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Extreme points and support points of conformal mappings

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Abstract: There are three types of results in this paper. The first, extending a representation theorem on a conformal mapping that omits two values of equal modulus. This was due to Brickman and Wilken. They constructed a representation as a convex combination with two terms. Our representation constructs convex combinations with unlimited number of terms. In the limit one can think of it as an integration over a probability space with the uniform distribution. The second result determines the sign of $\Re L(\overline{z}_0(f(z))^2)$ up to a remainder term which is expressed using a certain integral that involves the Löwner chain induced by $f(z)$, for a support point $f(z)$ which maximizes $\Re L$. Here L is a continuous linear functional on $H(U)$, the topological vector space of the holomorphic functions in the unit disk $U = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Such a support point is known to be a slit mapping and $f(z_0)$ is the tip of the slit $\mathbb{C} - f(U)$. The third demonstrates some properties of support points of the subspace S_n of S. S_n contains all the polynomials in S of degree n or less. For instance such a support point $p(z)$ has a zero of its derivative $p'(z)$ on ∂U.

Keywords: extreme points, support points, conformal mappings, schlicht functions

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1 Introduction

Let $S := \{f \in H(U) \,|\, f(0) = f'(0) - 1 = 0, f \text{ is injective on } U := \{z \in \mathbb{C} \,|\, |z| < 1\}\}$. This is the family of normalized conformal mappings on the open unit disk U. S is a normal family and a compact subspace of the holomorphic functions on U, $H(U)$. The topology is taken to be that of uniform convergence on compact subsets of U. This topology is locally convex on $H(U)$. We recall the following standard definitions.

Definition 1.1. Let X be a topological vector space over the field of complex numbers. Let Y be a subset of X. A point $x \in Y$ is called an extreme point of Y if it has no representation of the form

$$x = t \cdot y + (1 - t) \cdot z, \quad 0 < t < 1,$$

as a proper convex combination of two distinct points y and z in Y. A point $x \in Y$ is called a support point of Y if there is a continuous linear functional L on X, not constant on Y, such that

$$\Re\{L(x)\} \geq \Re\{L(y)\} \text{ for all } y \in Y.$$

In this paper we will give an extension of a result of L. Brickman and D. R. Wilken. This result whose elegant proof is essentially due to Brickman and Wilken, can be found in [1]. See also [2].

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Another property we will prove is that for a function $f \in S$ that maximizes $\Re\{L(g)\}$, $g \in S$ where L is a linear continuous functional on $H(U)$. We have for any natural number $n \in \mathbb{Z}^+$, and for any positive real number $t \in \mathbb{R}^+$:

$$\Re L\left\{\overline{z}_0 f(z)^2 e^{-t}\right\} + \Re L\left\{\int_t^\infty \left\{\frac{e^s f(z,s)(k(s)f(z,s))^2}{1-k(s)f(z,s)}\right\} ds\right\} + o(e^{-t}) \le 0.$$

where $f(z_0)$ is the tip of the monotone slit $\mathbb{C} - f(U)$, $|z_0| = 1$ and $f'(z_0) = 0$. $f(z,s)$, $(z \in U, s \in \mathbb{R}^+)$ is the Löwner chain generated by the support point $f(z)$. As a general reference we will use the book [3], especially Chapter 9, 275-287 and Chapter 3, 76-113.
In the final section we will prove that the properties of the support points f of S, such as that f' has a zero on the boundary ∂U, are inherited by much smaller subfamilies of S such as S_n, the spaces of all the polynomials in S of degree n or less ($n \in \mathbb{Z}^+$). Clearly the S_n's are less geometric than S. Nevertheless, the birth of the slit structure of the image is starting to be visible by their support points. An important part of geometric function theory is the solution of extremal problems, such as coefficient problems, integral means problems, distortion problems and many other extremal problems. In order to apply functional analytic tools it is natural to identify the extreme points of S and its support points. By the Krein-Milman theorem, there is an extreme point of S among the support points associated with each linear continuous functional on $H(U)$. Knowing properties of support points might allow restricting the search for a solution to a much smaller family of points in S, than the whole of S. This is one aspect of the importance of such results.

2 A simple extension of a result of Brickman and Wilken

Here is a result of Brickman and Wilken, [1].

Theorem (Brickman and Wilken, [1]). *If a function $f \in S$ omits two values of equal modulus, then f has the form $f = t \cdot f_1 + (1-t) \cdot f_2$, $0 < t < 1$, where f_1 and f_2 are distinct functions in S which omit open sets.*

The clever proof given by Brickman and Wilken considers the image of f, $D = f(U)$ which omits α and β, $\alpha \ne \beta$. They define an analytic single-valued branch of $\Psi(w) = \{(w-\alpha)(w-\beta)\}^{1/2}$ in D and prove that the two functions $w \pm \Psi(w)$ are univalent and have disjoint images of D. They normalize to get two conformal mappings later on that belong to S

$$\Psi_1(w) = \frac{w + \Psi(w) - \Psi(0)}{1 + \Psi'(0)}, \quad \Psi_2(w) = \frac{w - \Psi(w) + \Psi(0)}{1 - \Psi'(0)}.$$

Now, by the identity

$$(1 + \Psi'(0)) \cdot \Psi_1(w) + (1 - \Psi'(0)) \cdot \Psi_2(w) = 2 \cdot w,$$

and with the compositions $f_1 = \Psi_1 \circ f$, $f_2 = \Psi_2 \circ f$ they obtain two functions f_1 and f_2 in S that satisfy $f(z) = t \cdot f_1(z) + (1-t) \cdot f_2(z)$ for $z \in U$, where

$$t = \frac{1}{2}(1 + \Psi'(0)).$$

So far they made no use of the important assumption $|\alpha| = |\beta|$. Coming to prove that $0 < t < 1$ this assumption is needed. Indeed if $\alpha = r \cdot e^{i\theta}$ and $\beta = r \cdot e^{i\phi}$, where $0 < \theta - \phi < 2\pi$ (by $\alpha \ne \beta$ and $|\alpha| = |\beta|$) a simple computation gives

$$\Psi'(0) = \pm \cos \frac{1}{2}(\theta - \phi).$$

Hence $0 < t < 1$ and the elegant proof is done.

Immediate consequences (see [3], Corollary 1 and Corollary 2 on page 287) are that each extreme point of

S and each support point of S have the monotonic modulus property. We show how to get more information on f, based on the above nice proof. The two functions $w \pm \Psi(w)$ are analytic and injective in D. In fact this is true in every domain that is complementary to two disjoint slits that start respectively at α and at β and extend to infinity. We note that if $w \notin \{\alpha, \beta\}$ then also $w \pm \Psi(w) \notin \{\alpha, \beta\}$ (for $w \pm \Psi(w) = \alpha \Rightarrow (w-\alpha)^2 = (w-\alpha)(w-\beta) \Rightarrow w = \alpha$ or $w = \beta$). Hence the following 4 compositions are analytic, single-valued and injective in D and omit $\{\alpha, \beta\}$,

$$\begin{aligned} g_1(w) &= w + \Psi(w) + \Psi(w + \Psi(w)),\\ g_2(w) &= w + \Psi(w) - \Psi(w + \Psi(w)),\\ g_3(w) &= w - \Psi(w) + \Psi(w - \Psi(w)),\\ g_4(w) &= w - \Psi(w) - \Psi(w - \Psi(w)). \end{aligned}$$

These 4 functions have disjoint images (for $\xi + \Psi(\xi)$ and $\eta - \Psi(\eta)$ are disjoint, so $\eta = w_2 + \Psi(w_2)$, $\xi = w_1 + \Psi(w_1)$ give us the conclusion that g_1, g_3 are disjoint. Similarly $\eta = w_2 - \Psi(w_2)$, $\xi = w_1 + \Psi(w_1)$ show that g_1, g_4 are disjoint. Also $\xi + \Psi(\xi)$ is injective hence g_2, g_3 are disjoint because $w_1 + \Psi(w_1) \neq w_2 - \Psi(w_2)$, again because the disjointness of the functions of Brickman and Wilken.). Clearly we have

$$\sum_{j=1}^{4} g_j(w) = 4w.$$

We define for $1 \le j \le 4$ and $w \notin \{\alpha, \beta\}$,

$$h_j(w) = \frac{g_j w) - g_j(0)}{g_j'(0)},$$

then

$$\sum_{j=1}^{4} g_j'(0) \cdot h_j(w) = \sum_{j=1}^{4} g_j(w) - \sum_{j=1}^{4} g_j(0) = 4w,$$

and

$$\sum_{j=1}^{4} \frac{1}{4} g_j'(0) = 1 \quad \text{by} \quad \sum_{j=1}^{4} g_j'(w) = 4.$$

We conclude that if for $1 \le j \le 4$ we have $g_j'(0) > 0$ then

$$w = \sum_{j=1}^{4} \left(\frac{1}{4} g_j'(0) \right) h_j(w)$$

is a strict convex combination (no zero coefficients) of the h_j, $1 \le j \le 4$. Thus if $f \in S$ omits the values α, β so that $g_j'(0) > 0$ for $1 \le j \le 4$, then f has the following representation

$$f = \alpha_1 \cdot f_1 + \alpha_2 \cdot f_2 + \alpha_3 \cdot f_3 + \alpha_4 \cdot f_4,$$

where $0 < \alpha_j < 1$, $\sum_{j=1}^{4} \alpha_j = 1$ and f_j are distinct functions in S that omit non-empty open sets. Here, as in Brickman and Wilken's proof, $f_j = g_j \circ f$, $1 \le j \le 4$. So we need to prove that $g_j'(0) > 0$ for $1 \le j \le 4$. We, once more, will make a use in the assumption $|\alpha| = |\beta|$ (which was already used by Brickman and Wilken in the first step of our iteration). Let us compute $g_j'(0)$.

$$g_j'(w) = 1 \pm \Psi'(w) \hat{\mp} (1 \pm \Psi'(w)) \Psi'(w \pm \Psi(w)),$$

where $\hat{\mp}$ are signs not synchronized with the other sign changes in the expression.

$$g_j'(w) = (1 \pm \Psi'(w)) \cdot (1 \hat{\mp} \Psi'(w \pm \Psi(w))),$$

$$g_j'(0) = (1 \pm \Psi'(0)) \cdot (1 \hat{\mp} \Psi'(\pm \Psi(0))).$$

Now we have

$$\Psi(0) = (\alpha\beta)^{1/2}, \quad \Psi'(0) = -\left(\frac{\alpha + \beta}{2(\alpha\beta)^{1/2}} \right), \Psi'(\pm\Psi(0)) = -\left(\frac{\alpha^{1/2} \mp \beta^{1/2}}{2\{\mp(\alpha\beta)^{1/2}\}^{1/2}} \right).$$

Hence

$$1 \pm \Psi(0) = 1 \mp \left(\frac{\alpha+\beta}{2(\alpha\beta)^{1/2}}\right) = 2(\Psi'(\pm\Psi(0)))^2,$$

$$g_j'(0) = 2\Psi'(\pm\Psi(0))^2(1\widehat{\mp}\Psi'(\pm\Psi(0))).$$

We denote $A = \Psi'(\pm\Psi(0))$ and then we need $0 < 2 \cdot A^2 \cdot (1 \mp A)$. This happens when $-1 < A < 1$ and so $-1 < \Psi'(\pm\Psi(0)) < 1$. This means that

$$-1 < \left(\frac{\alpha^{1/2} \mp \beta^{1/2}}{2\{\mp(\alpha\beta)^{1/2}\}^{1/2}}\right) < 1,$$

for if $\alpha = re^{i\theta}$, $b = re^{i\phi}$, then:

$$\frac{e^{i\theta/2} \mp e^{i\phi/2}}{2\{\mp e^{i(\theta+\phi}/2\}^{1/2}} = \sin\left(\frac{\theta-\phi}{4}\right) \text{ or } \cos\left(\frac{\theta-\phi}{4}\right).$$

and we already know that this is indeed the case when $|\alpha| = |\beta|$. This proves the case $n = 2$ in our general theorem below.

Theorem 2.1. *If the function $f \in S$ omits two values of equal modulus, and if n is a natural number, $n \in \mathbb{Z}^+$, then f has the form*

$$f = \sum_{j=1}^{2^n} \alpha_j \cdot f_j,$$

where $0 < \alpha_j < 1$, $1 \le j \le 2^n$, $\sum_{j=1}^{2^n} \alpha_j = 1$ and where f_j, $1 \le j \le 2^n$ are different functions in S that omit (each) open non-empty sets

Proof. We denote $D = f(U)$ and we assume that $\alpha, \beta \notin D$, $\alpha \ne \beta$, $|\alpha| = |\beta|$. We define an analytic and single-valued function in D by $\Psi(w) = \{(w-\alpha)(w-\beta)\}^{1/2}$ and denote two more functions in D, $\Psi_1(w) = w + \Psi(w)$, $\Psi_2(w) = w - \Psi(w)$. We will define a sequence of n sequences of functions. In the j'th sequence there will be 2^j functions. The first sequence is: $g_{11} = \Psi_1, g_{12} = \Psi_2$. We now assume that $j > 1$ and that the $(j-1)$'st sequence is: $g_{(j-1)k}$, $k = 1, 2, \dots, 2^{j-1}$. Then the j'th sequence is:

$$\begin{cases} g_{jk}(w) & = (\Psi_1 \circ g_{(j-1)k})(w) \\ g_{j(k+2^{j-1})}(w) & = (\Psi_2 \circ g(_{(j-1)k})(w) \end{cases} \quad 1 \le k \le 2^{j-1}.$$

The functions in our series are injective ($g_{jk}(w_1) = g_{jk}(w_2) \Rightarrow (\Psi_l \circ g_{(j-1)k})(w_1) = (\Psi_l \circ g_{(j-1)k})(w_2)$ for $l = 1$ or $l = 2$. $\Rightarrow g_{(j-1)k}(w_1) = g_{(j-1)k}(w_2) \Rightarrow w_1 = w_2$ inductively). The functions are pairwise disjoint in each of the n sequences (functions within the same sequence). For if $g_{jk}(w_1) = g_{jk}(w_2)$ then there can be only two possibilities:

(i) $(\Psi_s \circ g_{(j-1)k})(w_1) = (\Psi_s \circ g_{(j-1)l})(w_2)$ where $s = 1$ or $s = 2$. But Ψ_l is injective and hence $g_{(j-1)k}(w_1) = g_{(j-1)l}(w_2)$ and we use induction.

(ii) $(\Psi_1 \circ g_{(j-1)k})(w_1) = (\Psi_2 \circ g_{(j-1)l})(w_2)$ but Ψ_1 and Ψ_2 are disjoint and hence again $g_{(j-1)k}(w_1) = g_{(j-1)l}(w_2)$. In particular for the n'th sequence we have: the functions g_{nk}, $1 \le k \le 2^n$ are analytic, single-valued, injective and disjoint in $D \subseteq \mathbb{C} - \{\alpha, \beta\}$. Also we have

$$\sum_{k=1}^{2^n} g_{nk}(w) = 2^n \cdot w.$$

For we can use inductive argument as follows

$$\sum_{k=1}^{2^n} g_{nk}(w) = \sum_{k=1}^{2^{n-1}} \{(\Psi_1 \circ g_{(n-1)k})(w) + (\Psi_2 \circ g_{(n-1)k})(w)\} =$$

$$= 2\sum_{k=1}^{2^{n-1}} g_{(n-1)k}(w) = 2\cdot(2^{n-1}\cdot w) = 2^n\cdot w.$$

We define for $1 \le j \le 2^n$,

$$h_j(w) = \frac{g_{nj}(w) - g_{nj}(0)}{g'_{nj}(0)},$$

and then

$$\sum_{j=1}^{2^n} g'_{nj}(0)\cdot h_j(w) = 2^n w, \text{ and } \sum_{j=1}^{2^n} 2^{-n} g'_{nj}(0) = 1,$$

where the second identity originates in

$$\sum_{j=1}^{2^n} g'_{nj}(w) = (2^n\cdot w)' = 2^n.$$

We conclude that if for $1 \le j \le 2^n$ we have $g'_{nj}(0) > 0$, then

$$w = \sum_{j=1}^{2^n} (2^{-n} g'_{nj}(0))\cdot h_j(w),$$

the usual convex combination with positive coefficients of the $h_j(w)$'s. If this is the case, we define $\alpha_j = 2^{-n} g'_{nj}(0)$, $f_j = h_j \circ f$, $1 \le j \le 2^n$ and we get the convex representation $f = \sum_{j=1}^{2^n} \alpha_j \cdot f_j$ that we were looking for. For it is obvious that each $f_j \in S$ and those functions omit open non-empty sets, for the h_j do, because the g_{nj}'s are disjoint. Thus we need to prove that $g'_{nj}(0) > 0$ for $1 \le j \le 2^n$. This follows by induction and by the assumption that $|\alpha| = |\beta|$, $\alpha \ne \beta$. We note the following

$$g'_{nj}(0) = \Psi'_s(g_{(n-1)j}(0))\cdot g'_{(n-1)j}(0) = \{1 \pm \Psi'(g_{(n-1)j}(0))\}\cdot g'_{(n-1)j}(0) =$$

$$= \left\{1 \pm \frac{2g_{(n-1)j}(0) - \alpha - \beta}{2\{(g_{(n-1)j}(0) - \alpha)(g_{(n-1)j}(0) - \beta)\}^{1/2}}\right\}\cdot g'_{(n-1)j}(0) > 0.$$

We elaborate a bit more this final part of the proof – the proof that $g'_{nj}(0) > 0$ is inductive (on n). It is convenient to denote $X_n = g_{nj}(0)$ (j is fixed) and the induction assumption is that $|X_n - \alpha| = |X_n - \beta|$. By $X_n = X_{n-1} \pm (X_n - \alpha)^{1/2}(X_n - \beta)^{1/2}$ we get

$$\begin{cases} |X_n - \alpha| = |X_{n-1} - \alpha|^{1/2}|(X_{n-1} - \alpha)^{1/2} \pm (X_{n-1} - \beta)^{1/2}|, \\ |X_n - \beta| = |X_{n-1} - \beta|^{1/2}|(X_{n-1} - \alpha)^{1/2} \pm (X_{n-1} - \beta)^{1/2}|, \end{cases}$$

and hence $|X_n - \alpha| = |X_n - \beta|$ for all n. Hence

$$\Psi'(g_{nj}(0)) = \frac{2X_n - \alpha - \beta}{2\{(X_n - \alpha)(X_n - \beta)\}^{1/2}} = \frac{(X_n - \alpha) + (X_n - \beta)}{2\{(X_n - \alpha)(X_n - \beta)\}^{1/2}},$$

and we conclude that indeed $-1 < \Psi'(g_{nj}(0)) < 1$. □

The construction in the proof of Theorem 2.1 applies to any natural number $n \in \mathbb{Z}^+$. A natural question is whether when $n \to \infty$ it converges to some kind of, say, an integral representation of the function $f \in S$ that omits $\{\alpha, \beta\}$, where as usual $\alpha \ne \beta$, $|\alpha| = |\beta|$. To start with, we inquire if a recursion such as the one we have $g_{k+1}(w) = w + \Psi(g_k(w))$ or $g_{k+1}(w) = w - \Psi(g_k(w))$ converges (the sign is chosen at each stage arbitrarily). We first try to solve for g in $g(w) = w + \Psi(g(w))$ or $g(w) = w - \Psi(g(w))$. We immediately note the following,

Proposition 2.2. *Let us consider the following functions that result by applying finitely many times recursions of the form*

$$g_0(w) = w,\ g_{k+1}(w) = w \pm \Psi(g_k(w)),\ k = 0, 1, 2, \ldots.$$

where at each step th sign + or – is chosen arbitrarily. Then all the resulting functions have a unique fixed-point which is the same for all of them. This fixed-point is the rational function

$$g(w) = \frac{w^2 - \alpha \cdot \beta}{2w - \alpha - \beta}.$$

Proof. Solving for $g = w \pm \{(g-\alpha)(g-\beta)\}^{1/2}$ amounts in the equation $(g-w)^2 = (g-\alpha)(g-\beta)$ regardless of the sign. This last equation is linear in g, $-2wg + w^2 = -(\alpha+\beta)g + \alpha\beta$ and it's (unique) solution is

$$\frac{w^2 - \alpha \cdot \beta}{2w - \alpha - \beta}.$$

The same is true when we solve the fixed-point equation of higher members of the recursion. For example, solving for $g = w \pm \Psi(w \pm \Psi(g))$, is independent of the sign choices. It leads to

$$(\alpha-\beta)^2(w^2 - \alpha\beta) = (\alpha-\beta)^2(2w - \alpha - \beta)g. \qquad \square$$

We conclude this section by noting that in passing with the sum of 2^n elements $\sum_{j=1}^{2^n} g'_{nj}(0)h_j(w) = 2^n \cdot w$ to the next sum, that of 2^{n+1} elements, $\sum_{j=1}^{2^{n+1}} g'_{(n+1)j}(0)\tilde{h}_j(w) = 2^{n+1} \cdot w$, each element in the former sum gave birth to two descendents $w + \Psi(g_{nj}(w))$ and $w - \Psi(g_{nj}(w))$. So in a sense, each of the elements in a particular sum (say the one with 2^n elements) developed from a well-defined chain of elements in the former (smaller) sums, in a way that resembles partial sums in s series development. When $n \to \infty$ we can interpret our recursive process as integrating all these multitude of elements that can be thought of as the values of a random variable over a probability space with the uniform distribution.

3 One more property of support points of *S*

We recall that the space $H(U)$ is a linear topological locally convex space. The normalized conformal mappings $S \subset H(U)$ is a compact topological subspace of $H(U)$. The topology is that of uniform convergence on compact subsets. If $f \in S$ is a support point of S that corresponds to the continuous linear functional L on $H(U)$, then by the definition $\Re L(f) = \max_{g \in S} \Re L(g)$. The complement of the image of f, $\Gamma = \mathbb{C} - f(U)$ is an analytic curve having the property of increasing modulus and having the $\pi/4$-property, i.e. the angle between the segment that connects the origin to the tip of Γ is at most $\pi/4$. Moreover, Γ has an asymptotic direction at ∞. There is a point $z_1 \in \partial U$ such that f is analytic on $\overline{U} - \{z_1\}$ and has a pole of order 2 at $z = z_1$. Also, if w_0 is the tip of the slit Γ, then there is a point $z_0 \in \partial U$ such that $w_0 = f(z_0)$, and $f'(z_0) = 0$. If the functional L is not constant on S (as we assume throughout) then $L(f^2) \neq 0$ as is well known. In fact this was used in order to prove that the slit has an asymptotic direction at infinity. See [4]. Also, in [3], Theorem 10.4 on page 307, and Theorem 10.5 on page 311. It is here that we go further and prove a family of inequalities that involve $\Re L(\overline{z}_0^j (f(z)^{j+1})$.

Theorem 3.1. *Let L be a continuous linear functional on $H(U)$ which is not constant on S. Let $f \in S$ satisfy the equation $\Re L(f) = \max_{g \in S} \Re L(g)$, and suppose that $|z_0| = 1$, $f'(z_0) = 0$. Then for any natural number $n \in \mathbb{Z}^+$, and for any positive real number $t \in \mathbb{R}^+$:*

$$\Re L\left\{\overline{z}_0 f(z)^2 e^{-t}\right\} + \Re L\left\{\int_t^{\infty}\left\{\frac{e^s f(z,s)(k(s)f(z,s))^2}{1 - k(s)f(z,s)}\right\} ds\right\} + o(e^{-t}) \leq 0.$$

Proof. Since $\Gamma = \mathbb{C} - f(U)$ is a slit, we can embed f inside a Löwner chain. We briefly recall this standard procedure (see Chapter 3 in [3], 76-92). One chooses a parametric representation of Γ, $w = \Psi(t)$, $0 \leq t < \infty$ so that $\Psi(0) = f(z_0)$, $\Psi(s) \neq \Psi(t)$ for $s \neq t$. Also, if Γ_t is the tail of Γ from $\Psi(t)$ to ∞, then $g(z,t)$ is the Riemann mapping of U onto $\mathbb{C} - \Gamma_t$ so that $g(0,t) = 0$, $g'(0,t) > 0$ and we have:

$$g(z,t) = e^t\left\{z + \sum_{n=2}^{\infty} b_n(t) z^n\right\}, \quad 0 \leq t < \infty.$$

We define

$$f(z,t) = g^{-1}(f(z),t) = e^{-t}\left\{z + \sum_{n=2}^{\infty} a_n(t)z^n\right\}.$$

Then $f(z,t)$ is called a Löwner chain and it satisfies:

$$\frac{\partial f(z,t)}{\partial t} = -f(z,t)\cdot\frac{1+k(t)f(z,t)}{1-k(t)f(z,t)},$$

$$f(z,0) \equiv z, \quad \forall z \in U,$$

$$\lim_{t\to\infty} e^t f(z,t) \equiv f(z), \quad \forall z \in U,$$

where the limit is uniform on compact subsets of U. The point $1/k(t) = \overline{k(t)}$ is that point on ∂U that is mapped by $f(z,t)$ onto the tip of Γ_t. We note that $e^t f(z,t) \in S$, $0 \le t < \infty$ and so:

(1) $\Re L(e^t f(z,t) - f(z)) \le 0$, $0 \le t < \infty$.

(2) On the other hand we have:

$$f(z) - e^t f(z,t) = \lim_{T\to\infty}\left\{e^T f(z,T) - e^t f(z,t)\right\} = \left[e^s f(z,s)\right]_{s=t}^{\infty} = \int_t^{\infty} h(z,s)ds,$$

where $\int h(z,s)ds = e^s f(z,s)$.

(3) By differentiation:

$$h(z,s) = \frac{\partial}{\partial s}\left\{e^s f(z,s)\right\} = e^s f(z,s) + e^s\frac{\partial f(z,s)}{\partial s} =$$

$$= e^s f(z,s) - e^s f(z,s)\frac{1+k(s)f(z,s)}{1-k(s)f(z,s)} = -e^s f(z,s)\frac{2k(s)f(z,s)}{1-k(s)f(z,s)}.$$

(4) From the equations in (1), (2) and (3) we conclude that:

$$\Re L\left(\int_t^{\infty}\left\{e^s f(z,s)\frac{k(s)f(z,s)}{1-k(s)f(z,s)}\right\}ds\right) \le 0.$$

We recall that $\lim_{s\to\infty} k(s) = \overline{z}_0$ and also $\lim_{s\to\infty} e^s f(z,s) = f(z)$ uniformly on compact subsets of U. So we can write $e^s f(z,s) = f(z) + \epsilon(s)$, where $\lim_{s\to\infty}\epsilon(s) = 0$. Also $k(s) = \overline{z}_0 + \delta(s)$, where $\lim_{s\to\infty}\delta(s) = 0$. Hence:

$$e^s f(z,s)\frac{k(s)f(z,s)}{1-k(s)(\overline{z}_0+\delta(s))e^{-s}(f(z)+\epsilon(s))f(z,s)} =$$

$$= (f(z)+\epsilon(s))\frac{(\overline{z}_0+\delta(s))e^{-s}(f(z)+\epsilon(s))}{1-(\overline{z}_0+\delta(s))e^{-s}(f(z)+\epsilon(s))} =$$

$$= \overline{z}_0 f(z)^2 e^{-s} + \left(\overline{z}_0(2\epsilon(s)f(z)+\epsilon(s)^2) + \delta(f(z)+\epsilon(s))^2\right)e^{-s} +$$

$$+ \frac{e^s f(z,s)(k(s)f(z,s))^2}{1-k(s)f(z,s)}.$$

Integrating between t and ∞ we obtain the identity:

$$\int_t^{\infty}\left\{e^s f(z,s)\frac{k(s)f(z,s)}{1-k(s)f(z,s)}\right\}ds = \overline{z}_0 f(z)^2 e^{-t} + \int_t^{\infty}\left\{\frac{e^s f(z,s)(k(s)f(z,s))^2}{1-k(s)f(z,s)}\right\}ds +$$

$$+ \int_t^{\infty}\left(\overline{z}_0(2\epsilon(s)f(z)+\epsilon(s)^2) + \delta(f(z)+\epsilon(s))^2\right)e^{-s}ds.$$

The last integral is $o(e^{-t})$ for $t \to \infty$. Hence using the inequality in (4), we obtain:

$$\Re L\left\{\overline{z}_0 f(z)^2 e^{-t}\right\} + \Re L\left\{\int_t^{\infty}\left\{\frac{e^s f(z,s)(k(s)f(z,s))^2}{1-k(s)f(z,s)}\right\} ds\right\} + o(e^{-t}) \le 0.$$

This proves the theorem. □

Remark 3.2. It is not possible to deduce from the inequality in Theorem 3.1 that

$$\Re L\left\{\overline{z}_0 f(z)^2 e^{-t}\right\} + o(e^{-t}) \le 0,$$

when $t \to \infty$ as the author wrongly thought in the first version of this paper. The author thanks the referee for his remark and insight on that matter. Here is a simple example that shows that such an inequality can not be true. Let $L(a_0 + a_1 z + a_2 z^2 + \ldots)$ be the third coefficient functional on $H(U)$. Then as is well known, L has two support points in S, $f_1(z) = z/(1-z)^2$ the Koebe function and its rotation $f_2(z) = -f_1(-z) = z/(1+z)^2$. For $f_1(z)$ we have $z_0 = -1$ $(f_1'(-1) = 0)$, and for $f_2(z)$, $z_0 = 1$. Clearly $L(f_1^2) = L(f_2^2) = 1$. Hence $\Re L(\overline{z}_0 f_1^2) = -1 \cdot 1 = -1 < 0$, but $\Re L(\overline{z}_0 f_2^2) = 1 \cdot 1 = 1 > 0$.

4 Properties of support points are inherited by less geometric families of mappings

If $f \in S$ is a support point of f that corresponds to the continuous linear functional L on $H(U)$, then $\Gamma = \mathbb{C} - f(U)$ is an analytic curve, called the slit of f. It starts at its tip w_0 and monotonically extends to infinity. The tip w_0 has a single pre-image z_0 on ∂U, $w_0 = f(z_0)$ and $f'(z_0) = 0$. In this section we will see that the tip of the slit, w_0, already appears in a support point of univalent polynomials.

Definition 4.1. Let $n \in \mathbb{Z}^+$ be a natural number. The family of all the polynomials in S, of degree n or less will be denoted by S_n. We note that S_n is a compact subspace of $H(U)$.

Example 4.2. To demonstrate Definition 4.1 we note that $S_1 = \{z\}$ and $S_2 = \{z + \alpha z^2 \mid |\alpha| \le 1/2\}$.

Here is our result.

Theorem 4.3. *Let $n > 1$ be a natural number. Let $L \in H(U)'$ be a continuous linear functional. Then either L is constant on S_n, or, if L is not constant on S_n, then if $p \in S_n$ solves the following extremal problem $|L(p)| = \max_{f \in S_n} |L(f)|$, then $p'(z)$ has a zero on ∂U.*

Proof. Let us suppose that L is not constant on S_n. Then there are two polynomials $q_2, q_1 \in S_n$ such that $q_2 - q_1 \notin \mathbb{C}$ and $L(q_2) \ne L(q_1)$. Using the normalization of elements in S_n we have $q_1(0) = q_2(0) = 0$ and $q_1'(0) = q_2'(0) = 1$. Hence $q(0) = q'(0) = 0$ and $L(q) = L(q_2 - q_1) = L(q_2) - L(q_1) \ne 0$. We proceed with a Rouche's type of principle for injectivity.

Lemma 4.4. *If $f(z) = z + a_2 z^2 + \ldots \in S$ is analytic in a neighborhood of $\overline{U}$ such that $f'(z)$ does not have zero on ∂U, then for any function $g(z)$ analytic in a neighborhood of $\overline{U}$ there exists a $\delta > 0$ (depending on g), so that if $|w_0| < \delta$, then $f(z) + w_0 \cdot g(z)$ is injective on $\overline{U}$.*

Proof of Lemma 4.4.
Let us fix w_0. We denote $F(z) = f(z) + w_0 \cdot g(z)$. Then for any $z, w \in \overline{U}$ we have:

$$|F(z) - F(w)| = |f(z) - f(w) + w_0(g(z) - g(w))| =$$

$$= |f(z) - f(w)| \times \left|1 + w_0\left(\frac{g(z) - g(w)}{f(z) - f(w)}\right)\right|,$$

where for $z = w$ we agree to interpret

$$\frac{g(z)-g(w)}{f(z)-f(w)} = \frac{g'(z)}{f'(z)}.$$

Since f is in fact injective on $\overline{U}$ (because $f'(z)$ does not vanish on U) we deduce that for $z \neq w$, $f(z) - f(w) \neq 0$ and so it is sufficient to prove that

$$\frac{g(z)-g(w)}{f(z)-f(w)},$$

is bounded on $\overline{U} \times \overline{U}$. We write the following identity:

$$\frac{g(z)-g(w)}{f(z)-f(w)} = \left(\frac{g(z)-g(w)}{z-w}\right) \Big/ \left(\frac{f(z)-f(w)}{z-w}\right).$$

So it is sufficient to prove that:

$$\max_{\overline{U}\times\overline{U}} \left|\frac{g(z)-g(w)}{z-w}\right| \leq M < \infty$$

and

$$\min_{\overline{U}\times\overline{U}} \left|\frac{f(z)-f(w)}{z-w}\right| \geq \epsilon > 0.$$

For the minimum. If the estimate is false then there is a sequence $(z_k, w_k) \in \overline{U} \times \overline{U}$ so that

$$\frac{f(z_k)-f(w_k)}{z_k - w_k} \to 0.$$

The set $\overline{U} \times \overline{U}$ is compact in $\mathbb{C} \times \mathbb{C}$ and so we may assume that $z_k \to a$ and $w_k \to b$ and we get in the case $a \neq b$ the equation

$$\frac{f(a)-f(b)}{a-b} = 0,$$

which contradicts the injectivity of f in $\overline{U}$. If $a = b$ we get a contradiction to the assumption that $f'(z)$ does not have a zero in $\overline{U}$. For the maximum. Using arguments similar to those above we get $a, b \in \overline{U}$ such that

$$\left|\frac{g(a)-g(b)}{a-b}\right| = \infty.$$

If $a \neq b$ this contradicts the fact that g is analytic in a neighborhood of $\overline{U}$. If $a = b$ this contradicts the fact that g' is analytic in a neighborhood of $\overline{U}$. The proof of Lemma 4.4 is now completed. □

We now conclude the proof of Theorem 4.3 as follows. Using Lemma 4.4 there exists an $\epsilon > 0$ such that $p(z) + \epsilon e^{i\theta} q(z) \in S_n$, for any $0 \leq \theta < 2\pi$. We use here the assumption that $p'(z)$ does not have a zero on $\overline{U}$. Since p is solving the extremal problem for L we conclude that $|L(p)| + \epsilon|L(q)| \leq |L(p)|$ (one needs to choose properly the θ). By $\epsilon > 0$ it follows that $L(q) = 0$. This contradicts the assumption on q. Hence $p'(z)$ must have a zero on ∂U. □

Remark 4.5. (1) In particular for the coefficients functionals, $L(p) = p^{(j)}(0)/j!$ where j is in $1 < j \leq n$ we note that $L(z^j) = 1 \neq 0$, so we can use in Theorem 4.3 $q(z) = z^j$. We conclude that if $p(z)$ maximizes $|p^{(j)}(0)/j!|$ then $p'(z)$ must have a zero on ∂U.
(2) If, as in Theorem 4.3, $p'(e^{i\theta}) = 0$, then necessarily $p''(e^{i\theta}) \neq 0$ for $p \in S_n$. Thus p' has only simple zeros on ∂U and at least one. The reason that $p''(e^{i\theta}) \neq 0$ for $p \in S_n$, is that otherwise $p(z)$ is locally at $e^{i\theta}$ equivalent to $(z - e^{i\theta})^3$. This means that it folds the tangent line to ∂U at $e^{i\theta}$ by more than π radians hence it can not be univalent in a U-neighborhood of $e^{i\theta}$.

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