general, the consumption of prey by predator throughout its past history governs the present birth rate of the predator, in other words, time delay due to gestation is a common example. Obviously, delay differential equations exhibit much more complicated dynamics than ordinary differential equations (see [6-14]). For example, Wang and Chen [15] considered the following predator prey system with

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stage structure for the predator population:

$$\begin{aligned}
\dot{x}(t) &= x(t)\big(r - ax(t - \tau_1) - \mu y_2(t)\big), \\
\dot{y}_1(t) &= \eta\mu x(t - \tau_2)y_2(t - \tau_2) - \beta y_1(t) - dy_1(t), \\
\dot{y}_2(t) &= \beta y_1(t) - ey_2(t).
\end{aligned} \tag{1.1}$$

The authors studied the asymptotic behavior of system (1.1). When a time delay due to gestation of the predators and a time delay from a crowding effect of the prey are incorporated, we establish conditions for the permanence of the populations and sufficient conditions for the existence of globally stable positive equilibrium of system (1.1).

Beddington [16] and DeAngelis et al. [17] established a famous B-D functional response that is a predator dependent functional response. In many cases, predators need to search for food and share or compete for food. Therefore, the stage-structured predator-prey models incorporating Beddington-DeAngelis functional response better reflect the ecology. Chen et al. [18] discussed the stability of the boundary solution of a nonautonomous predator-prey model with the Beddington-DeAngelis functional response, which reflects the dynamics of interacting predators and prey in a fluctuating environment. Xia et al. [19] considered stability and traveling waves in a Beddington-DeAngelis type stage-structured predator-prey reaction-diffusion systems with nonlocal delays and harvesting. Chen et al. [20] discussed the extinction of a two species nonautonomous competitive system with Beddington-DeAngelis functional response and the effect of toxic substances. Khajanchi [21] investigated the dynamic behavior of a Beddington-DeAngelis type stage structured predator-prey model:

$$\begin{aligned}
\dot{x}(t) &= x(t)\big(r - \frac{r}{k}x(t)\big) - \frac{\mu x(t)y_2(t)}{1 + bx(t) + cy_2(t)}, \\
\dot{y}_1(t) &= \frac{\eta\mu x(t)y_2(t)}{1 + bx(t) + cy_2(t)} - \beta y_1(t) - dy_1(t), \\
\dot{y}_2(t) &= \beta y_1(t) - ey_2(t).
\end{aligned} \tag{1.2}$$

By analyzing the above system, conditions for positivity, boundedness, uniform persistence, existence of positive equilibria with their local stability have been established. Also, the author showed the existence of Hopf bifurcation when the conversion parameter k_1 passes the critical value. Finally, the conditions for the occurrence of global stability for the unique interior equilibrium point were derived.

In the real world, refuge is a strategy to reduce the risk of predation. It is clear that the existence of refuge can have a significant impact on the coexistence of predator species and prey species. In recent years, many papers [22-27] have proposed and analyzed predator-prey models incorporating prey refuges. Recently, Wei and Fu [28] discussed the Hopf bifurcation and stability for predator-prey systems with Beddington-DeAngelis type functional response and stage structure for prey incorporating refuge

$$\begin{aligned}
\dot{x}_1(t) &= ax_2(t) - rx_1(t) - bx_1(t), \\
\dot{x}_2(t) &= bx_1(t) - cx_2(t) - \alpha x_2^2(t) - \frac{\beta(1 - m)x_2(t)y(t)}{a_1 + b_1(1 - m)x_2(t) + c_1y(t)}, \\
\dot{y}(t) &= \frac{d\beta(1 - m)x_2(t - \tau)y(t - \tau)}{a_1 + b_1(1 - m)x_2(t - \tau) + c_1y(t - \tau)} - \gamma y(t).
\end{aligned} \tag{1.3}$$

By using the characteristic equations, the local stability of each feasible equilibrium of model (1.3) was discussed, and the existence of a Hopf bifurcation at the coexistence equilibrium was established.

Motivated by the works [21] and [28], a Beddington-DeAngelis predator-prey model with stage structure for predator and time delay incorporating prey refuge is investigated in this paper. The proposed model is as follows:

$$\begin{aligned}
\dot{x}(t) &= x(t)(r - ax(t)) - \frac{\mu(1 - m)x(t)y_2(t)}{1 + b(1 - m)x(t) + cy_2(t)}, \\
\dot{y}_1(t) &= \frac{\eta\mu(1 - m)x(t - \tau)y_2(t - \tau)}{1 + b(1 - m)x(t - \tau) + cy_2(t - \tau)} - \beta y_1(t) - dy_1(t), \\
\dot{y}_2(t) &= \beta y_1(t) - ey_2(t),
\end{aligned} \tag{1.4}$$

where $x(t)$, $y_1(t)$ and $y_2(t)$ denote the densities of prey species, immature predator species and mature predator species at time t, respectively; r is the intrinsic growth rate of prey species; a is the intraspecific

competition rate of prey species; d and e are the death rates of the immature and mature predator species respectively; $\mu(1 - m)$ is the capturing rate of the mature predator; η is the conversion rate of nutrients into the production of predator species; τ denotes the time delay due to the gestation of the mature predator; $m \in [0, 1)$ is refuge rate to prey; the predator species consumes the prey species with Beddington-DeAngelis functional response incorporating prey refuge $\frac{\mu(1-m)x(t)y_2(t)}{1+b(1-m)x(t)+cy_2(t)}$, and $\frac{\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1+b(1-m)x(t-\tau)+cy_2(t-\tau)}$ denotes the growth rate of predator which are pregnant at time $t - \tau$.

The initial conditions for system (1.4) take the form

$$
\begin{aligned}
&x(\theta) = \phi(\theta), \quad y_1(\theta) = \psi_1(\theta), \quad y_2(\theta) = \psi_2(\theta), \\
&\phi(\theta) \geq 0, \qquad \psi_1(\theta) \geq 0, \qquad \psi_2(\theta) \geq 0, \qquad \theta \in [-\tau, 0), \\
&\phi(0) > 0, \qquad \psi_1(0) > 0, \qquad \psi_2(0) > 0,
\end{aligned}
\tag{1.5}
$$

where $(\phi(\theta), \psi_1(\theta), \psi_2(\theta)) \in C([-\tau, 0], \mathbb{R}_+^3)$, which is the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}_+^3$, where $\mathbb{R}_+^3 = \{(x_1, x_2, x_3) : x_i \geq 0, i = 1, 2, 3\}$.

The rest of this paper is organized as follows. The boundedness and local stability of the equilibrium and the existence of Hopf bifurcation at positive equilibrium of system (1.4) are derived in the next section. In Section 3, we study the permanence of system (1.4). In Section 4, the global stability of system (1.4) are investigated. In Section 5, the influence of refuge rate on the densities to predator species and prey species is discussed. We end this paper with some examples and a briefly discussion.

2 Boundedness, Local stability and Hopf bifurcation

In this section, we study the boundedness and local stability of the equilibrium as well as the existence of Hopf bifurcation at positive equilibrium of system (1.4). It is obvious that solutions of model (1.4) with initial conditions (1.5) are positive for all $t \geq 0$. The result is a direct consequence of Nagumo's theorem [29].

2.1 Boundedness

Theorem 2.1. *Every solution of system* (1.4) *with initial conditions* (1.5) *is bounded for all $t \geq 0$ and all of these solutions are ultimately bounded.*

Proof. Let $V(t) = \eta x(t - \tau) + y_1(t) + y_2(t)$, and calculating the derivative of $V(t)$ with respect to t along the positive solution of system (1.4), we have

$$
\begin{aligned}
\dot{V}(t) &= \eta\dot{x}(t) + \dot{y}_1(t) + \dot{y}_2(t) \\
&= \eta\left[x(t-\tau)(r - ax(t-\tau)) - \frac{\mu(1-m)x(t-\tau)y_2(t-\tau)}{1+b(1-m)x(t-\tau)+cy_2(t-\tau)}\right] \\
&\quad + \frac{\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1+b(1-m)x(t-\tau)+cy_2(t-\tau)} - \beta y_1(t) - dy_1(t) \\
&\quad + \beta y_1(t) - ey_2(t) \\
&= \eta x(t-\tau)[r - ax(t-\tau)] - dy_1(t) - ey_2(t).
\end{aligned}
\tag{2.1}
$$

For a small positive constant $s \leq \min\{d, e\}$,

$$
\begin{aligned}
\dot{V}(t) + sV(t) &= (s - d)y_1(t) + (s - e)y_2(t) + \eta x(t-\tau)[s + r - ax(t-\tau)] \\
&\leq \eta x(t-\tau)[s + r - ax(t-\tau)].
\end{aligned}
\tag{2.2}
$$

Hence there exists a positive constant $M = \frac{\eta(s+r)^2}{4a}$ such that

$$
\dot{V}(t) + sV(t) \leq M,
\tag{2.3}
$$

that is

$$
V(t) \leq \left(V(0) - \frac{M}{s}\right)e^{-st} + \frac{M}{s}.
\tag{2.4}
$$

Thus $V(t)$ is ultimately bounded, that is, each solution $z(t) = (x(t), y_1(t), y_2(t))$ of system (1.4) is ultimately bounded. The proof is complete.

2.2 Equilibria

Obviously, system (1.4) always has a trivial equilibrium $E_0(0, 0, 0)$ and a predator-extinction equilibrium $E_1(r/a, 0, 0)$. Further, if the following holds:

$$\mu\eta > be\left(1 + \frac{d}{\beta}\right), \quad 0 \le m < 1 - \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]}, \tag{H1}$$

then model (1.4) has a unique coexistence equilibrium $E^*(x^*, y_1^*, y_2^*)$, where

$$x^* = \frac{-K + \sqrt{\Delta}}{2}, \quad y_1^* = \frac{e}{\beta}y_2^*, \quad y_2^* = \frac{(1 - m)[\mu\eta - be(1 + \frac{d}{\beta})]x^* - e(1 + \frac{d}{\beta})}{ce(1 + \frac{d}{\beta})} \tag{2.5}$$

with

$$K = \frac{1}{a}\left(\frac{(1 - m)[\mu\eta - be(1 + \frac{d}{\beta})]}{\eta c} - r\right), \quad \Delta = K^2 + \frac{4e}{ac\eta}\left(1 + \frac{d}{\beta}\right).$$

Let $\widetilde{E} = (\widetilde{x}, \widetilde{y}_1, \widetilde{y}_2)$ be any arbitrary equilibrium, then the variational matrix of system (1.4) at $\widetilde{E}$ is given by

$$J = \begin{bmatrix} r - 2a\widetilde{x} - \dfrac{\mu(1 - m)\widetilde{y}_2(1 + c\widetilde{y}_2)}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2} & 0 & -\dfrac{\mu(1 - m)\widetilde{x}[1 + b(1 - m)\widetilde{x}]}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2} \\ \dfrac{\eta\mu(1 - m)\widetilde{y}_2(1 + c\widetilde{y}_2)e^{-\lambda\tau}}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2} & -(\beta + d) & \dfrac{\eta\mu(1 - m)\widetilde{x}[1 + b(1 - m)\widetilde{x}]e^{-\lambda\tau}}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2} \\ 0 & \beta & -e \end{bmatrix}$$

and the characteristic equation becomes

$$\begin{aligned} &(\lambda + \beta + d)(\lambda + e)\left(\lambda - r + 2a\widetilde{x} + \frac{\mu(1 - m)\widetilde{y}_2(1 + c\widetilde{y}_2)}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2}\right) + \frac{\mu(1 - m)\widetilde{y}_2(1 + c\widetilde{y}_2)}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2} \\ &\times\frac{\beta\eta\mu(1 - m)\widetilde{x}[1 + b(1 - m)\widetilde{x}]}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2}e^{-\lambda\tau} - \left(\lambda - r + 2a\widetilde{x} + \frac{\mu(1 - m)\widetilde{y}_2(1 + c\widetilde{y}_2)}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2}\right) \\ &\times\frac{\beta\eta\mu(1 - m)\widetilde{x}[1 + b(1 - m)\widetilde{x}]}{[1 + b(1 - m)\widetilde{x} + c\widetilde{y}_2]^2}e^{-\lambda\tau} = 0. \end{aligned} \tag{2.6}$$

2.3 $E_0 = (0, 0, 0)$

First, we analyze the stability of equilibrium E_0.

Theorem 2.2. *The trivial equilibrium $E_0(0, 0, 0)$ of system (1.4) is unstable.*

Proof. The characteristic equation (2.6) takes the form at the trivial equilibrium $E_0(0, 0, 0)$

$$(\lambda - r)(\lambda + \beta + d)(\lambda + e) = 0. \tag{2.7}$$

It is readily seen that Eq.(2.7) has a positive root, thus the equilibrium E_0 is always unstable. The proof is complete.

2.4 $E_1 = (r/a, 0, 0)$

After that, we consider the stability of equilibrium E_1.

Theorem 2.3. *If the following holds:*

$$\mu\eta > be\left(1 + \frac{d}{\beta}\right), \quad 1 - \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]} < m < 1, \tag{H2}$$

then the predator-extinction equilibrium $E_1(r/a, 0, 0)$ of system (1.4) is locally asymptotically stable; if ($H1$) holds, then E_1 is unstable.

Proof. The characteristic equation (2.6) at predator-extinction equilibrium E_1 becomes

$$(\lambda + r)\left[(\lambda + \beta + d)(\lambda + e) - \frac{\beta\eta\mu r(1 - m)}{a + br(1 - m)}e^{-\lambda\tau}\right] = 0. \tag{2.8}$$

Clearly, the equation $\lambda + r = 0$ has one negative real root, which implies that all other roots of Eq.(2.8) are determined by

$$\lambda^2 + h_1\lambda + h_2 + h_3 e^{-\lambda\tau} = 0, \tag{2.9}$$

where $h_1 = e + \beta + d > 0$, $h_2 = e(\beta + d)$, $h_3 = -\frac{\beta\eta\mu r(1 - m)}{a + br(1 - m)}$.

When $\tau = 0$, Eq.(2.9) turns to

$$\lambda^2 + h_1\lambda + h_2 + h_3 = 0. \tag{2.10}$$

According to ($H2$), we have $h_2 + h_3 > 0$. By the Routh-Hurwitz criterion, the boundary equilibrium E_1 is locally asymptotically stable. If ($H1$) holds, then Eq.(2.10) has at least a positive real root, thus the predator-extinction equilibrium E_1 is unstable.

For $\tau > 0$, we investigate the existence of purely imaginary roots of (2.9). If $i\omega_1(\omega_1 > 0)$ is a solution of (2.9) if and only if ω_1 satisfies

$$-\omega_1^2 + h_1\omega_1 i + h_2 + h_3\left(\cos(\tau\omega_1) - i\sin(\tau\omega_1)\right) = 0.$$

Separating the real and imaginary parts, we obtain

$$\begin{aligned} h_1\omega_1 &= h_3\sin(\tau\omega_1), \\ \omega_1^2 - h_2 &= h_3\cos(\tau\omega_1), \end{aligned} \tag{2.11}$$

which implies

$$\omega_1^4 + (h_1^2 - 2h_2)\omega_1^2 + h_2^2 - h_3^2 = 0. \tag{2.12}$$

Note that

$$\begin{aligned} h_1^2 - 2h_2 &= (e + \beta + d)^2 - 2e(\beta + d) = e^2 + (\beta + d)^2 > 0, \\ h_2 - h_3 &= e(\beta + d) + \frac{\beta\eta\mu r(1 - m)}{a + br(1 - m)} > 0, \end{aligned}$$

and $h_2 + h_3 > 0$, then $h_2^2 - h_3^2 > 0$. Hence (2.9) has no positive real roots. By Theorem 3.4.1 in [30], if ($H2$) holds, then all the roots of (2.9) have negative real parts for all $\tau \geq 0$, this implies that the boundary equilibrium E_1 is locally asymptotically stable for all $\tau \geq 0$. The proof is complete.

2.5 $E^* = (x^*, y_1^*, y_2^*)$

Further, we analyze the stability of equilibrium E^*.

For the positive equilibrium $E^*(x^*, y_1^*, y_2^*)$, the characteristic equation (2.6) reduces to

$$\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0 + (q_1\lambda + q_0)e^{-\lambda\tau} = 0, \tag{2.13}$$

where

$$\begin{aligned} p_2 &= \beta + d + e + A + 2ax^* - r, \\ p_1 &= Q + (\beta + d + e)(A + 2ax^* - r), \\ p_0 &= Q(A + 2ax^* - r), \\ q_1 &= -B, \\ q_0 &= -B(2ax^* - r), \end{aligned}$$

with

$$Q = e(\beta + d), \quad A = \frac{\mu(1-m)y_2^\star(1+cy_2^\star)}{[1+b(1-m)x^\star + cy_2^\star]^2},$$

$$B = \frac{\beta\eta\mu(1-m)x^\star[1+b(1-m)x^\star]}{[1+b(1-m)x^\star + cy_2^\star]^2} = \frac{Q[e(1+\frac{d}{\beta})+be(1+\frac{d}{\beta})(1-m)x^\star]}{\mu\eta(1-m)x^\star}.$$

When $\tau = 0$, Eq.(2.13) turns to

$$\lambda^3 + p_2\lambda^2 + (p_1 + q_1)\lambda + p_0 + q_0 = 0. \tag{2.14}$$

From $(H1)$, we derive that $B = \dfrac{Q[e(1+\frac{d}{\beta})+be(1+\frac{d}{\beta})(1-m)x^\star]}{\mu\eta(1-m)x^\star} < Q$, thus

$$p_2 = \beta + d + e + (A + 2ax^\star - r),$$
$$p_1 + q_1 = Q - B + (\beta + d + e)(A + 2ax^\star - r),$$
$$p_0 + q_0 = QA + (Q-B)2ax^\star - (Q-B)r,$$
$$p_2(p_1+q_1) - (p_0+q_0) = (\beta+d+e+A+2ax^\star-r)(Q-B) + (A+2ax^\star-r)$$
$$\times[(\beta+d)(\beta+d+A+2ax^\star-r) + e(e+A+2ax^\star-r)]$$
$$+QA + (Q+B)2ax^\star - (Q+B)r.$$

Denote

$$r_1 = \frac{QA}{Q+B} + 2ax^\star, \quad r_2 = A + 2ax^\star, \quad r_3 = \frac{QA}{Q-B} + 2ax^\star,$$

$$r_4 = r_1 + \frac{(\beta+d+e+r_2-r)(Q-B)}{Q+B} + \frac{(r_2-r)[(\beta+d)(\beta+d+r_2-r)+e(e+r_2-r)]}{Q+B}.$$

We see that if $r < \min\{r_2, r_4\}$ and $(H1)$ hold, then $p_2 > 0, p_1+q_1 > 0, p_0+q_0 > 0$ and $p_2(p_1+q_1)-(p_0+q_0) > 0$. Therefore, by the Routh-Hurwitz criterion, the positive equilibrium $E^\star$ is locally asymptotically stable.

For $\tau > 0$, if $i\omega(\omega > 0)$ is a solution of (2.13) if and only if ω satisfies

$$-\omega^3 i - p_2\omega^2 + p_1\omega i + p_0 + (q_1\omega i + q_0)(\cos(\tau\omega) - i\sin(\tau\omega)) = 0.$$

Separating the real and imaginary parts, we have

$$\omega^3 - p_1\omega = q_1\omega\cos(\tau\omega) - q_0\sin(\tau\omega),$$
$$p_2\omega^2 - p_0 = q_1\omega\sin(\tau\omega) + q_0\cos(\tau\omega), \tag{2.15}$$

which implies

$$\omega^6 + (p_2^2 - 2p_1)\omega^4 + (p_1^2 - 2p_0p_2 - q_1^2)\omega^2 + p_0^2 - q_0^2 = 0, \tag{2.16}$$

where

$$p_2^2 - 2p_1 = (\beta+d+e+A+2ax^\star-r)^2 - 2Q - 2(\beta+d+e)(A+2ax^\star-r)$$
$$= (\beta+d)^2 + e^2 + (A+2ax^\star-r)^2 > 0,$$
$$p_1^2 - 2p_0p_2 - q_1^2 = [Q+(\beta+d+e)(A+2ax^\star-r)]^2 - 2Q(A+2ax^\star-r)$$
$$\times(\beta+d+e+A+2ax^\star-r) - B^2$$
$$> (A+2ax^\star-r)^2[(\beta+d)^2+e^2] > 0,$$
$$p_0^2 - q_0^2 = [QA+(Q-B)2ax^\star-(Q-B)r] \times [QA+(Q+B)2ax^\star-(Q+B)r].$$

Obviously, if $r < r_1$ and $(H1)$ hold, then $p_0^2 - q_0^2 > 0$, this implies that (2.16) has no positive real roots. Therefore, by Theorem 3.4.1 in [30], if $r < r_1$ and $(H1)$ are satisfied, then all the roots of (2.16) have negative real parts for all $\tau \geq 0$. Hence the positive equilibrium $E^\star = (x^\star, y_1^\star, y_2^\star)$ is locally asymptotically stable for all $\tau \geq 0$.

If $r_1 < r < \min\{r_2, r_4\}$ holds, which implies $p_0^2 - q_0^2 < 0$, then there exists a unique positive root ω_0 satisfying (2.16). From (2.15), we have

$$\cos(\tau\omega_0) = \frac{q_1\omega_0^4 + (p_2q_0 - p_1q_1)\omega_0^2 - p_0q_0}{q_1^2\omega_0^2 + q_0^2}. \tag{2.17}$$

Denote

$$\tau_{0n} = \frac{1}{\omega_0}\arccos\frac{q_1\omega_0^4 + (p_2q_0 - p_1q_1)\omega_0^2 - p_0q_0}{q_1^2\omega_0^2 + q_0^2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \cdots. \tag{2.18}$$

By Theorem 3.4.1 in Kuang [30], we see that if $p_0^2 - q_0^2 < 0$ hold, then E^* remains stable for $\tau < \tau_0 := \tau_{00}$.

We now claim that

$$\left\{\frac{d(\text{Re}\lambda)}{d\tau}\right\}_{\tau=\tau_0} > 0.$$

This shows that there exists at least one eigenvalue with a positive real part for $\tau > \tau_0$. Moreover, the conditions for the existence of a Hopf bifurcation [31] are then satisfied yielding a periodic solution. To this end, differentiating Eq.(2.13) with respect to τ, it follows that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{3\lambda^2 + 2p_2\lambda + p_1}{\lambda(\lambda^3 + p_2\lambda^2 + p_1\lambda + p_0)} + \frac{q_1}{\lambda(q_1\lambda + q_0)} - \frac{\tau}{\lambda}.$$

Hence, a direct calculation shows that

$$\text{sgn}\left\{\frac{d(\text{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} = \text{sgn}\left\{\text{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0}$$

$$= \text{sgn}\left\{-\frac{(p_1 - 3\omega_0^2)(\omega_0^2 - p_1) + 2p_2(p_0 - p_2\omega_0^2)}{(\omega_0^3 - p_1\omega_0)^2 + (p_0 - p_2\omega_0^2)^2} - \frac{q_1^2}{q_1^2\omega_0^2 + q_0^2}\right\}.$$

We derive from (2.15) that

$$(\omega_0^3 - p_1\omega_0)^2 + (p_0 - p_2\omega_0^2)^2 = q_1^2\omega_0^2 + q_0^2.$$

Therefore, we yield

$$\text{sgn}\left\{\frac{d(\text{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} = \text{sgn}\left\{\frac{3\omega_0^4 + 2(p_2^2 - 2p_1)\omega_0^2 + p_1^2 - 2p_0p_2 - q_1^2}{q_1^2\omega_0^2 + q_0^2}\right\} > 0.$$

Thus, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0$, $\tau = \tau_0$. Now, let us summarize our results as follows:

Theorem 2.4. (i) *If $(H1)$ and $r < r_1$ hold, then the positive equilibrium E^* of system (1.4) is locally asymptotically stable for all $\tau \geq 0$.*

(ii) *If $(H1)$ and $r_1 < r < \min\{r_2, r_4\}$ hold, then there exists a $\tau_0 > 0$ such that E^* is locally asymptotically stable when $\tau \in [0, \tau_0)$. Furthermore, system (1.4) undergoes a Hopf bifurcation at E^* when $\tau = \tau_0$.*

Remark (i) *Let $\tau = 0$ and $m = 0$ in (1.4), then Theorem 2.3 of this paper is equivalent to Theorem 3 of [21]. Obviously, we generalize the conclusion of boundary equilibrium in [21], and show that prey refuge affect the stability of the boundary equilibrium. Further, the global stability of the boundary equilibrium will be studied in Section 4 .*

(ii) *Notice that the conditions of positive equilibrium E^* is locally asymptotically stable in [21] is very complicated. Let $\tau = 0$ and $m = 0$ in model (1.4), compare Theorem 2.4 of this paper with Lemma 4 of [21], we find that the conditions of our positive equilibrium locally stable are more extensive and concise than that of [21].*

3 Permanence

Lemma 3.1. *Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.4) with initial conditions (1.5). Assume that $(H1)$ holds, then*

$$\limsup_{t\to+\infty} x(t) \leq L_1, \quad \limsup_{t\to+\infty} y_1(t) \leq L_2, \quad \limsup_{t\to+\infty} y_2(t) \leq L_3, \tag{3.1}$$

where

$$L_1 = \frac{r}{a}, \quad L_2 = \frac{\beta\eta\mu(1-m)L_1 - e(\beta+d)[1 + b(1-m)L_1]}{c\beta(\beta+d)}, \quad L_3 = \frac{\beta}{e}L_2.$$

Proof. It follows from (H_1) that $L_2 = \frac{\beta\eta\mu(1-m)L_1 - e(\beta+d)[1+b(1-m)L_1]}{c\beta(\beta+d)} > 0$. Hence, there exists an enough small positive constant ε such that

$$L_{2\varepsilon} \overset{def}{=} \frac{\beta\eta\mu(1-m)(L_1+\varepsilon) - e(\beta+d)[1+b(1-m)(L_1+\varepsilon)]}{c\beta(\beta+d)} > 0,$$

$$L_{3\varepsilon} \overset{def}{=} \frac{\beta}{e}L_{2\varepsilon} > 0.$$

Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.4) with initial conditions (1.5). From the first equation of system (1.4), it follows that

$$\dot{x}(t) = x(t)(r - ax(t)) - \frac{\mu(1-m)x(t)y_2(t)}{1 + b(1-m)x(t) + cy_2(t)} \tag{3.2}$$
$$\leq x(t)(r - ax(t)).$$

Applying Lemma 2.3 in [32] to (3.2), it immediately follows that

$$\limsup_{t\to+\infty} x(t) \leq \frac{r}{a} \overset{def}{=} L_1. \tag{3.3}$$

Then for above $\varepsilon > 0$ sufficiently small there exists a $T_1 > 0$ such that if $t > T_1$, $x(t) \leq L_1 + \varepsilon$. We derive from the second and the third equations of system (1.4) that for $t > T_1 + \tau$,

$$\dot{y}_1(t) \leq \frac{\eta\mu(1-m)(L_1+\varepsilon)y_2(t-\tau)}{1 + b(1-m)(L_1+\varepsilon) + cy_2(t-\tau)} - \beta y_1(t) - dy_1(t), \tag{3.4}$$
$$\dot{y}_2(t) = \beta y_1(t) - ey_2(t).$$

Consider the following auxiliary equations:

$$\dot{u}_1(t) = \frac{\eta\mu(1-m)(L_1+\varepsilon)u_2(t-\tau)}{1 + b(1-m)(L_1+\varepsilon) + cu_2(t-\tau)} - \beta u_1(t) - du_1(t), \tag{3.5}$$
$$\dot{u}_2(t) = \beta u_1(t) - eu_2(t).$$

Using a similar argument as that in the proof of Lemma 2.4 in [33], it follows from (3.5) that

$$\lim_{t\to+\infty} u_1(t) = L_{2\varepsilon}, \quad \lim_{t\to+\infty} u_2(t) = L_{3\varepsilon}. \tag{3.6}$$

By comparison, we obtain that

$$\limsup_{t\to+\infty} y_1(t) \leq L_{2\varepsilon}, \quad \limsup_{t\to+\infty} y_2(t) \leq L_{3\varepsilon}. \tag{3.7}$$

Let $\varepsilon \to 0$, it follows that

$$\limsup_{t\to+\infty} y_1(t) \leq L_2, \quad \limsup_{t\to+\infty} y_2(t) \leq L_3. \tag{3.8}$$

The proof is complete.

Lemma 3.2. *Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.4) with initial conditions (1.5), if (H_1) and*

$$r > \frac{\mu(1-m)L_3}{1+cL_3}, \quad \mu\eta > \frac{e(\beta+d)[1+b(1-m)l_1]}{\beta(1-m)l_1} \tag{3.9}$$

hold, then

$$\liminf_{t\to+\infty} x(t) \geq l_1, \quad \liminf_{t\to+\infty} y_1(t) \geq l_2, \quad \liminf_{t\to+\infty} y_2(t) \geq l_3, \tag{3.10}$$

where

$$l_1 = \frac{r(1+cL_3) - \mu(1-m)L_3}{a(1+cL_3)}, \quad l_2 = \frac{\beta\eta\mu(1-m)l_1 - e(\beta+d)[1+b(1-m)l_1]}{c\beta(\beta+d)},$$

$$l_3 = \frac{\beta}{e}l_2.$$

Proof. It follows from (H_1) and condition (3.9) that $l_1 = \frac{r(1+cL_3)-\mu(1-m)L_3}{a(1+cL_3)} > 0$, $l_2 = \frac{\beta\eta\mu(1-m)l_1-e(\beta+d)[1+b(1-m)l_1]}{c\beta(\beta+d)} > 0$. Hence, there exists an enough small positive constant ε such that

$$l_{1\varepsilon} = \frac{r(1+c(L_3+\varepsilon))-\mu(1-m)(L_3+\varepsilon)}{a(1+c(L_3+\varepsilon))} > 0,$$
$$l_{2\varepsilon} = \frac{\beta\eta\mu(1-m)(l_1-\varepsilon)-e(\beta+d)[1+b(1-m)(l_1-\varepsilon)]}{c\beta(\beta+d)} > 0,$$
$$l_{3\varepsilon} = \frac{\beta}{e}l_{2\varepsilon} > 0.$$

Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.4) with initial conditions (1.5). For above $\varepsilon > 0$ sufficiently small, it follows from Lemma 3.1 that there exists a $T_2 > 0$ such that if $t > T_2$, $y_1(t) \le L_2 + \varepsilon$, $y_2(t) \le L_3 + \varepsilon$. Hence, we derive from the first equation of system (1.4) that for $t > T_2$,

$$\dot{x}(t) \ge x(t)\left(r - ax(t) - \frac{\mu(1-m)(L_3+\varepsilon)}{1+c(L_3+\varepsilon)}\right).$$

According to condition (3.9) and similar to the proof of Lemma 3.1, we have

$$\liminf_{t\to+\infty} x(t) \ge \frac{r(1+cL_3)-\mu(1-m)L_3}{a(1+cL_3)} \overset{def}{=} l_1 > 0. \tag{3.11}$$

For above $\varepsilon > 0$ sufficiently small, there exists a $T_3 \ge T_2$ such that if $t > T_3$, $x(t) \ge l_1 - \varepsilon$. Therefore, it follows from the second and the third equations of system (1.4) that for $t > T_3 + \tau$,

$$\dot{y}_1(t) \ge \frac{\eta\mu(1-m)(l_1-\varepsilon)y_2(t-\tau)}{1+b(1-m)(l_1-\varepsilon)+cy_2(t-\tau)} - \beta y_1(t) - dy_1(t),$$
$$\dot{y}_2(t) = \beta y_1(t) - ey_2(t). \tag{3.12}$$

Consider the following auxiliary equations:

$$\dot{u}_1(t) = \frac{\eta\mu(1-m)(l_1-\varepsilon)u_2(t-\tau)}{1+b(1-m)(l_1-\varepsilon)+cu_2(t-\tau)} - \beta u_1(t) - du_1(t),$$
$$\dot{u}_2(t) = \beta u_1(t) - eu_2(t). \tag{3.13}$$

Using a similar argument as that in the proof of Lemma 2.4 in [33], it follows from (3.13) that

$$\lim_{t\to+\infty} u_1(t) \ge l_{2\varepsilon}, \quad \lim_{t\to+\infty} u_1(t) \ge l_{3\varepsilon}. \tag{3.14}$$

By comparison, we obtain that

$$\liminf_{t\to+\infty} y_1(t) \ge l_{2\varepsilon}, \quad \liminf_{t\to+\infty} y_2(t) \ge l_{3\varepsilon}. \tag{3.15}$$

Let $\varepsilon \to 0$, we conclude that

$$\liminf_{t\to+\infty} y_1(t) \ge l_2, \quad \liminf_{t\to+\infty} y_2(t) \ge l_3. \tag{3.16}$$

The proof is complete.

As a direct corollary of Lemmas 3.1 and 3.2 we have the following theorem.

Theorem 3.1. *Assume that* $(H1)$ *and* (3.9) *hold, then system* (1.4) *is permanent.*

4 Global stability

In this section, we study the global stability of the predator-extinction equilibrium E_1 and the global attractivity of the coexistence equilibrium $E^\star$ of system (1.4). The strategy of proofs is to use Lyapunov functionals and the LaSalle invariance principle.

Theorem 4.1. *If* $(H2)$ *holds, then the predator-extinction equilibrium* $E_1(r/a, 0, 0)$ *of system* (1.4) *is globally asymptotically stable.*

Proof. Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.4) with initial conditions (1.5). Denote $x_0 = r/a$.

Define

$$V_{11}(t) = x - x_0 - x_0 \ln \frac{x}{x_0} + \xi_1 y_1 + \xi_2 y_2, \tag{4.1}$$

where $\xi_1 = \frac{1+b(1-m)x_0}{\eta}$, $\xi_2 = \frac{\mu(1-m)x_0}{e}$.

Calculating the derivative of V_{11} along positive solutions of system (1.4), it follows that

$$\frac{d}{dt}V_{11}(t) = \left(1 - \frac{x_0}{x}\right)\left[x(t)(r - ax(t)) - \frac{\mu(1-m)x(t)y_2(t)}{1 + b(1-m)x(t) + cy_2(t)}\right]$$
$$+ \xi_1 \left[\frac{\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1 + b(1-m)x(t-\tau) + cy_2(t-\tau)} - \beta y_1(t) - dy_1(t)\right] \tag{4.2}$$
$$+ \xi_2[\beta y_1(t) - ey_2(t)].$$

Substituting $r = ax_0$ into (4.2), we obtain that

$$\frac{d}{dt}V_{11}(t) = -\frac{r}{x_0}(x(t) - x_0)^2 - [1 + b(1-m)x_0]\frac{\mu(1-m)x(t)y_2(t)}{1 + b(1-m)x(t) + cy_2(t)}$$
$$+ \mu(1-m)x_0y_2(t) - \frac{\mu(1-m)cx_0y_2^2(t)}{1 + b(1-m)x(t) + cy_2(t)}$$
$$+ \xi_1\frac{\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1 + b(1-m)x(t-\tau) + cy_2(t-\tau)} - \xi_1\beta y_1(t) \tag{4.3}$$
$$- \xi_1 dy_1(t) + \xi_2\beta y_1(t) - \xi_2 ey_2(t).$$

Define

$$V_1(t) = V_{11}(t) + \xi_1\eta\mu(1-m)\int_{t-\tau}^{t}\frac{x(s)y_2(s)}{1 + b(1-m)x(s) + cy_2(s)}ds. \tag{4.4}$$

We derive from (4.3) and (4.4) that

$$\frac{d}{dt}V_1(t) = -\frac{r}{x_0}(x(t) - x_0)^2 - \frac{\mu(1-m)cx_0y_2^2(t)}{1 + b(1-m)x(t) + cy_2(t)} - \left[\frac{(\beta + d)[a + br(1-m)]}{\eta a} - \frac{\beta\mu r(1-m)}{ae}\right]y_1(t). \tag{4.5}$$

If $(H2)$ holds, it then follows from (4.5) that $\dot{V}_1(t) \leq 0$. By Theorem 5.3.1 in [34], solutions limit to $\mathcal{M}$, the largest invariant subset of $\{\dot{V}_1(t) = 0\}$. Obviously, we see from (4.5) that $\dot{V}_1(t) = 0$ if and only if $x = x_0, y_2 = 0$. Noting that $\mathcal{M}$ is invariant, for each element in $\mathcal{M}$, we have $x(t) = x_0, y_2(t) = 0$. It therefore follows from the third equation of system (1.4) that $0 = \dot{y}_2(t) = \beta y_1(t)$, which yields $y_1(t) = 0$. Therefore, $\dot{V}_1(t) = 0$ if and only if $(x, y_1, y_2) = (x_0, 0, 0)$. Accordingly, the global asymptotic stability of E_1 follows from LaSalle invariance principle. This completes the proof. $\blacksquare$

Theorem 4.2. *Assume that $(H1)$ and (3.9) are satisfied. If the following holds:*

$$a(l_1 + x^\star) - r \geq \overline{\rho}_1, \quad l_1 x^\star + b(1-m)l_1 x^{\star 2} \geq \overline{\rho}_2, \tag{4.6}$$

where $\overline{\rho}_1 = \frac{2\mu(1-m)cL_3y_2^\star(1+cy_2^\star)+\mu c(1-m)x^\star y_2^\star[b(1-m)L_1+cL_3]}{2[1+b(1-m)l_1+cl_3][1+b(1-m)x^\star+cy_2^\star]}, \overline{\rho}_2 = \frac{x^\star y_2^\star[b(1-m)L_1+cL_3]}{2}$, and L_i, l_i $(i = 1, 2, 3)$ are defined as those in Theorem 3.1, then the positive equilibrium $E^\star(x^\star, y_1^\star, y_2^\star)$ of system (1.4) is globally attractive.

Proof. Let $(x(t), y_1(t), y_2(t))$ be any positive solution of system (1.4) with initial conditions (1.5). Define

$$V_{21}(t) = x - x^\star - x^\star\ln\frac{x}{x^\star} + k_1\left(y_1 - y_1^\star - y_1^\star\ln\frac{y_1}{y_1^\star}\right) + k_2\left(y_2 - y_2^\star - y_2^\star\ln\frac{y_2}{y_2^\star}\right), \tag{4.7}$$

where $k_1 = \frac{1+b(1-m)x^\star+cy_2^\star}{\eta}$, $k_2 = \frac{k_1(\beta+d)}{\beta}$.

Calculating the derivative of V_{11} along positive solutions of system (1.4), it follows that

$$\frac{d}{dt}V_{21}(t) = \left(1 - \frac{x^\star}{x}\right)\left[x(t)(r - ax(t)) - \frac{\mu(1-m)x(t)y_2(t)}{1 + b(1-m)x(t) + cy_2(t)}\right]$$
$$+ k_1\left(1 - \frac{y_1^\star}{y_1}\right)\left[\frac{\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1 + b(1-m)x(t-\tau) + cy_2(t-\tau)} - \beta y_1(t) - dy_1(t)\right] \tag{4.8}$$
$$+ k_2\left(1 - \frac{y_2^\star}{y_2}\right)[\beta y_1(t) - ey_2(t)].$$

Substituting $r = ax^* + \frac{\mu(1-m)y_2^*}{1+b(1-m)x^*+cy_2^*}$ into (4.8), we derive that

$$
\begin{aligned}
\frac{d}{dt}V_{21}(t) = &\left(1 - \frac{x^*}{x}\right)\left[x(r - ax) - x^*(r - ax^*) + \frac{\mu(1-m)x^*y_2^*}{1+b(1-m)x^*+cy_2^*}\right] \\
&-\mu(1-m)\left[[1+b(1-m)x^*+cy_2^*] - \frac{x^*[1+b(1-m)x+cy_2]}{x}\right. \\
&\left.+\frac{c(x^*y_2 - xy_2^*)}{x}\right] \times \frac{x(t)y_2(t)}{1+b(1-m)x(t)+cy_2(t)} + k_1\left(1 - \frac{y_1^*}{y_1}\right) \\
&\times\left[\frac{\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1+b(1-m)x(t-\tau)+cy_2(t-\tau)} - dy_1(t) - \beta y_1(t)\right] \\
&+k_2\left(1 - \frac{y_2^*}{y_2}\right)[\beta y_1(t) - ey_2(t)]
\end{aligned}
$$

$$
\begin{aligned}
= &\left(1 - \frac{x^*}{x}\right)\left[x(r - ax) - x^*(r - ax^*) + \frac{\mu(1-m)x^*y_2^*}{1+b(1-m)x^*+cy_2^*}\right] \\
&-\mu(1-m)[1+b(1-m)x^*+cy_2^*] \times \frac{x(t)y_2(t)}{1+b(1-m)x(t)+cy_2(t)} \\
&-\frac{\mu(1-m)cx(t)y_2(t)(x^*y_2 - xy_2^*)}{x[1+b(1-m)x(t)+cy_2(t)]} + \frac{k_1\eta\mu(1-m)x(t-\tau)y_2(t-\tau)}{1+b(1-m)x(t-\tau)+cy_2(t-\tau)} \qquad (4.9) \\
&-k_1\eta\mu(1-m)\frac{y_1^*x(t-\tau)y_2(t-\tau)}{y_1[1+b(1-m)x(t-\tau)+cy_2(t-\tau)]} + k_1(\beta+d)y_1^* \\
&-k_2\beta y_1^*\frac{y_2^*}{y_1^*}\frac{y_1(t)}{y_2(t)} + k_2ey_2^*.
\end{aligned}
$$

Define

$$
\begin{aligned}
V_2(t) = V_{21}(t) + k_1\eta\mu(1-m)\int_{t-\tau}^{t}&\left[\frac{x(s)y_2(s)}{1+b(1-m)x(s)+cy_2(s)}\right. \\
&-\frac{x^*y_2^*}{1+b(1-m)x^*+cy_2^*} - \frac{x^*y_2^*}{1+b(1-m)x^*+cy_2^*} \qquad (4.10) \\
&\left.\times\ln\frac{[1+b(1-m)x^*+cy_2^*]x(s)y_2(s)}{x^*y_2^*[1+b(1-m)x(s)+cy_2(s)]}\right]ds.
\end{aligned}
$$

It follows from (4.9) and (4.10) that

$$
\begin{aligned}
\frac{d}{dt}V_2(t) = &\left(1 - \frac{x^*}{x}\right)\left[x(r - ax) - x^*(r - ax^*) + \frac{\mu(1-m)x^*y_2^*}{1+b(1-m)x^*+cy_2^*}\right] \\
&-\frac{\mu(1-m)cx(t)y_2(t)(x^*y_2 - xy_2^*)}{x[1+b(1-m)x(t)+cy_2(t)]} - k_1\eta\mu(1-m)\frac{x^*y_2^*}{1+b(1-m)x^*+cy_2^*} \\
&\times\frac{y_1^*[1+b(1-m)x^*+cy_2^*]x(t-\tau)y_2(t-\tau)}{x^*y_2^*y_1[1+b(1-m)x(t-\tau)+cy_2(t-\tau)]} + k_1(\beta+d)y_1^* \qquad (4.11) \\
&-k_2\beta y_1^*\frac{y_2^*}{y_1^*}\frac{y_1(t)}{y_2(t)} + k_2ey_2^* + k_1\eta\mu(1-m)\frac{x^*y_2^*}{1+b(1-m)x^*+cy_2^*} \\
&\times\ln\frac{[1+b(1-m)x(t)+cy_2(t)]x(t-\tau)y_2(t-\tau)}{x(t)y_2(t)[1+b(1-m)x(t-\tau)+cy_2(t-\tau)]}.
\end{aligned}
$$

Noting that

$$
k_2ey_2^* = k_2\beta y_1^* = k_1(\beta+d)y_1^* = k_1\eta\mu(1-m)\frac{x^*y_2^*}{1+b(1-m)x^*+cy_2^*} = \mu(1-m)x^*y_2^*,
$$

and

$$
\begin{aligned}
\left(1 - \frac{x^*}{x}\right)\frac{\mu(1-m)x^*y_2^*}{1+b(1-m)x^*+cy_2^*} = &\mu(1-m)x^*y_2^*\left(1 - \frac{x^*[1+b(1-m)x(t)+cy_2(t)]}{x(t)[1+b(1-m)x^*+cy_2^*]}\right) \\
&+\frac{\mu(1-m)cx^*y_2^*(x^*y_2 - xy_2^*)}{x(t)[1+b(1-m)x^*+cy_2^*]},
\end{aligned}
$$

we derive from (4.11) that

$$\frac{d}{dt}V_2(t) = \frac{(x-x^\star)^2}{x}[r - a(x+x^\star)] - \frac{\mu(1-m)cx(t)y_2(t)(x^\star y_2 - xy_2^\star)}{x(t)[1 + b(1-m)x(t) + cy_2(t)]}$$

$$+ \frac{\mu(1-m)cx^\star y_2^\star(x^\star y_2 - xy_2^\star)}{x(t)[1 + b(1-m)x^\star + cy_2^\star]} - \mu(1-m)x^\star y_2^\star$$

$$\times \left[\frac{x^\star[1 + b(1-m)x + cy_2]}{x^\star[1 + b(1-m)x^\star + cy_2^\star]} - 1 - \ln \frac{x^\star[1 + b(1-m)x + cy_2]}{x^\star[1 + b(1-m)x^\star + cy_2^\star]} \right]$$

$$- \mu(1-m)x^\star y_2^\star \times \left[\frac{y_2^\star y_1(t)}{y_1^\star y_2(t)} - 1 - \ln \frac{y_2^\star y_1(t)}{y_1^\star y_2(t)} \right]$$

$$- \mu(1-m)x^\star y_2^\star \times \left[\frac{y_1^\star[1 + b(1-m)x^\star + cy_2^\star]x(t-\tau)y_2(t-\tau)}{x^\star y_2^\star y_1[1 + b(1-m)x(t-\tau) + cy_2(t-\tau)]} - 1 \right.$$

$$\left. - \ln \frac{y_1^\star[1 + b(1-m)x^\star + cy_2^\star]x(t-\tau)y_2(t-\tau)}{x^\star y_2^\star y_1[1 + b(1-m)x(t-\tau) + cy_2(t-\tau)]} \right]$$

$$\leq -\frac{(x-x^\star)^2}{x}[a(x+x^\star) - r - \rho_1]$$

$$- \frac{\mu(1-m)c(y-y_2^\star)^2[xx^\star + b(1-m)xx^{\star 2} - \rho_2]}{x(t)[1 + b(1-m)x(t) + cy_2(t)][1 + b(1-m)x^\star + cy_2^\star]} - \mu(1-m)x^\star y_2^\star$$

$$\times \left[\frac{x^\star[1 + b(1-m)x + cy_2]}{x^\star[1 + b(1-m)x^\star + cy_2^\star]} - 1 - \ln \frac{x^\star[1 + b(1-m)x + cy_2]}{x^\star[1 + b(1-m)x^\star + cy_2^\star]} \right] \qquad (4.12)$$

$$- \mu(1-m)x^\star y_2^\star \times \left[\frac{y_2^\star y_1(t)}{y_1^\star y_2(t)} - 1 - \ln \frac{y_2^\star y_1(t)}{y_1^\star y_2(t)} \right]$$

$$- \mu(1-m)x^\star y_2^\star \times \left[\frac{y_1^\star[1 + b(1-m)x^\star + cy_2^\star]x(t-\tau)y_2(t-\tau)}{x^\star y_2^\star y_1[1 + b(1-m)x(t-\tau) + cy_2(t-\tau)]} - 1 \right.$$

$$\left. - \ln \frac{y_1^\star[1 + b(1-m)x^\star + cy_2^\star]x(t-\tau)y_2(t-\tau)}{x^\star y_2^\star y_1[1 + b(1-m)x(t-\tau) + cy_2(t-\tau)]} \right],$$

where $\rho_1 = \frac{2\mu(1-m)cy_2 y_2^\star(1 + cy_2^\star) + \mu c(1-m)x^\star y_2^\star[b(1-m)x + cy_2]}{2[1 + b(1-m)x(t) + cy_2(t)][1 + b(1-m)x^\star + cy_2^\star]}$, $\rho_2 = \frac{x^\star y_2^\star[b(1-m)x + cy_2]}{2}$.

Hence, if (4.6) holds, it then follows from (4.12) that $\dot{V}_1(t) \leq 0$, with equality if and only if $x = x^\star$, $\frac{y_1^\star[1 + b(1-m)x^\star + cy_2^\star]x(t-\tau)y_2(t-\tau)}{x^\star y_2^\star y_1[1 + b(1-m)x(t-\tau) + cy_2(t-\tau)]} = \frac{y_2^\star y_1(t)}{y_1^\star y_2(t)} = 1$. We now look for the invariant subset $\mathcal{M}$ within the set $\mathbf{M} = \{(x, y_1, y_2) : x = x^\star, \frac{y_1^\star[1 + b(1-m)x^\star + cy_2^\star]x(t-\tau)y_2(t-\tau)}{x^\star y_2^\star y_1[1 + b(1-m)x(t-\tau) + cy_2(t-\tau)]} = \frac{y_2^\star y_1(t)}{y_1^\star y_2(t)} = 1\}$. Since $x = x^\star$ on $\mathcal{M}$ and due to $0 = \dot{x}(t) = x^\star(r - ax^\star) - \frac{\mu(1-m)x^\star y_2(t)}{1 + b(1-m)x^\star + cy_2(t)}$, we obtain $y_2(t) = y_2^\star$. From the third equation of model (1.4) that $0 = \dot{y}_2(t) = \beta y_1(t) - ey_2^\star$, which yields $y_1 = y_1^\star$. Therefore, the only invariant set in $\mathbf{M}$ is $\mathcal{M} = \{(x^\star, y_1^\star, y_2^\star)\}$. Using the LaSalle invariance principle, then $E^\star$ of system (1.4) is globally attractive.

5 The influence of prey refuge

In this section, we investigate the influence of prey refuge. Under the condition ($H1$), let us compute the derivative along the positive equilibrium $E^\star$ with respect to m, that is

$$\frac{dx^\star}{dm} = \frac{[\mu\eta - be(1 + \frac{d}{\beta})]x^\star}{a\eta c\sqrt{\Delta}} > 0,$$

$$\frac{dy_2^\star}{dm} = \frac{[\mu\eta - be(1 + \frac{d}{\beta})]x^\star\left[(1-m)[\mu\eta - be(1 + \frac{d}{\beta})] - a\eta c\sqrt{\Delta}\right]}{a\eta c^2 e(1 + \frac{d}{\beta})\sqrt{\Delta}},$$

$$\frac{dy_1^\star}{dm} = \frac{[\mu\eta - be(1 + \frac{d}{\beta})]x^\star\left[(1-m)[\mu\eta - be(1 + \frac{d}{\beta})] - a\eta c\sqrt{\Delta}\right]}{\beta a\eta c^2(1 + \frac{d}{\beta})\sqrt{\Delta}}.$$

Denote

$$\mathfrak{R}_1 = \frac{\eta c r^2 + 4ae(1 + \frac{d}{\beta})}{2r[\mu\eta - be(1 + \frac{d}{\beta})]} > \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]} = \mathfrak{R}_2.$$

Due to the existence of E^*, which implies that x^* is a strictly increasing function of m, that is, increasing the constant amount of prey refuge m leads to the increase of prey densities. When $0 < m < 1 - \mathfrak{R}_1$, $\frac{dy_i^*}{dm} > 0$ ($i = 1, 2$), it then yields that both y_1^* and y_2^* are strictly increasing functions on $m \in (0, 1 - \mathfrak{R}_1)$; when $1 - \mathfrak{R}_1 < m < 1 - \mathfrak{R}_2$, $\frac{dy_i^*}{dm} < 0$ ($i = 1, 2$), which implies that both y_1^* and y_2^* are strictly decreasing functions on $m \in (1 - \mathfrak{R}_1, 1 - \mathfrak{R}_2)$; when $m = 1 - \mathfrak{R}_1$, the predator species reaches its maximum, and when $m = 1 - \mathfrak{R}_2$, the predator species goes to extinction.
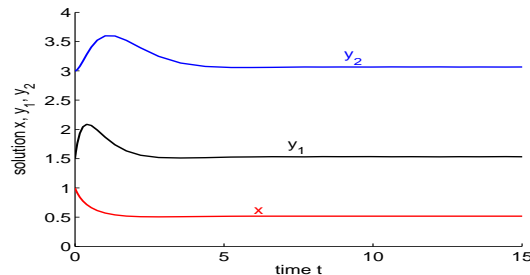
Figure 1: $m = \tau = 0, E^* = (0.5168, 1.5330, 3.0660)$ of system (1.4) is locally asymptotically stable.
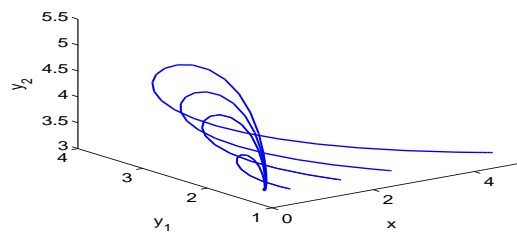
Figure 2: $m = \tau = 0, E^* = (0.5168, 1.5330, 3.0660)$ of system (1.4) with different initial values is locally asymptotically stable.

6 Numerical simulations

Example 6.1. In (1.4), let $m = \tau = 0$, then system (1.4) is degenerated into model (2.1) of [21]. Let $r = 6$, $a = 3, \mu = 20, b = d = e = 1, \eta = \beta = 2$ and $c = 4$, then system (1.4) has a positive equilibrium $E^* = (0.5168, 1.5330, 3.0660)$. By calculation, $\mu\eta = 40 > 1.5 = be(1 + \frac{d}{\beta})$, $0 < 0.9805 = 1 - \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]}$, $r < 7.3835 = r_2$ and $r < 17.7279 = r_4$. Thus E^* is locally asymptotically stable.
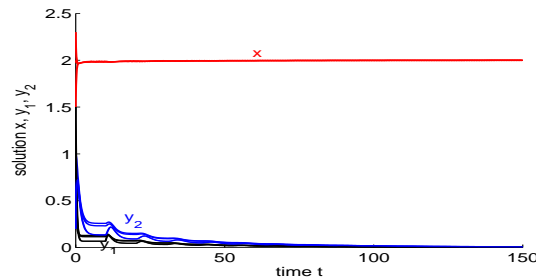
Figure 3: $\mu = 1, \tau = 10, E_1 = (2, 0, 0)$ of system (1.4) is globally asymptotically stable.
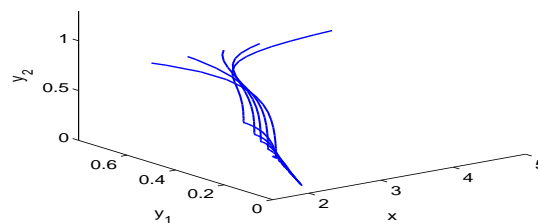

Figure 4: $\mu = 1, \tau = 10, E_1 = (2, 0, 0)$ of system (1.4) with different initial values is globally asymptotically stable.

Note that $\eta\beta(1 + bx^\star) = 6.0671 > 0.4778 = \frac{b^2 x^\star y_2^\star (\beta + d + e)}{1 + cy_2^\star}$. Therefore, condition $(S3) : \eta\beta(1 + bx^\star) <$ $\min\left\{ \frac{e(\beta + d)(1 + bx^\star + cy_2^\star)^2}{\mu x^\star}, \frac{b^2 x^\star y_2^\star (\beta + d + e)}{1 + cy_2^\star} \right\}$ in [21] which ensures the local stability of the positive equilibrium $E^\star$ is not satisfied. This implies that our conditions are weaker that those in [21] (see Figs. 1 and 2).

Example 6.2. In (1.4), let $r = 6, a = 3, m = 0.2, b = d = e = 1, \eta = \beta = 2$ and $c = 4$, then $be(1 + \frac{d}{\beta}) = 1.5$.

(i) If $\mu = 1, \tau = 10$, it is easy to show that $\mu\eta = 2 > be(1 + \frac{d}{\beta}), 1 - m = 0.8 < 1.5 = \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]}$ and the predator-extinction equilibrium $E_1 = (2, 0, 0)$. By Theorem 4.1, E_1 is globally asymptotically stable (see Figs. 3 and 4).

(ii) If $\mu = 20, \tau = 10$, it is easy to show that $\mu\eta = 40 > be(1 + \frac{d}{\beta}), m < 0.9805 = 1 - \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]},$ $r < 7.9443 = r_1$ and the positive equilibrium $E^\star = (0.7953, 1.9162, 3.8323)$. By Theorem 2.4, $E^\star$ is locally asymptotically stable (see Figs. 5 and 6).

(iii) If $\mu = 36$, it is easy to show that $\mu\eta = 72 > be(1 + \frac{d}{\beta}), m < 0.9894 = 1 - \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]}, r > 5.2798 = r_1,$ $r < 6.2736 = r_2, r < 8.8358 = r_4$, and the positive equilibrium $E^\star = (0.1302, 0.4868, 0.9735)$. Further, by calculation, we have $\tau_0 = 0.9429$. By Theorem 2.4, when $\tau < \tau_0$, $E^\star$ is locally asymptotically stable (see Fig. 7); when $\tau > \tau_0$, the positive equilibrium $E^\star$ of model (1.4) is unstable, which yields a periodic solution (see Fig. 8).

(iv) If $\mu = 2, \tau = 2$, it is easy to show that $\mu\eta = 4 > be(1 + \frac{d}{\beta}), m < 0.7 = 1 - \frac{ae(1 + \frac{d}{\beta})}{r[\mu\eta - be(1 + \frac{d}{\beta})]}, a(l_1 + x^\star) - r = 5.5962 > 0.6417 = \bar{\rho}_1, l_1 x^\star + b(1 - m)l_1 x^{\star 2} = 9.5580 > 1.2781 = \bar{\rho}_2, r > 0.25 = \frac{\mu(1 - m)L_3}{1 + cL_3},$ $\mu\eta > 2.4783 = \frac{e(\beta + d)[1 + b(1 - m)l_1]}{\beta(1 - m)l_1}$ and the positive equilibrium $E^\star = (1.9487, 0.1998, 0.3996)$. By Theorem 4.2, $E^\star$ is globally attractive (see Fig. 9).
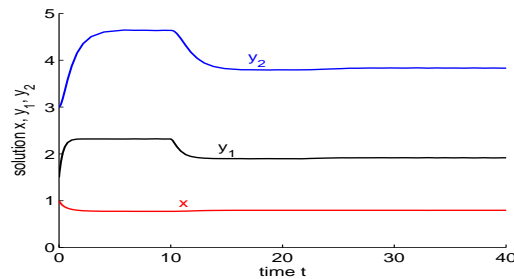
Figure 5: $\mu = 20, \tau = 10, E^* = (0.7953, 1.9162, 3.8323)$ of system (1.4) is locally asymptotically stable.
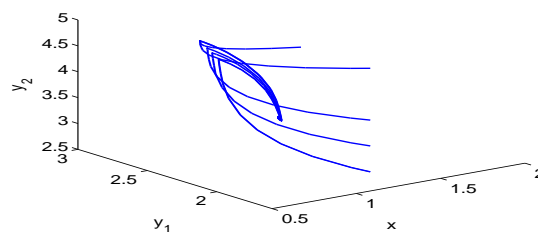

Figure 6: $\mu = 20, \tau = 10, E^* = (0.7953, 1.9162, 3.8323)$ of system (1.4) with different initial values is locally asymptotically stable.
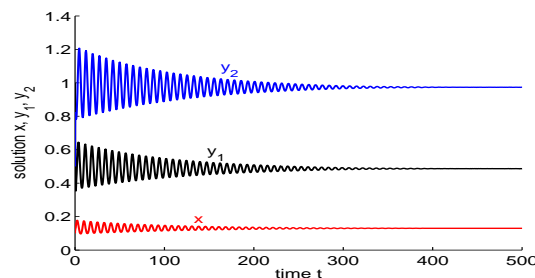

Figure 7: $\tau = 0.9 < 0.9429 = \tau_0, \mu = 36, E^* = (0.1302, 0.4868, 0.9735)$ of system (1.4) is locally asymptotically stable, system (1.4) undergoes a Hopf bifurcation at E^* when τ_0.

Example 6.3. In (1.4), let $r = 10$, $\mu = 6$, $c = e = 0.5$, $a = b = d = 1$, $\eta = \beta = 2$ and $\tau = 0$. By simple computations, we find $\mu\eta = 10 > 0.75 = be(1 + \frac{d}{\beta})$, $1 - \mathfrak{R}_1 = 0.4432$, $1 - \mathfrak{R}_2 = 0.9919$. Note that, if $m \in [0, 0.9919)$, by calculation, we can obtain $m < 1 - \frac{ae(1+\frac{d}{\beta})}{r[\mu\eta - be(1+\frac{d}{\beta})]}$, $r < \min\{r_2, r_4\}$, then E^* is locally asymptotically stable when $\tau = 0$. Our simulations show that the constant prey refuge m plays an important role on the coexistence of prey-predator population (see Figs. 10 and 11). When m is small and increasing, the predator density increases, due to the fact that predators have sufficient preys available for predation, even though the refuge increases (see Fig. 12). But, as m crosses its threshold value, the predator density decreases with increasing m, the predators was unable to catch preys to sustain themselves and ultimately goes to extinction due to starvation (see Fig. 13).
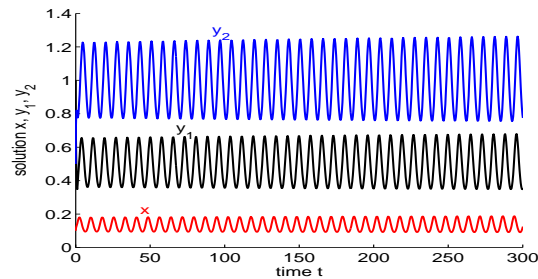
Figure 8: $\tau = 0.96 > 0.9429 = \tau_0, \mu = 36, E^* = (0.1302, 0.4868, 0.9735)$ of system (1.4) is locally asymptotically stable, system (1.4) undergoes a Hopf bifurcation at E^* when τ_0.
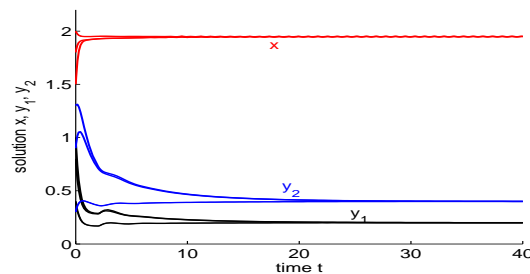

Figure 9: $\mu = 2, \tau = 2, E^* = (1.9487, 0.1998, 0.3996)$ of system (1.4) with different initial values is globally attractive.
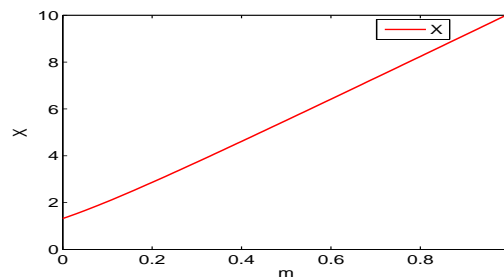

Figure 10: Bifurcation diagram of the prey species of model (1.4), m as the bifurcation parameter.

7 Conclusion

In this paper we investigate the influence of prey refuge on the dynamics of a Beddington-DeAngelis predator-prey system with stage structure for predator and time delay. Sufficient conditions are derived to ensure the predator-extinction and the locally asymptotically stability of positive equilibrium. Compared with [21] we get more precise conditions. Also, we find that time delay can cause a stable equilibrium to become unstable one, even Hopf bifurcation to occur, when time delay passes through some critical values. Furthermore, the persistence is investigated. After that, we study the global stability of the predator-extinction and positive equilibrium by constructing some suitable Lyapunov functionals.
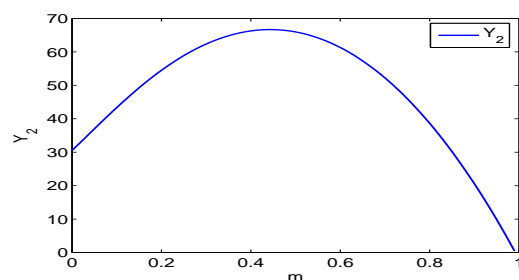
Figure 11: Bifurcation diagram of the predator species of model (1.4), m as the bifurcation parameter.
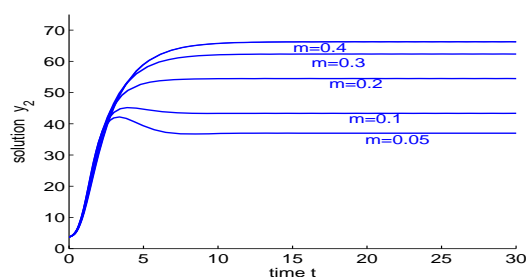

Figure 12: $\tau = 0, E^*$ is locally asymptotically stable. Change the values of the refuge m from zero to $1 - \mathfrak{R}_1 = 0.4432$, the predator species will increase.
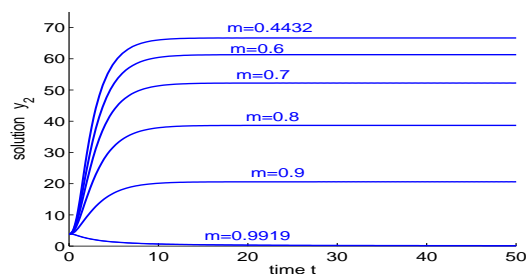

Figure 13: $\tau = 0, E^*$ is locally asymptotically stable. Change the values of the refuge m from $1 - \mathfrak{R}_1 = 0.4432$ to $0.9919 = 1 - \mathfrak{R}_2$, the predator species will decrease.

Also, we discuss the influence of prey refuge on the densities of predator species and prey species. When prey refuge m in the interval $(0, 1 - \mathfrak{R}_1)$, the density of predators will increase with prey refuge m, due to predator species having sufficient food for their predation with sufficiently small prey refuge m. Predator population attains its maximum when the prey refuge $m = 1 - \mathfrak{R}_1$. In case of larger values of m ($m > 1 - \mathfrak{R}_1$), this implies that predators species are less likely to catch prey, and the predator species deceases with the increasing of prey refuge m. Eventually, the predator population will be extinct when prey refuge $m = 1 - \mathfrak{R}_2$.

Competing interests The authors declare that there is no conflict of interests.

Authors' Contributions All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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