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Augmented, free and tensor generalized digroups

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Abstract: The concept of generalized digroup was proposed by Salazar-Díaz, Velásquez and Wills-Toro in their paper "Generalized digroups" as a non trivial extension of groups. In this way, many concepts and results given in the category of groups can be extended in a natural form to the category of generalized digroups. The aim of this paper is to present the construction of the free generalized digroup and study its properties. Although this construction is vastly different from the one given for the case of groups, we will use this concept, the classical construction for groups and the semidirect product to construct the tensor generalized digroup as well as the semidirect product of generalized digroups. Additionally, we give a new structural result for generalized digroups using compatible actions of groups and an equivariant map from a group set to the group corresponding to notions of associative dialgebras and augmented racks.

Keywords: Digroups, Free and tensor groups, semidirect product, group actions

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1 Introduction

The digroup structure is introduced by M. Kinyon [2], R. Felipe [3] and K. Liu [4] as a non trivial extension of the concept of group, with the purpose of giving an answer to the so called Coquecigrue problem which is supposed to provide a generalization of the third Lie theorem for Leibniz algebras, see [5].

A slightly different structure studied in [1] is called generalized digroup. It doesn't request bilateral inverses for its elements. This concept is corresponding to what is called Digroups_1 in [6].

For digroups with bar units that generate bilateral inverses (see [7]) several authors propose different generalizations of the notion of digroup. For instance, in [8], J. D. H. Smith shows that any digroup with bilateral inverses is equivalent to what he calls a $(4+2)$-diquasigroup (Theorem 10.8). His proof uses digroups generated by two groups that act in a commutative way over a set. This idea is similar to a work developed in [9] which leads to express associative dialgebras in terms of bimodules over associative algebras and equivariant maps.

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In addition, in [2], M. Kinyon proves that any digroup generates a rack and it is natural to think that it can be extended to generalized digroups. Since any rack can be generated by a group acting over a set, with the action commuting with the conjugation and an equivariant map, that gives another motivation to explore what we call here *augmented generalized digroups*, a construction that provides another characterization of generalized digroups.

Due to the fact that augmented racks give set theorical solutions to the quantum Yang-Baxter equation (see [10]), and that augmented generalized digroups can be defined, it is possible to study relations between the Yang-Baxter equations and generalized digroups that could procure solutions. These ideas are being explored by the authors in a work in progress.

Finding free structures is a central problem in abstract algebra. For the case of dimonoids we can find constructions in several works, for example see [6, 8, 11–13]. The free generalized digroup is exhibited in the present work and it is done following Loday's ideas for free dimonoids (see [5]).

The semidirect product of groups (see [15]) induces our definition of tensor generalized digroup and its representations. The cyclic generalized digroup and generalized semidirect product result naturally from the discussion involved.

The paper is organized as follows. In Section 2, we review the basic theory and the notions of subdigroup and normality in the sense of [1] and we introduce the definitions of anti-homomorphisms and involutions over generalized digroups.

We finish Section 2 with the relation of generalized digroups with associative dialgebras and racks, we define augmented generalized digroups, and we also show that each generalized digroup can be expressed in such a way.

Section 3 is dedicated to study the construction of the free generalized digroup and to show some properties of this new structure.

In the last section, we introduce the notions of generalized tensor digroups and generating sets and we finish with the concept of the semidirect product of generalized digroups.

2 Some results about Generalized Digroups

In this section we briefly recall some definitions and results about generalized digroups, for a deeper study see [1]. We also review some properties and we introduce the notion of anti-homomorphism and involution for this structure. Finally we introduce the notion of augmented generalized digroups.

2.1 General results

We summon up the definition, some basic properties and a way to look up generalized digroups.

Definition 1. *A set D is a* **generalized digroup** *if it has two binary associative operations $\vdash$ and $\dashv$ over D, such that they satisfy the following conditions:*

1. $x \vdash (y \dashv z) = (x \vdash y) \dashv z$
2. $x \dashv (y \dashv z) = x \dashv (y \vdash z)$ *and* $(x \vdash y) \vdash z = (x \dashv y) \vdash z$
3. *There exists (at least) an element e in D, such that for all x in D, $x \dashv e = x = e \vdash x$.*
4. *For a fixed, but otherwise arbitrary, bar-unit e, we have that for each x in D there exist $x_{r_e}^{-1}$ and $x_{l_e}^{-1}$ in D (the right-inverse of x and the left-inverse of x, respectively) such that $x \vdash x_{r_e}^{-1} = e$ and $x_{l_e}^{-1} \dashv x = e$.*

It is clear that a group $(G, \cdot)$ can be seen as a generalized digroup by considering $\vdash = \cdot = \dashv$.

The elements that satisfy condition 3. are called **bar-units** and the set of them, denoted by E, is called the **halo** of D. For any bar-unit $\xi \in E$, we denote the sets of left and right inverses with respect to ξ by G_l^ξ and G_r^ξ, respectively.

A generalized digroup that consists only of bar units or that is a group is called a **trivial generalized digroup**, here we exhibit a non trivial generalized digroup.

Example 1. *Let* $(D, \vdash, \dashv)$ *be a generalized digroup defined by*

$\dashv$	x_0	x_1	x_2	x_3
x_0	x_0	x_0	x_2	x_2
x_1	x_1	x_1	x_3	x_3
x_2	x_2	x_2	x_0	x_0
x_3	x_3	x_3	x_1	x_1

$\vdash$	x_0	x_1	x_2	x_3
x_0	x_0	x_1	x_2	x_3
x_1	x_0	x_1	x_2	x_3
x_2	x_3	x_2	x_1	x_0
x_3	x_3	x_2	x_1	x_0

For this generalized digroup we have that the halo is $E = \{x_0, x_1\}$

Among many basic properties that are proved in [1], we state a couple of them that are being used later.

Proposition 1. *Let D be a generalized digroup and let e be a fixed bar unit.*

1. *For a given $x \in D$, we have that*

$$(x_{l_e}^{-1})_{l_e}^{-1} = (x_{r_e}^{-1})_{l_e}^{-1} = x \vdash e \text{ and } (x_{r_e}^{-1})_{r_e}^{-1} = (x_{l_e}^{-1})_{r_e}^{-1} = x \vdash e.$$

2. *For x, y in D, the inverses of the products are $(x \star y)_{l_e}^{-1} = y_{l_e}^{-1} \dashv x_{l_e}^{-1}$ and $(x \star y)_{r_e}^{-1} = y_{r_e}^{-1} \vdash x_{l_e}^{-1}$, where $\star$ represents any of the products.*

It is easy to see that the halo also corresponds to any of the following sets

$$E = \left\{x_{l_e}^{-1} \vdash x | x \in D\right\} = \left\{x \dashv x_{l_e}^{-1} | x \in D\right\} = \left\{x \in D | x_{l_e}^{-1} = x_{r_e}^{-1} = e\right\}. \tag{1}$$

Since the proof of the following theorem comes from the results given in [1], we omit it.

Theorem 1. *Let $(D, \vdash, \dashv)$ be a generalized digroup. For any $\xi \in E$, $(G_l^\xi, \dashv)$ and $(G_r^\xi, \vdash)$ are isomorphic groups with unit ξ. Moreover, it is true that for any $\xi, \zeta \in E$*

1. $G_l^\xi = \xi \dashv G_l^\zeta$ *and* $G_r^\zeta = G_r^\xi \vdash \xi$,
2. $G_l^\xi \cong G_l^\zeta \cong G_r^\xi \cong G_r^\zeta$,
3. $\xi \dashv D = G_l^\xi$ *and* $D \vdash \xi = G_r^\xi$, *wich implies* $G_l^\xi \dashv D = G_l^\xi$ *and* $D \vdash G_r^\xi = G_r^\xi$,

As it is shown in [1] a characterization of D is given below

$$D = \dot{\bigcup_{\xi \in E}} G_l^\xi = \dot{\bigcup_{\xi \in E}} G_r^\xi$$

In order to describe another way of looking at a generalized digroup let's recall that if D and D' are generalized digroups, a map $\phi : D \to D'$ is a **generalized digroup homomorphism** if for any $x, y \in D$

$$\phi(x \dashv y) = \phi(x) \dashv \phi(y) \quad \text{and} \quad \phi(x \vdash y) = \phi(x) \vdash \phi(y).$$

In addition, if ϕ is a bijection, then ϕ is a generalized isomorphism and D is isomorphic to D'.

Theorem 2. *Let D be a generalized digroup and let E be the set of bar units. For any $e \in E$, we have that E is a G_l^e-set with respect to the action defined by*

$$a \bullet_l \xi = a \vdash \xi \dashv a^{-1}, \quad \forall a \in G_l^e, \forall \xi \in E.$$

Moreover, $G^e_l \times E$ is a generalized digroup with operations

$$(a, \alpha) \vdash (b, \beta) := (a \dashv b, a \bullet_l \beta) \text{ and } (a, \alpha) \dashv (b, \beta) := (a \dashv b, \alpha)$$

The second characterization of generalized digroups is an extension of the results of M. Kinyon (see [2]) and F. Ongay (see [16]).

Theorem 3. *Let D, E and G^ξ_l be as in Theorem 2, then the map $\varphi_l : D \longrightarrow G^\xi_l \times E$, defined by $\varphi_l(x) = (\xi \dashv x, x \dashv x^{-1}_{l_\xi})$, provides a generalized digroup isomorphism with inverse function $\varphi_l^{-1} : G^\xi_l \times E \longrightarrow D$, given by $(a, \alpha) \mapsto \alpha \dashv a$.*

That is, a generalized digroup can be seen as a cartesian product between a G-set E and the group G, with set of bar units $\{e\} \times E$. With respect to such decomposition of a generalized digroup, next theorem describes a generalized digroup homomorphism.

Theorem 4. *Let $\Psi : D \longrightarrow D'$ be a generalized digroup homomorphism. Then, there exists an unique homomorphism $\Psi' : G^\xi_l \times E \longrightarrow G^{\xi'}_l \times E'$ such that the diagram*

$$\begin{array}{ccc} D & \xrightarrow{\Psi} & D' \\ {\scriptstyle \varphi_l}\downarrow & & \downarrow{\scriptstyle \varphi'_l} \\ G^l_\xi \times E & \xrightarrow[\Psi']{} & G^{\xi'}_l \times E' \end{array}$$

commutes, where $\Psi' \equiv (\varphi, \mu)$, with

1. *the map $\varphi : G^\xi_l \longrightarrow G^{\xi'}_l$, where $\varphi(a) = \xi' \dashv \Psi(a)$, is a group homomorphism.*
2. *the map $\mu : E \longrightarrow E'$, defined as $\mu(\alpha) = \Psi(\alpha)$ is an equivariant map, i.e.*

$$\mu(x \bullet \alpha) = \Psi(x) \bullet \mu(\alpha) \text{ and } \mu(a \bullet \alpha) = \varphi(a) \bullet \mu(\alpha),$$

for all $\alpha \in E$, all $x \in D$ and all $a \in G^\xi_l$.

Now we extend the notion of anti-homomorphism and involution, known in group theory, for generalized digroups.

Definition 2. *A map $\psi : D \to D'$ is called a generalized digroup **anti-homomorphism** if for any $x, y \in D$,*

$$\psi(x \dashv y) = \psi(y) \vdash \psi(x) \text{ and } \psi(x \vdash y) = \psi(y) \dashv \psi(x).$$

*An anti-homomorphism $x \mapsto x^\star$ from a digroup to itself is called an **involution** if it is its own inverse, i.e. $(x^\star)^\star = x$, for all $x \in D$.*

A similar result as Theorem 4 can be stated for generalized digroup anti-homomorphisms, we only have to check that the corresponding function $\Psi' = (\phi, \mu)$ is in fact an anti-homomorphism.

Some basic properties about involution are given in the following proposition.

Proposition 2. *Let D be a generalized digroup and let $\star$ be an involution over D.*

1. *For any $e \in E$, $e^\star$ is a bar unit and therefore $E = E^\star$.*
2. *Given $x \in D$ and $e \in E$, we have that $(x^{-1}_{l_e})^\star = (x^\star)^{-1}_{r_{e^\star}}$ and $(x^{-1}_{l_e})^\star = (x^\star)^{-1}_{l_{e^\star}}$.*

The next example shows that we don't always have the equality $e^\star = e$.

Example 2. *Let G be the group of all the Möbius transformations of the form $z \mapsto e^{i\theta} z$ acting on the unit circle S^1. Then the set $D = G \times S^1$ can be endowed as a generalized digroup by considering the products*

$$(g, z) \vdash (h, w) = (gh, g(w)) \text{ and } (g, z) \dashv (h, w) = (gh, h^{-1}(z)).$$

Now, we consider the involution $\star : D \to D$ *defined by* $(g, z)^\star = (g^{-1}, -z)$*. As we can see, the halo of* D *is* $\{id\} \times S^1$*. And so, for every* $z \in S^{-1}$, $(id, z)^\star = (id, -z) \neq (id, z)$*.*

We recall the concept of a **generalized subdigroup** of a generalized digroup D, denoted by $S \leq D$, a subset, such that with the products in D restricted to it, is a generalized digroup. It is proven in [1], section 3., that S must satisfy that there are two subsets Γ and Δ of it such that Δ is Γ-invariant respect to the action defined in Theorem 2, i.e. $\Gamma \bullet \Delta = \Delta$.

It is also useful to bring back the definition of normality, where for a generalized digroup $(D; \vdash, \dashv)$ and a subdigroup N, the latter is a **normal generalized subdigroup** of D, denoted by $N \trianglelefteq D$, if $x \vdash N = N \dashv x$, for any $x \in D$. Moreover, we have that

Lemma 1. *Let* D *be a generalized digroup and let* N *be a generalized subdigroup of* D*. Then* $N \trianglelefteq D$ *iff there exist a normal subgroup* Γ *of* G *and a set* $\Delta \subseteq E$ *such that* $N \cong \Gamma \times \Delta$ *and* $G \bullet \Delta = \Delta$*.*

2.2 Augmented generalized digroups

This subsection is dedicated to the new characterizacion of generalized digroups mentioned in the introduction.

In the proposition we show that any generalized digroup is equivalent to a generalized digroup generated by compatible actions and an equivariant map. We call the structure that satisfies the hypothesis **an augmented generalized digroup** and we denote it by $(G, X, \lambda, \rho, \pi)$.

Proposition 3. *Let* G *be a group, with unit* 1_G*, and let* X *be a* G*-set under the left and rigth compatible actions* λ *and* ρ*, i.e.* λ_g *commutes with* ρ_h*, for any* $g, h \in G$*. If there is an equivariant map* $\pi : X \to G$ *with respect to both actions such that* $\pi(X)$ *generates* G*, then* $(X, \vdash, \dashv)$ *is a generalized digroup with the operations*

$$x \vdash y = \lambda_{\pi(x)}(y) := \pi(x) \bullet_\lambda y \quad \text{and} \quad x \dashv y = \rho_{\pi(y)}(x) := x \bullet_\rho \pi(y).$$

In this case, $E = \{x \star y \mid \pi(x) \cdot \pi(y) = 1_G\}$*, where* $\star$ *represents either* $\vdash$ *or* $\dashv$*.*

Proof. The products $\vdash$ and $\dashv$ are associative because λ and ρ are actions. From the compatibility of the actions, condition 1 in the definition of generalized digroup follows. The equivariance of π, with respect to the actions λ and ρ, implies that

$$(x \vdash y) \vdash z = \pi(\pi(x) \bullet_\lambda y) \bullet_\lambda z = \big(\pi(x)\pi(y)\big) \bullet_\lambda z = \pi(x \bullet_\rho \pi(y)) \bullet_\lambda z = (x \dashv y) \vdash z$$

and therefore $(x \vdash y) \vdash z = (x \dashv y) \vdash z$, for all $x, y, z \in X$.

Similarly, $x \dashv (y \dashv z) = x \dashv (y \vdash z)$, for all $x, y, z \in X$; and condition 2 is satisfied.

Because $\pi(X)$ generates G, there are $x, y \in X$ such that $\pi(x) \cdot \pi(y) = 1_G$, and therefore $E \neq \emptyset$. Since $\pi(\xi) = 1_G$, then ξ is a bar unit and we have that E is the halo of X. Indeed, if $\pi(x) \cdot \pi(y) = 1_G$, then

$$\begin{aligned}(x \vdash y) \vdash z &= (\pi(x) \bullet_\lambda y) \vdash z = \pi(\pi(x) \bullet_\lambda y) \bullet_\lambda z \\ &= (\pi(x) \cdot \pi(y)) \bullet_\lambda z = 1_G \bullet_\lambda z = z\end{aligned}$$

and

$$\begin{aligned}z \dashv (x \vdash y) &= z \dashv \pi(\pi(x) \bullet_\lambda y) = z \bullet_\rho \pi(\pi(x) \bullet_\lambda y) \\ &= z \bullet_\rho (\pi(x) \cdot \pi(y)) = z \bullet_\rho 1_G = z\end{aligned}$$

The proof for $x \dashv y$ is analogous.

To get the inverses, let's assume that z is the inverse of w with respect to $\xi = x \vdash y$, then $w \dashv z = x \vdash y$, this is, $w \bullet_\rho \pi(z) = \pi(x) \bullet_\lambda y$ and so $(w \bullet_\rho \pi(z)) \bullet_\rho \pi(z)^{-1} = (\pi(x) \bullet_\lambda y) \bullet_\rho \pi(z)^{-1}$. Thus, $w = \pi(x) \bullet_\lambda y \bullet_\rho (\pi(z))^{-1}$.

In a similar way, we get the expressions for the inverses with respect to the bar units $\xi, \eta \in E$, with $\xi = x \vdash y$ and $\eta = x \dashv y$:

$$z_{l_\xi}^{-1} = \pi(x) \bullet_\lambda y \bullet_\rho (\pi(z))^{-1}, \quad z_{r_\xi}^{-1} = (\pi(z))^{-1} \bullet_\lambda \pi(x) \bullet_\lambda y,$$

$$z_{l_\eta}^{-1} = x \bullet_\rho \pi(y) \bullet_\rho (\pi(z))^{-1} \quad \text{and} \quad z_{r_\eta}^{-1} = (\pi(z))^{-1} \bullet_\lambda x \bullet_\rho \pi(y).$$

□

We show that any generalized digroup can be seen as in the previous proposition. Using that $G_l^e = e \dashv D$ and $G_r^e = D \vdash e$, for all $e \in E$ (see Theorem 1), and $E \dashv x = x \vdash E$, since E is a normal generalized subdigroup, we obtain the following result.

Theorem 5. *Let D be a generalized digroup. There exist a group G, two compatible actions over D, λ and ρ, and an equivariant map $\pi : D \to G$ such that D can be seen as an augmented generalized digroup $(G, D, \lambda, \rho, \pi)$.*

Proof. Given the factorization of a generalized digroup in terms of the set E and the group G_l^ξ, for an arbitrary $\xi \in E$, we have that $G := D/E$ is a group isomorphic to all groups of inverses and $[x] = E \dashv x = x \vdash E$, for any $x \in D$.

If we define the maps $\lambda : G \times D \to D$ and $\rho : D \times G \to D$ by

$$\lambda_{[x]}(y) = x \vdash y \quad \text{and} \quad \rho_{[x]}(y) = y \dashv x, \quad \forall x, y \in D,$$

we have that λ and ρ are compatible actions from G on D by identities 1. and 2. in Definition 1.

Finally, the projection map $\pi : D \to G$, defined by $\pi(x) = [x]$, is an equivariant map with respect to both actions and $\pi(D)$ generates G. It's easy to see for any $x, y \in D$ that

$$x \vdash y = \lambda_{\pi(x)}(y) \quad \text{and} \quad x \dashv y = \rho_{\pi(y)}(x).$$

□

Now we are going to show some basic properties respect to augmented generalized digroups.

In Theorem 4, for any generalized digroup homomorphism $\Psi : D \to D'$, the map $\varphi : G_\xi^l \to G_{\xi'}^l$, defined by $\varphi(\xi \dashv x) = \xi' \dashv \Psi(\xi \dashv x)$, is a group homomorphism. Since $\Psi(E) \subset E'$, then $\varphi(\xi \dashv x) = \xi' \dashv \Psi(x)$, for all $x \in D$, and therefore φ is a group homomorphism equivariant with respect to the generalized digroup homomorphism Ψ. In particular, if $\xi' = \Psi(\xi)$, then $\varphi(\xi) = \Psi(\xi)$ and $\varphi(\xi \dashv x) = \varphi(\xi) \dashv \Psi(x)$, for all $x \in D$.

Because $x \vdash E = E \dashv x$ and $G_\xi^l = \{\xi \dashv x | x \in D\}$, for all $\xi \in E$, we have that $[x] = E \dashv x$ and therefore $\varsigma([x]) = [\Psi(x)]$, for all $x \in D$, defines a group homomorphism from G to G'.

Moreover, this digroup homomorphism satisfies

$$\Psi\big(\lambda_{[x]}(y)\big) = \lambda'_{\varsigma[x]}\big(\Psi(y)\big) \quad \text{and} \quad \Psi\big(\rho_{[x]}(y)\big) = \rho'_{\varsigma[x]}\big(\Psi(y)\big),$$

for all $x, y \in D$. From these considerations we obtain the following result.

Theorem 6. *Let D, D' be generalized digroups and $\Psi : D \longrightarrow D'$ be a generalized digroup homomorphism. If $(G, D, \lambda, \rho, \pi)$ and $(G', D', \lambda', \rho', \pi')$ are the augmented generalized digroups obtained in Theorem 5, for D and D', respectively, then there is a group homomorphism $\varsigma : G \to G'$ such that for all $x \in D$ and all $g \in G$,*

1. *The homomorphism Ψ is ς-equivariant with respect to the left action λ and to the right action ρ i.e.*

$$\Psi(g \bullet_\lambda x) = \varsigma(g) \bullet_{\lambda'} \Psi(x) \;\; \textit{and} \;\; \Psi(x \bullet_\rho g) = \Psi(x) \bullet_{\rho'} \varsigma(g).$$

2. *The homomorphism ς is equivariant with respect to the maps π and π', i.e. $\varsigma(\pi(x)) = \pi'(\Psi(x))$.*

The previous result induces the following definition

Definition 3. *Let $(G, X, \lambda, \rho, \pi)$ and $(H, Y, \tau, \varrho, \Pi)$ be augmented generalized digroups. A pair (φ, Ψ), where $\varphi : \pi(X) \to \Pi(Y)$ and $\Psi : X \to Y$ are maps, is called an* ***augmented generalized digroup homomorphism*** *if*

1. *The map φ induces a group homomorphism $\widehat{\varphi} : G \to H$.*

2. *The map φ is Ψ-equivariant with respect to the maps π and Π, i.e. $\varphi(\pi(x)) = \Pi(\Psi(x))$, for all $x \in X$.*
3. *The map Ψ is φ-equivariant with respect to the group actions λ, ρ, τ, ϱ and the maps π, Π, i.e. for all $x, y \in X$, $\Psi(\pi(x) \bullet_\lambda y) = \varphi(\pi(x)) \bullet_\tau \Psi(y)$ and $\Psi(x \bullet_\rho \pi(y)) = \Psi(x) \bullet_\varrho \varphi(\pi(y))$.*

Lemma 2. *Let $(G, X, \lambda, \rho, \pi)$ and $(H, Y, \tau, \varrho, \Pi)$ be augmented generalized digroups. If $\Psi : X \to Y$ is a generalized digroup homomorphism, then there is a map $\varphi : \pi(X) \to \Pi(Y)$, such that (φ, Ψ) is an augmented generalized digroup homomorphism.*

Proof. Let $\varphi : \pi(X) \to \Pi(Y)$ be a map defined by $\varphi(\pi(x)) := \Pi(\Psi(x))$, for all $x \in X$. From the digroup axioms and generalized digroup homomorphism characterization (see Theorem 4) it follows that (φ, Ψ) is an augmented generalized digroup homomorphism, $G \cong G_l^{\xi}$ and $H \cong G_l^{\xi'}$, since the following diagram commutes

$$\begin{array}{ccc} \pi(X) & \xrightarrow{\varphi} & \pi(Y) \\ {\scriptstyle\pi}\uparrow & & \uparrow{\scriptstyle\Pi} \\ X & \xrightarrow{\Psi} & Y \\ {\scriptstyle\varphi_l}\downarrow & & \downarrow{\scriptstyle\varphi_l'} \\ G_\xi^l \times E & \xrightarrow[\Psi']{} & G_l^{\xi'} \times E' \end{array}$$

□

We finish this section reviewing the characterization of digroups given by J. D. H. Smith in [8], section 10, via diquasigroup algebras (see Proposition 10.5, Corollary 10.7 and Theorem 10.8). J. D. H. Smith uses a digroup generated by two groups with compatible actions over a same set and a fixed element with respect to both actions. For generalized digroups we don't assume the existence of a fixed point.

Let $(G, 1_G)$ and $(H, 1_H)$ be groups, and let X be a left G-set and a rigth H-set, where the left G-action λ and the right H-action ρ commute. Then $(G \times X \times H, \dashv, \vdash)$, with binary maps $\dashv$ and $\vdash$ defined by

$$(g_1, x_1, h_1) \dashv (g_2, x_2, h_2) = (g_1 g_2, x \bullet_\rho h_2, h_1 h_2)$$

and

$$(g_1, x_1, h_1) \vdash (g_2, x_2, h_2) = (g_1 g_2, g_1 \bullet_\lambda x, h_1 h_2),$$

for all $g_1, g_2 \in G$, $x_1, x_2 \in X$ and $h_1, h_2 \in H$, is a generalized digroup.

The halo is $E = \{1_G\} \times X \times \{1_H\} \cong X$, and for any $(g, x, h) \in G \times X \times H$ the inverses with respect to the bar unit $(1_G, z, 1_H)$ are:

$$(g, x, h)_{l_z}^{-1} = (g^{-1}, z \bullet_\rho h^{-1}, h^{-1}) \quad \text{and} \quad (g, x, h)_{r_z}^{-1} = (g^{-1}, g^{-1} \bullet_\lambda z, h^{-1}).$$

Therefore the groups of inverses are isomorphic to the direct product $G \times H$.

The actions λ and ρ induce a left action $\widehat{\lambda}$ and a right action $\widehat{\rho}$, from $G \times H$ on the set $G \times X \times H$ by

$$(g_1, h_1) \bullet_{\widehat{\lambda}} (g_2, x_2, h_2) = (g_1 g_2, g_1 \bullet_\lambda x, h_1 h_2)$$

and

$$(g_1, x_1, h_1) \bullet_{\widehat{\rho}} (g_2, h_2) = (g_1 g_2, x \bullet_\rho h_2, h_1 h_2).$$

The projection map $\widehat{\pi}$, given by $(g, x, h) \mapsto (g, h)$, is equivariant with respect to the actions $\widehat{\lambda}$ and $\widehat{\rho}$.

With these actions and the projection map, we have that $(G \times X \times H, \dashv, \vdash)$ is a generalized digroup generated by the actions $\widehat{\lambda}$ and $\widehat{\rho}$, and the surjective equivariant map $\widehat{\pi}$, i.e. $(G \times X \times H, \dashv, \vdash)$ is an augmented generalized digroup.

J. D. H. Smith uses Kinyon's results to prove Theorem 10.8. Now we give a simple, direct and independent proof of this result for generalized digroup.

Let $(D, \vdash, \dashv)$ be a generalized digroup, since for any $\xi \in E$, we have that for the isomorphic groups G_l^ξ and G_l^ξ, the set D is a left G_l^ξ-set and a rigth G_r^ξ set, with commutative actions defined by $\lambda_g(x) := g \vdash x$ and $\rho_h(y) := y \dashv h$, for any $g \in G_l^\xi$, $h \in G_r^\xi$ and $x, y \in D$, respectively, then $(G_l^\xi \times D \times G_r^\xi)$ is a generalized digroup isomorphic to D, with isomorphism given by $x \mapsto (\xi \dashv x, x, x \vdash \xi)$.

3 The free generalized digroup

In [5] J.-L. Loday constructs the free dimonoid. Later, using a free semigroup $FS[X]$ and the word lenght map A. V. Zhuchok in [11] constructs another free dimonoid which is isomorphic to the one defined by J. - L. Loday. Both dimonoids don't extend to digroups since A. V. Zhuchok in [12], Theorem 4, p. 833, shows that it's impossible to adjoin a set of bar units (halo) to a Loday's free dimonoid. In this section, we exhibit the free generalized digroup $FD(X)$ for any set X. The basic ideas in the construction are related to the articles [1, 7, 9], and the book *Dialgebras and related operads* (see [14]).

In addition we present several properties of $FD(X)$ and relate it with the characterization theorems and augmented generalized digroups.

For the construction of free generalized digroup we use classical results for free structures (see [17]).

Let X be a set and $F(X)$ the free group generated by X, that is, the set of all words in $X^{\pm}$, see [15] for details on this definition. The proof of the following statement is straightforward and therefore we omit it. However, the result is central in this section.

Proposition 4. *Let $FD(X) := F(X) \times X \times F(X)$ with the binary maps $\dashv$ and $\vdash$ defined for all $x, y \in X$ and $u, v, a, b \in F(X)$ by*

$$(u, x, a) \dashv (v, y, b) = (u, x, avyb) \quad \text{and} \quad (u, x, a) \vdash (v, y, b) = (uxav, y, b).$$

Then $(FD(X), \dashv, \vdash)$ is a generalized digroup with halo

$$E(X) = \{(v, y, b) \mid vyb = e\} = \{(v, y, b) \mid v = (yb)^{-1}\} = \{(v, y, b) \mid b = (vy)^{-1}\}$$

and inverses with respect to the bar unit (v, y, b)

$$(u, x, a)^{-1}_{l_{(v,y,b)}} = (v, y, ba^{-1}x^{-1}u^{-1}) = (v, y, b(uxa)^{-1})$$

and

$$(u, x, a)^{-1}_{r_{(v,y,b)}} = (a^{-1}x^{-1}u^{-1}v, y, b) = ((uxa)^{-1}v, y, b),$$

where the empty word e is the unit of the free group $F(X)$ and w^{-1} is the inverse of w in $F(X)$.

Remark 1. *The inverses $(u, x, a)^{-1}_{l_{(v,y,b)}}$ and $(u, x, a)^{-1}_{r_{(v,y,b)}}$ coincide iff $(u, x, a) \in E$. In this case*

$$(u, x, a)^{-1}_{l_{(v,y,b)}} = (u, x, a)^{-1}_{r_{(v,y,b)}} = (v, y, b).$$

We want to show now that $FD(X)$ is the free generalized digroup in X, i.e. it's the free element in the generalized digroup category. First, note that the natural immersion $X \hookrightarrow FD(X)$ is given by $x \mapsto (e, x, e)$.

Theorem 7. *(**Universal property**) Let D be a generalized digroup and let X be a set. For each function $f : X \to D$, there exists a homomorphism $\Psi : FD(X) \to D$ that extends f, that is, such that the following diagram commutes.*

$$\begin{array}{ccc} X & \hookrightarrow & FD(X) \\ & \searrow & \downarrow \\ & & D \end{array}$$

Proof. Let E be the halo of D and $\xi \in E$ fixed. For any $x \in X$ there is a bar unit $\eta \in E$ such that $f(x) \in G_l^{\eta}$, (recall that $D = \dot{\bigcup}_{\zeta \in E} G_l^{\zeta}$).

Since G_l^{η} is a group, the inverse of $f(x)$ in G_l^{η} is $f(x)_{l_\eta}^{-1} \in G_l^{\eta}$, and therefore we use the convention $f(x)^{-1} := f(x)_{l_\eta}^{-1}$. We define a function $\varphi : F(X) \to G_l^{\xi}$ in the following way: for any $x \in X$, $\varphi(x) = \xi \dashv f(x) \in G_l^{\xi}$, $\varphi(x^{-1}) = \xi \dashv f(x)^{-1} = \big(f(x)\big)_{l_\xi}^{-1} \in G_l^{\xi}$, and for any $w = x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \cdots x_{i_n}^{\delta_n} \in F(X)$, with $x_{i_j} \in X$ and $\delta_j = \pm 1$,

$$\varphi(w) = \xi \dashv f(x_{i_1})^{\delta_1} \dashv f(x_{i_2})^{\delta_2} \dashv \cdots \dashv f(x_{i_n})^{\delta_n}.$$

Then $\varphi : F(X) \to G_l^{\xi}$ is a group homomorphism.

Now, we define $\Psi : FD(X) \to D$ by $\Psi((u, x, a)) = \varphi(u) \vdash f(x) \dashv \varphi(a)$, for all $(u, x, a) \in FD(X)$. The map Ψ is a homomorphism. Indeed,

$$\Psi\left((u,x,a) \dashv (v,y,b)\right) = \Psi\left((u,x,avyb)\right) = \varphi(u) \vdash f(x) \dashv \varphi(avyb)$$

and

$$\begin{aligned}\Psi\left((u,x,a)\right) \dashv \Psi\left((v,y,b)\right) &= \left(\varphi(u) \vdash f(x) \dashv \varphi(a)\right) \dashv \left(\varphi(v) \vdash f(y) \dashv \varphi(b)\right)\\ &= \varphi(u) \vdash f(x) \dashv \left(\varphi(a) \dashv \varphi(v) \dashv \varphi(y) \dashv \varphi(b)\right)\\ &= \varphi(u) \vdash f(x) \dashv \varphi(avyb).\end{aligned}$$

Similarly,

$$\Psi((u,x,a) \vdash (v,y,b)) = \varphi(uxav) \vdash f(y) \dashv \varphi(b)$$

and

$$\Psi((u,x,a)) \vdash \Psi((v,y,b)) = \varphi(uaxv) \vdash f(y) \dashv \varphi(b).$$

□

Definition 4. *The generalized digroup $FD(X)$ is called the* ***free generalized digroup*** *on X.*

The bar units and the inverses in the free generalized digroup have the following properties

Remark 2. *For $FD(X)$ we have that*

1. *If $y \in X$ and $v \in F(X)$ then*

$$\begin{aligned}\Psi\left((v,y,y^{-1}v^{-1})\right) &= \varphi(v) \vdash f(y) \dashv \varphi(y^{-1}v^{-1})\\ &= \varphi(v) \vdash f(y) \dashv \left(\varphi(y^{-1}) \dashv \varphi(v^{-1})\right)\\ &= \varphi(v) \vdash f(y) \dashv f(y)^{-1} \dashv \left(\varphi(v)\right)^{-1}\\ &= \left(\varphi(v) \vdash f(y)\right) \dashv \left(\varphi(v) \vdash f(y)\right)^{-1} \in E.\end{aligned}$$

   Similarly, $\Psi\left((b^{-1}y^{-1}, y, b)\right) = \left(f(y) \dashv \varphi(b)\right)^{-1} \vdash \left(f(y) \dashv \varphi(b)\right) \in E$. Therefore, $\Psi(E(X)) \subseteq E$.
2. *For $(v,y,b) \in E(X)$, we have that $\Psi\left((v,y,b)\right) = \xi \in E$ iff $f(y) \in G_l^{\xi}$.*
3. *For an element $(u,x,a) \in FD(X)$,*

$$\begin{aligned}\Psi\left((u,x,a)_{r_{(b^{-1}y^{-1},y,b)}}^{-1}\right) &= \Psi(a^{-1}x^{-1}u^{-1}b^{-1}y^{-1},y,b)\\ &= \left(\varphi(a^{-1}) \dashv \varphi(x^{-1}) \dashv \varphi(u^{-1})\right) \vdash \left(\varphi(b^{-1}) \vdash \varphi(y^{-1}) \vdash f(y) \dashv \varphi(b)\right)\\ &= \left(\Psi((u,x,a))\right)^{-1} \vdash \Psi\left((b^{-1}y^{-1},y,b)\right)\\ &= \left(\Psi((u,x,a))\right)_{r_{\Psi((b^{-1}y^{-1},y,b))}}^{-1}.\end{aligned}$$

   Similarly, $\Psi\left((u,x,a)_{l_{(b^{-1}y^{-1},y,b)}}^{-1}\right) = \left(\Psi((u,x,a))\right)_{l_{\Psi((b^{-1}y^{-1},y,b))}}^{-1}$

We want to describe the sets in Definition 6 from [1] for the generalized digroup $FD(X)$.

Proposition 5. *For any bar unit $(v,y,b) \in E(X)$, the group of left inverses $G_{(v,y,b)}^{l}$ is isomorphic to $F(X)$.*

Proof. Since

$$G_l^{(v,y,b)} = \left\{(v, y, b(uxa)^{-1}) \mid (u, x, a) \in FD(X)\right\},$$

then let's define $\phi : G^l_{(v,y,b)} \to F(X)$ by $\phi((v, y, b(uxa)^{-1})) = uxa$, the map ϕ is a bijective homomorphism, with inverse function given by $w \mapsto (v, y, bw^{-1})$, which gives us the result. □

Remark 3. *According to the previous result, we can see that $G^l_{(v,y,b)} = \{(v, y, bw^{-1}) \mid w \in F(X)\}$. Moreover, Theorem 1 implies that $G^r_{(v,y,b)}$ is isomorphic to $F(X)$, for all $(v, y, b) \in E(X)$.*

We describe now some actions from the group $F(X)$ to the free generalized digroup $FD(X)$.

First, we define the maps

$$\begin{aligned} \lambda : F(X) \times FD(X) &\to FD(X) \\ (w, (u, x, a)) &\mapsto w \bullet_\lambda (u, x, a) := (wu, x, a) \end{aligned}$$

$$\begin{aligned} \rho : F(X) \times FD(X) &\to FD(X) \\ (w, (u, x, a)) &\mapsto (u, x, a) \bullet_\rho w := (u, x, aw^{-1}) \end{aligned}$$

and $\gamma : F(X) \times FD(X) \to FD(X)$ the conjugate composition of λ and ρ, i.e

$$\gamma_w((u, x, a)) = (\lambda_w \circ \rho_{w^{-1}})((u, x, a)) = (wu, x, aw^{-1})$$

From these definitions it is simple to see that

Theorem 8. *The free generalized digroup $FD(X)$ is a $F(X)$-set with respect to the actions λ, ρ and γ. Besides, λ and ρ are compatible.*

After defining the action γ, we can see how it works out with the function Ψ, say,

$$\begin{aligned} \Psi(\gamma_w(u, x, a)) &= \Psi(wu, x, aw^{-1}) = \varphi(wu) \vdash f(x) \dashv \varphi(aw^{-1}) \\ &= \big(\varphi(w) \dashv \varphi(u)\big) \vdash f(x) \dashv \big(\varphi(a) \dashv \varphi(w^{-1})\big) \\ &= \big(\varphi(w) \vdash \varphi(u)\big) \vdash f(x) \dashv \big(\varphi(a) \dashv (\varphi(w))^{-1}\big) \\ &= \varphi(w) \vdash \big(\varphi(u) \vdash f(x) \dashv \varphi(a)\big) \dashv (\varphi(w))^{-1} \\ &= \gamma_{\varphi(w)}(\Psi(u, x, a)) \end{aligned}$$

Thus, we can conclude

Lemma 3. *The map Ψ is φ-equivariant with respect to the actions λ, ρ and γ.*

Now, given a projection map $\Pi : FD(X) \to F(X)$ defined by $(u, x, a) \mapsto uxa$, we have that Π is surjective and equivariant with respect to the actions λ and ρ. Moreover, for all $(u, x, a),\ (v, y, b) \in FD(X)$,

$$\begin{aligned} (u, x, a) \vdash (v, y, b) &= \Pi\big((u, x, a)\big) \bullet_\lambda (v, y, b), \\ (u, x, a) \dashv (v, y, b) &= (u, x, a) \bullet_\rho \Pi\big((v, y, b)\big) \end{aligned}$$

and therefore

Theorem 9. *The augmented generalized digroup $(F(X),\ FD(X),\ \lambda,\ \rho,\ \Pi)$ is the free generalized digroup $FD(X)$.*

Going back to the units of the free generalized digroup, we classify them in two groups. For that, note that elements of the form (e, z, z^{-1}) and (z^{-1}, z, e) are bar units.

We denote by ξ_{wz} the bar unit $(w, z, (wz)^{-1})$ and by ξ_{zc} the bar unit $((zc)^{-1}, z, c)$. Thus $\xi_{ez} = (e, z, z^{-1})$ and $\xi_{ze} = (z^{-1}, z, e)$.

Definition 5. *The bar units of $FD(X)$ of the form either (e, z, z^{-1}) or (z^{-1}, z, e) are called* ***basic bar units*** *of the generalized digroup. We define the sets $E^l_B := \{(e, z, z^{-1})\}$ and $E^r_B := \{(z^{-1}, z, e)\}$, and they are called left basic free halo and right basic free halo, respectively.*

It is simple to see that X, E^l_B and E^l_B are equipotent sets.

Remark 4. *Note that*

$$\rho_z(e,z,e)=(e,z,z^{-1})=\gamma_z(z^{-1},z,e)\ \text{ and }\ \lambda_{z^{-1}}(e,z,e)=(z^{-1},z,e)=\gamma_{z^{-1}}(e,z,z^{-1}).$$

Using the characterization of the bar units, we can express the inverses in terms of it.

If we denote $(u,x,a)^{-1}_{l_{wz}} := (u,x,a)^{-1}_{l_{(w,z,(wz)^{-1})}}$ and $(u,x,a)^{-1}_{r_{wz}} := (u,x,a)^{-1}_{r_{(w,z,(wz)^{-1})}}$, we have that

Lemma 4. *For $a,u,w\in F(X)$ and $x,z\in X$,*

$$(u,x,a)^{-1}_{l_{wz}}=\gamma_{uxa}(u,x,a)^{-1}_{r_{wz}}$$

and

$$(u,x,a)^{-1}_{r_{wz}}=\gamma_{(uxa)^{-1}}(u,x,a)^{-1}_{l_{wz}}.$$

For the halo $E(X)$, we consider the following notation $E^l(X)=\{(w,z,(wz)^{-1})\,|\,z\in X \text{ and } w\in F(X)\}$ and $E^r(X)=\{((zc)^{-1},z,c)\,|\,z\in X \text{ and } c\in F(X)\}$.

Lemma 5. *If in the free generalized digroup $FD(X)$ we denote the action γ by $\cdot$, we can see that*
1. $F(X)\cdot E^l_B=E^l$
2. $F(X)\cdot E^r_B=E^r$
3. $F(X)\cdot E(X)=E(X)$, *i.e. $E(X)$ is invariant under the action γ.*

We review involutions in free generalized digroups.

Proposition 6. *Let's consider the map $^\star : FD(X)\to FD(X)$, with $(u,x,a)\mapsto(u,x,a)^\star=(a^\iota,x,u^\iota)$, where $\iota: F(X)\to F(X)$ is the function such that*

$$\left(z_{q_1}^{\delta_{q_1}}z_{q_2}^{\delta_{q_2}}\cdots z_{q_r}^{\delta_{q_r}}\right)^\iota=z_{q_r}^{\delta_{q_r}}z_{q_{r-1}}^{\delta_{q_{r-1}}}\cdots z_{q_1}^{\delta_{q_1}}.$$

Then * is an involution.

Proof. It is clear that $\left((u,x,a)^\star\right)^\star=(u,x,a)$. Moreover, since $z^\iota=z$, for any $z\in X^\pm$, then

$$((u,x,a)\dashv(v,y,b))^\star=(v,y,b)^\star\vdash(u,x,a)^\star$$

and

$$((u,x,a)\vdash(v,y,b))^\star=(v,y,b)^\star\dashv(u,x,a)^\star.$$

□

If we consider the group action η of $F(X)$ on $FD(X)$, defined by $w\bullet_\eta(u,x,a)=(wu,x,aw^\iota)$, we have that η is invariant under the involution $\star$ since

$$\begin{aligned}\left(\eta_w(u,x,a)\right)^\star&=(wu,x,aw^\iota)^\star=((aw^\iota)^\iota,x,(wu)^\iota)\\&=((w^\iota)^\iota a^\iota,x,u^\iota w^\iota)=(wa^\iota,x,u^\iota w^\iota)\\&=\eta_w(a\iota,x,u^\iota)=\eta_w\left((u,x,a)^\star\right).\end{aligned}$$

Moreover, because $\Pi\left((u,x,a)^\star\right)=(a^\iota xu^\iota)=\left(\Pi(u,x,a)\right)^\iota$, then $\Pi\circ\star=\iota\circ\Pi$

Remark 5. *For the sets E^l_B and E^r_B, the map $\star: E^l_B\to E^r_B$ is a bijection, since $(e,z,z^{-1})^\star=(z^{-1},z,e)$, for all $z\in X$.*

As we have in subsection 2.1, every involution * defined on a generalized digroup $G\times E$ can be projected to a group involution $\iota: G\to G$. In the following theorem we describe a way to extend a group involution on $F(X)$ to an involution on $FD(X)$. Its proof comes directly from calculations, then we omit it.

Theorem 10. *Let $\iota : F(X) \to F(X)$ be a involution over the free group generated by a set X. The involution ι can be extended to an involution $^* : FD(X) \to FD(X)$, by $(u, x, a)^* := (a^\iota, x^\iota, u^\iota)$ if and only if $x^\iota \in X$, for all $x \in X$.*

4 The tensor generalized digroup

In this section we use the construction of the semidirect product of groups given in [15] in order to define the tensor generalized digroup and the concept of generating set for a generalized digroup. It is a surprise to prove that the same construction of the semidirect product of groups works for generalized digroups.

It is well known that if G is a group, and $\phi : G \to Aut(G')$ is a representation of G, with $\phi_g = \phi(g)$, then $G \times G'$ can be endowed with group structure as follows

$$(n, h)(n', h') = (n\phi_h(n'), hh').$$

This product is called, in the literature, the *semidirect product of G and G′* and it is denoted by $G \rtimes_\phi G'$. As we see above, $G \rtimes_\phi G'$ can also be considered as a generalized digroup by taking

$$(n, h) \vdash (n', h') = (n, h) \dashv (n', h') = (n, h)(n', h').$$

This form of constructing generalized digroups from group structures is called *the trivial form*.

The following theorem gives us a non trivial form to provide the semidirect product $G \rtimes_\phi G$ with generalized digroup structure. At the end of this section we use this idea to define the tensor generalized digroups.

Theorem 11. *Let G be a group. Then, the group $G \rtimes_\phi G$ is a generalized digroup with the following operations*

$$(u_1, u_2) \vdash (v_1, v_2) = (u_1u_2v_1, v_2) \tag{2}$$

and

$$(u_1, u_2) \dashv (v_1, v_2) = (u_1, u_2v_1v_2). \tag{3}$$

Since the binary operations $\vdash$ and $\dashv$ do not depend on the representation ϕ, we use the notation $G \otimes G$ to refer to $G \rtimes_\phi G$ as a generalized digroup in a nontrivial form and $u \otimes v$ instead of (u, v).

Proof. The proof that $\vdash$ and $\dashv$ are associative binary operations is taken from direct calculations. Now, we verify the condition given in the Definition 5 of [1]. In fact, let $u_1 \otimes u_2$, $v_1 \otimes v_2$ and $w_1 \otimes w_2$ in $G \otimes G$, then

$$\begin{aligned} u_1 \otimes u_2 \vdash (v_1 \otimes v_2 \dashv w_1 \otimes w_2) &= u_1 \otimes u_2 \vdash v_1 \otimes v_2w_1w_2 \\ &= u_1u_2v_1 \otimes v_2w_1w_2 \\ &= u_1u_2v_1 \otimes v_2 \dashv w_1 \otimes w_2 \\ &= (u_1 \otimes u_2 \vdash v_1 \otimes v_2) \dashv w_1 \otimes w_2. \end{aligned}$$

Besides,

$$\begin{aligned} u_1 \otimes u_2 \dashv (v_1 \otimes v_2 \dashv w_1 \otimes w_2) &= u_1 \otimes u_2v_1v_2w_1w_2 \\ &= u_1 \otimes u_2 \dashv v_1v_2w_1 \otimes w_2 \\ &= u_1 \otimes u_2 \dashv (v_1 \otimes v_2 \vdash w_1 \otimes w_2). \end{aligned}$$

and

$$\begin{aligned} (u_1 \otimes u_2 \vdash v_1 \otimes v_2) \vdash w_1 \otimes w_2 &= u_1u_2v_1v_2w_1 \otimes w_2 \\ &= u_1 \otimes u_2v_1v_2 \vdash w_1 \otimes w_2 \\ &= (u_1 \otimes u_2 \dashv v_1 \otimes v_2) \vdash w_1 \otimes w_2. \end{aligned}$$

It is not hard to verify that the set of bar units of $G \otimes G$, i.e., the halo of $G \otimes G$, is $E = \{u_1 \otimes u_2 \mid u_1u_2 = 1\}$. For $\xi = u \otimes u^{-1} \in E$ and $u_1 \otimes u_2 \in G \otimes G$, we have that

$$u_1 \otimes u_2 \vdash u_2^{-1}u_1^{-1}u \otimes u^{-1} = \xi,$$

and

$$u \otimes u^{-1}u_2^{-1}u_1^{-1} \dashv u_1 \otimes u_2 = \xi.$$

Therefore,

$$(u_1 \otimes u_2)_{r_\xi}^{-1} = u_2^{-1}u_1^{-1}u \otimes u^{-1} \text{ and } (u_1 \otimes u_2)_{l_\xi}^{-1} = u \otimes u^{-1}u_2^{-1}u_1^{-1}.$$

Moreover,

$$(u_1 \otimes u_2)_{r_\xi}^{-1} = u_2^{-1} \otimes u_1^{-1} \vdash u \otimes u^{-1} = u_2^{-1} \otimes u_1^{-1} \vdash \xi$$

and

$$(u_1 \otimes u_2)_{l_\xi}^{-1} = u \otimes u^{-1} \dashv u_2^{-1} \otimes u_1^{-1} = \xi \dashv u_2^{-1} \otimes u_1^{-1}.$$

□

Definition 6. *The generalized digroup in the theorem is called the* ***tensor generalized digroup*** *of G.*

Note that if in the previous theorem we define, for every $u \in G$, the bijection $\varphi_u : G \to G$, as $\varphi_u(v) = uv$, then we can rewrite the equations (2) and (3) as follows

$$u_1 \otimes u_2 \vdash v_1 \otimes v_2 = (u_1\varphi_{u_2}(v_1)) \otimes v_2 \tag{4}$$

and

$$u_1 \otimes u_2 \dashv v_1 \otimes v_2 = u_1 \otimes (\varphi_{u_2}(v_1)v_2). \tag{5}$$

Let us consider the following interesting fact. If G is obtained by the quotient of the free group $F(X)$ by the relations $R_i(X) = e$, $i = 1, 2, \ldots, k$, then the tensor generalized digroup $G \otimes G$ can be described, in an informal way, as the quotient of the tensor generalized digroup $F(X) \otimes F(X)$ by the relations $R_i(X) \otimes R_j(X) = e \otimes e$, $i, j = 1, 2, \ldots, k$, like in the case of group presentations, see [15] for more details.

Consider the following definition.

Definition 7. *For the free generalized digroup $FD(X)$, we define the* ***fiber*** *of $FD(X)$ at the distinguished element y of X as the subset $F^y(X) = F(X) \times \{y\} \times F(X)$ of $FD(X)$.*

Proposition 7. *The subset $F^y(X)$ is a normal generalized subdigroup of $FD(X)$. Moreover, it is isomorphic to $F(X) \otimes F(X)$.*

Proof. The proof that $\vdash$ and $\dashv$ are associative binary operations on $F^y(X)$ comes from direct calculations. Besides, the bar units of $F^y(X)$ must be of the form $(b^{-1}y^{-1}, y, b)$ or $(v, y, y^{-1}v^{-1})$, i.e. (v, y, b) such that vyb is the empty word, and so they belong to $F^y(X)$. We end the proof that $F^y(X)$ is a generalized subdigroup noting that for every bar unit (v, y, b),

$$(u, y, a)_{l_{(v,y,b)}}^{-1} = (v, y, ba^{-1}y^{-1}u^{-1}) \in F^y(X)$$

and

$$(u, y, a)_{r_{(v,y,b)}}^{-1} = (a^{-1}y^{-1}u^{-1}v, y, b) \in F^y(X).$$

The normality of $F^y(X)$ is taken directly from the following fact. For every $(s, y, d) \in F^y(X)$ and all (u, x, a), (w, z, c) in $FD(X)$, we have that:

$$(u, x, a) \vdash (s, y, d) \dashv (w, z, c) = (uxas, y, dwzc) \in F^y(X).$$

In fact, $F^y(X)$ satisfies a stronger condition than normality.

Now, let $\varphi^y : F^y(X) \to F(X) \otimes F(X)$ be the function defined by

$$\varphi^y((u, y, a)) = uy \otimes a.$$

Since,

$$\begin{aligned}\varphi^y((u,y,a)\vdash(w,y,c)) &= \varphi^y\left((uyaw,y,c)\right)\\ &= uyawy\otimes c\\ &= (uy\otimes a)\vdash(wy\otimes c)\\ &= \varphi^y((u,y,a))\vdash\varphi^y((w,y,c)),\end{aligned}$$

and

$$\begin{aligned}\varphi^y((u,y,a)\dashv(w,y,c)) &= \varphi^y\left((u,y,awyc)\right)\\ &= uy\otimes awyc\\ &= (uy\otimes a)\dashv(wy\otimes c)\\ &= \varphi^y((u,y,a))\dashv\varphi^y((w,y,c))\end{aligned}$$

the map φ^y is a generalized digroup homomorphism.

Consider the function $\phi_y : F(X)\otimes F(X)\to F^y(X)$ given by $\phi_y(u\otimes a) = (uy^{-1}, y, a)$. It is not hard to see that this function is a generalized digroup homomorphism, and that ϕ_y is the inverse function of φ^y. □

Let X be a non empty set and let $X^{\otimes}$ be the set of all tensors of the form $x^{\epsilon}\otimes 1$, with $x\in X$ and $\epsilon\in\{1,-1\}$. Here, 1 means that there is nothing in the corresponding position, so it acts as the identity in the free group $F(X)$.

Let $u\otimes v\in F(X)\otimes F(X)$, with $u = x_{i_1}^{\epsilon_1}\cdots x_{i_p}^{\epsilon_p}$ and $v = x_{j_1}^{\delta_1}\cdots x_{j_k}^{\delta_k}$. From the definition of $\vdash$ and $\dashv$,

$$\begin{aligned}u\otimes v &= \left(x_{i_1}^{\epsilon_1}\otimes 1\vdash\cdots\vdash x_{i_p}^{\epsilon_p}\otimes 1\right)\dashv\left(x_{j_1}^{\delta_1}\otimes 1\dashv\cdots\dashv x_{j_k}^{\delta_k}\otimes 1\right)\\ &= \left(x_{i_1}^{\epsilon_1}\otimes 1\vdash\cdots\vdash x_{i_{p-1}}^{\epsilon_{p-1}}\otimes 1\right)\vdash x_{i_p}^{\epsilon_p}\otimes 1\dashv\left(x_{j_1}^{\delta_1}\otimes 1\dashv\cdots\dashv x_{j_k}^{\delta_k}\otimes 1\right).\end{aligned}$$

The previous calculation motivates the following proposition.

Proposition 8 (Generating set). *Let X be a subset of a generalized digroup $(D,\vdash,\dashv)$. For X^- we mean the set of all inverses, right and left, with respect to all bar units in D. In other words, if E denotes the halo of D, then*

$$X_{l,r}^- = \bigcup_{\xi\in E}\{x_{l_\xi}^{-1}, x_{r_\xi}^{-1} \mid x\in X\}.$$

Thus the set $\langle X\rangle$ of all elements of D of the form

$$\left(g_1\vdash\cdots\vdash g_p\right)\vdash y\dashv\left(h_1\dashv\cdots\dashv h_k\right),$$

where g_t, h_n and y are in $X^{\pm} = X\cup X_{l,r}^{-1}$, for every $t = 1,2,\ldots,p$ and $n = 1,2,\ldots,k$, is a subdigroup of D.

Proof. In order to simplify the notation we use $u\vdash y\dashv v$, with $u = g_1\vdash\cdots\vdash g_p$ and $v = h_1\dashv\cdots\dashv h_k$ to represent the elements of $\langle X\rangle$.

Since

$$\begin{aligned}(u\vdash y\dashv v)\vdash(u'\vdash z\dashv v') &= (u\vdash y\vdash v)\vdash(u'\vdash z\dashv v')\\ &= (u\vdash y\vdash v\vdash u')\vdash z\vdash v'\end{aligned}$$

and

$$\begin{aligned}(u\vdash y\dashv v)\dashv(u'\vdash z\dashv v') &= (u\vdash y\dashv v)\dashv(u'\dashv z\dashv v')\\ &= u\vdash y\dashv(v\dashv u'\dashv z\dashv v'),\end{aligned}$$

$\vdash$ and $\dashv$ define binary operations on $\langle X\rangle$.

Besides, due to the fact that for every bar unit ξ in D, ξ can be represented as

$$\xi = y\vdash y_{r_\xi}^{-1}\vdash x_{l_\xi}^{-1}\dashv x,$$

for every $x,y\in X$, the set $\langle X\rangle$ contains the halo of D.

Let $\xi = y\vdash y_{r_\xi}^{-1}\vdash x_{l_\xi}^{-1}\dashv x$ and let $u\vdash y\dashv v\in\langle X\rangle$, with $u = g_1\vdash\cdots\vdash g_p$ and $v = h_1\dashv\cdots\dashv h_p$. From Proposition 1,

$$\begin{aligned}(u\vdash y\dashv v)_{l_\xi}^{-1} &= v_{l_\xi}^{-1}\dashv(u\vdash y)_{l_\xi}^{-1}\\ &= v_{l_\xi}^{-1}\dashv(y_{l_\xi}^{-1}\dashv u_{l_\xi}^{-1})\\ &= y\vdash y_{r_\xi}^{-1}\vdash v_{l_\xi}^{-1}\dashv y_{l_\xi}^{-1}\dashv u_{l_\xi}^{-1}.\end{aligned}$$

Again, from Proposition 1, we have that

$$v_{l_\xi}^{-1} = (h_k)_{l_\xi}^{-1} \dashv \cdots \dashv (h_1)_{l_\xi}^{-1} \quad \text{and} \quad u_{l_\xi}^{-1} = (g_p)_{l_\xi}^{-1} \dashv \cdots \dashv (g_1)_{l_\xi}^{-1}.$$

If there exists i, such that $h_i = x_{l_\xi}^{-1}$ or $h_i = x_{r_\xi}^{-1}$, then from Proposition 1,

$$\begin{aligned} v_{l_\xi}^{-1} &= (h_k)_{l_\xi}^{-1} \dashv \cdots \dashv (h_i)_{l_\xi}^{-1} \dashv \cdots \dashv (h_k)_{l_\xi}^{-1} \\ &= (h_k)_{l_\xi}^{-1} \dashv \cdots \dashv (\xi \dashv x) \dashv \cdots \dashv (h_k)_{l_\xi}^{-1} \\ &= (h_k)_{l_\xi}^{-1} \dashv \cdots \dashv x \dashv \cdots \dashv (h_k)_{l_\xi}^{-1} \end{aligned}$$

The same conclusion is obtained for the case $u_{l_\xi}^{-1}$. As a consequence, $(u \vdash y \dashv v)_{l_\xi}^{-1}$ belongs to $\langle X \rangle$.
In a similar way, from Proposition 1, we have that

$$\begin{aligned} (u \vdash y \dashv v)_{r_\xi}^{-1} &= v_{r_\xi}^{-1} \vdash (u \vdash y)_{r_\xi}^{-1} \\ &= v_{r_\xi}^{-1} \vdash (y_{r_\xi}^{-1} \vdash u_{r_\xi}^{-1}) \\ &= v_{r_\xi}^{-1} \vdash y_{r_\xi}^{-1} \vdash u_{r_\xi}^{-1} \dashv x_{r_\xi}^{-1} \dashv x. \end{aligned}$$

And, we also prove that $(u \vdash y \dashv v)_{r_\xi}^{-1}$ is in $\langle X \rangle$. □

Definition 8. *The subdigroup $\langle X \rangle$ is called* ***the generalized subdigroup generated by*** *X. In the case that $\langle X \rangle = D$, we say that D is generated by X and if X has one element, then D is called a* ***cyclic generalized digroup****.*

As a consequence, $X^{\otimes}$ is a generating set of $F(X) \otimes F(X)$. This fact is not surprising because $F(X) \otimes F(X)$ is a subdigroup of $FD(X)$ and $FD(X)$ is free on X.

Now, we continue with the construction of the semidirect product of generalized digroups, but first we consider

Definition 9. *Let $(D, \vdash, \dashv)$ and $(D', \vdash', \dashv')$ be two generalized digroups. We define* **Aut(D)** *as the set of all bijective generalized digroup homomorphisms from D onto itself.*

A ***representation*** *of D' on D is a function $\varphi : D' \to Aut(D)$, with $\varphi(u') = \varphi_{u'}$, such that, for every u', v' in D' and every $w \in D$,*

$$\varphi_{u' \vdash' v'}(w) = \varphi_{u'}(\varphi_{v'}(w)) = (\varphi_{u'} \circ \varphi_{v'})(w)$$

and

$$\varphi_{u' \dashv' v'}(w) = \varphi_{u'}(\varphi_{v'}(w)) = (\varphi_{u'} \circ \varphi_{v'})(w)$$

Note that if, in the previous definition, we make $\dashv = \vdash$ and D' is a group with the operation $\vdash$, then φ is a group representation in the usual sense. Another important fact is that for every $u' \in D'$, $\varphi_{u'}^{-1}$ is also a generalized digroup homomorphism. It is because, if $a = \varphi_{u'}(u)$ and $b = \varphi_{u'}(v)$, then

$$\varphi_{u'}^{-1}(a \vdash b) = \varphi_{u'}^{-1}(\varphi_{u'}(u) \vdash \varphi_{u'}(v)) = u \vdash v = \varphi_{u'}^{-1}(a) \vdash_{u'}^{-1} (v). \tag{6}$$

Let ξ' be a bar unit of D' and let φ be a representation of D' on D, then for every $u' \in D$ we have that

$$\varphi_{u'} = \varphi_{\xi' \vdash' u'} = \varphi_{\xi'} \circ \varphi_{u'} \text{ and } \varphi_{u'} = \varphi_{u' \dashv' \xi'} = \varphi_{u'} \circ \varphi_{\xi'}.$$

Therefore,

$$\varphi_{\xi'} = id_D. \tag{7}$$

Proposition 9. *If $f \in Aut(D)$, then $f(E) \subset E$, where E is the halo of D. Moreover, for every $u \in D$ and every $f \in Aut(D)$,*

$$f(u_{r_\xi}^{-1}) = (f(u))_{r_{f(\xi)}}^{-1} \tag{8}$$

and

$$f(u_{l_\xi}^{-1}) = (f(u))_{l_{f(\xi)}}^{-1}. \tag{9}$$

Proof. Let ξ be a bar unit of D and $f \in Aut(D)$. Since for all $u \in D$ there exists $v \in D$ such that $u = f(v)$, we have that

$$f(\xi) \vdash u = f(\xi) \vdash f(v) = f(\xi \vdash v) = f(v) = u$$

and

$$u \dashv f(\xi) = f(v) \dashv f(\xi) = f(u \dashv \xi) = f(v) = u,$$

thus, $f(\xi)$ is in the halo of D.

Besides, let $v = f(u^{-1}_{r_\xi})$, then $f^{-1}(v) = u^{-1}_{r_\xi}$. Therefore, $u \vdash f^{-1}(v) = \xi$. So, by applying f on the two sides of the previous equation, we have that

$$f(u) \vdash v = f(\xi).$$

Since $f(\xi)$ is also a bar unit of D, then v is a right inverse of $f(u)$ with respect to $f(\xi)$. In other words,

$$v = (f(u))^{-1}_{r_{f(\xi)}}.$$

Following a similar argument, we prove (9). □

Theorem 12. *Let $(D, \vdash, \dashv)$ and $(D', \vdash', \dashv')$ be two generalized digroups and let φ be a representation from D' into $Aut(D)$, then $D \times D'$ with the binary operations*

$$(u_1, u'_1) \triangleright (u_2, u'_2) = (u_1 \vdash \varphi_{u'_1}(u_2), u'_1 \vdash' u'_2)$$

and

$$(u_1, u'_1) \triangleleft (u_2, u'_2) = (u_1 \dashv \varphi_{u'_1}(u_2), u'_1 \dashv' u'_2)$$

is a generalized digroup denoted $D \rtimes_\varphi D'$.

Moreover, for every bar unit ξ' of D', the subset $D \times \{\xi'\}$ is a normal generalized subdigroup of $D \rtimes_\varphi D'$. If E denotes the halo of D, then $E \times D'$ is a generalized subdigroup of $D \rtimes_\varphi D'$.

Proof. Let (u_1, u'_1), (u_2, u'_2) and (u_3, u'_3) in $D \rtimes_\varphi D'$, then

$$\begin{aligned}(u_1, u'_1) \triangleright ((u_2, u'_2) \triangleright (u_3, u'_3)) &= (u_1, u'_1) \triangleright (u_2 \vdash \varphi_{u'_2}(u_3), u'_2 \vdash' u'_3)\\ &= (u_1 \vdash \varphi_{u'_1}(u_2 \vdash \varphi_{u'_2}(u_3)), u'_1 \vdash' (u'_2 \vdash' u'_3))\\ &= (u_1 \vdash (\varphi_{u'_1}(u_2) \vdash \varphi_{u'_1}(\varphi_{u'_2}(u_3))), u'_1 \vdash' (u'_2 \vdash' u'_3))\\ &= ((u_1 \vdash \varphi_{u'_1}(u_2)) \vdash \varphi_{u'_1 \vdash' u'_2}(u_3), (u'_1 \vdash' u'_2) \vdash' u'_3)\\ &= ((u_1 \vdash \varphi_{u'_1}(u_2), u'_1 \vdash' u'_2) \triangleright (u_3, u'_3)\\ &= ((u_1, u'_1) \triangleright (u_2, u'_2)) \triangleright (u_3, u'_3).\end{aligned}$$

The associativity of $\triangleleft$ comes in a similar way. Now, we verify the conditions given in the Definition 5 of [1]. In fact,

$$\begin{aligned}(u_1, u'_1) \triangleright ((u_2, u'_2) \triangleleft (u_3, u'_3)) &= (u_1, u'_1) \triangleright (u_2 \dashv \varphi_{u'_2}(u_3), u'_2 \dashv' u'_3)\\ &= (u_1 \vdash \varphi_{u'_1}(u_2 \dashv \varphi_{u'_2}(u_3)), u'_1 \vdash' (u'_2 \dashv' u'_3))\\ &= (u_1 \vdash (\varphi_{u'_1}(u_2) \dashv \varphi_{u'_1}(\varphi_{u'_2}(u_3))), u'_1 \vdash' (u'_2 \dashv' u'_3))\\ &= ((u_1 \vdash \varphi_{u'_1}(u_2)) \dashv \varphi_{u'_1 \vdash' u'_2}(u_3), (u'_1 \vdash' u'_2) \dashv' u'_3)\\ &= ((u_1 \vdash \varphi_{u'_1}(u_2), u'_1 \vdash' u'_2) \triangleleft (u_3, u'_3)\\ &= ((u_1, u'_1) \triangleright (u_2, u'_2)) \triangleleft (u_3, u'_3).\end{aligned}$$

We also have that,

$$\begin{aligned}(u_1, u'_1) \triangleleft ((u_2, u'_2) \triangleleft (u_3, u'_3)) &= (u_1, u'_1) \triangleleft (u_2 \dashv \varphi_{u'_2}(u_3), u'_2 \dashv' u'_3)\\ &= (u_1 \dashv \varphi_{u'_1}(u_2 \dashv \varphi_{u'_2}(u_3)), u'_1 \dashv' (u'_2 \dashv' u'_3))\\ &= (u_1 \dashv (\varphi_{u'_1}(u_2) \dashv \varphi_{u'_1}(\varphi_{u'_2}(u_3))), u'_1 \dashv' (u'_2 \dashv' u'_3))\\ &= (u_1 \dashv (\varphi_{u'_1}(u_2) \vdash \varphi_{u'_1}(\varphi_{u'_2}(u_3))), u'_1 \dashv' (u'_2 \vdash' u'_3))\\ &= (u_1 \dashv \varphi_{u'_1}(u_2 \vdash \varphi_{u'_2}(u_3)), u'_1 \dashv' (u'_2 \vdash' u'_3))\\ &= (u_1, u'_1) \triangleleft (u_2 \vdash \varphi_{u'_2}(u_3), u'_2 \vdash' u'_3)\\ &= (u_1, u'_1) \triangleleft ((u_2, u'_2) \triangleright (u_3, u'_3)).\end{aligned}$$

We end this part of the proof with the following,

$$\begin{aligned}((u_1,u_1')\rhd((u_2,u_2'))\rhd(u_3,u_3') &= ((u_1\vdash\varphi_{u_1'}(u_2),u_1'\vdash' u_2')\rhd(u_3,u_3')\\ &= ((u_1\vdash\varphi_{u_1'}(u_2))\vdash\varphi_{u_1'\vdash' u_2'}(u_3),(u_1'\vdash' u_2')\vdash' u_3')\\ &= ((u_1\dashv\varphi_{u_1'}(u_2))\vdash\varphi_{u_1'\vdash' u_2'}(u_3),(u_1'\dashv' u_2')\vdash' u_3')\\ &= ((u_1\dashv\varphi_{u_1'}(u_2))\vdash\varphi_{u_1'\dashv' u_2'}(u_3),(u_1'\dashv' u_2')\vdash' u_3')\\ &= ((u_1\dashv\varphi_{u_1'}(u_2),u_1'\dashv' u_2')\rhd(u_3,u_3')\\ &= ((u_1,u_1')\lhd(u_2,u_2'))\rhd(u_3,u_3').\end{aligned}$$

From (7) and a direct calculation, the bar units on $D\rtimes_\varphi D'$ are of the form (ξ,ξ'), where ξ and ξ' are bar units in D and D', respectively.

For a bar unit $\widehat{\xi}=(\xi,\xi')\in D\rtimes_\varphi D'$ and $(u,u')\in D\rtimes_\varphi D'$, we have from (8) that

$$(u,u')^{-1}_{r_{\widehat{\xi}}}=\big((\varphi^{-1}_{u'}(u))^{-1}_{r_{\varphi^{-1}_{u'}(\xi)}},(u')^{-1}_{r_{\xi'}}\big).$$

In fact,

$$\begin{aligned}(u,u')\vdash\big((\varphi^{-1}_{u'}(u))^{-1}_{r_{\varphi^{-1}_{u'}(\xi)}},(u')^{-1}_{r_{\xi'}}\big) &= \big(u\vdash\varphi_{u'}(\varphi^{-1}_{u'}(u))^{-1}_{r_{\varphi^{-1}_{u'}(\xi)}}),u'\vdash'(u')^{-1}_{r_{\xi'}}\big)\\ &= (u\vdash\varphi_{u'}(\varphi^{-1}_{u'}(u^{-1}_{r_\xi}),\xi')\\ &= (u\vdash u^{-1}_{r_\xi},\xi')\\ &= (\xi,\xi').\end{aligned}$$

Besides, since

$$\varphi_{(u')^{-1}_{l_{\xi'}}}\circ\varphi_{u'}=\varphi_{(u')^{-1}_{l_{\xi'}}\dashv' u'}=\varphi_{\xi'}=id_D,$$

then

$$\varphi_{(u')^{-1}_{l_{\xi'}}}=\varphi^{-1}_{u'}.$$

Therefore,

$$(u,u')^{-1}_{l_{\widehat{\xi}}}=\big((\varphi^{-1}_{u'}(u))^{-1}_{l_\xi},(u')^{-1}_{l_{\xi'}}\big).$$

The last equality is true because

$$\begin{aligned}\big((\varphi^{-1}_{u'}(u))^{-1}_{l_\xi},(u')^{-1}_{l_{\xi'}}\big)\dashv(u,u') &= \big(\varphi^{-1}_{u'}(u))^{-1}_{l_\xi}\dashv\varphi_{(u')^{-1}_{l_{\xi'}}}(u),\xi'\big)\\ &= \big(\varphi^{-1}_{u'}(u))^{-1}_{l_\xi}\dashv\varphi^{-1}_{u'}(u),\xi'\big)\\ &= (\xi,\xi').\end{aligned}$$

In order to simplify the notation, we use $D^{\xi'}=D\times\{\xi'\}$. Let ξ be a bar unit in D and let $(v,\xi')\in D^{\xi'}$ and (u,u') in $D\rtimes_\varphi D'$, then

$$\begin{aligned}(u,u')\rhd(v,\xi')\lhd(u,u)^{-1}_{l_{(\xi,\xi')}} &= (u,u')\rhd(v,\xi')\lhd\big((\varphi^{-1}_{u'}(u))^{-1}_{l_\xi},(u')^{-1}_{l_{\xi'}}\big)\\ &= \big(u\vdash\varphi_{u'}(v),u'\vdash'\xi'\big)\lhd\big((\varphi^{-1}_{u'}(u))^{-1}_{l_\xi},(u')^{-1}_{l_{\xi'}}\big)\\ &= \big((u\vdash\varphi_{u'}(v))\dashv\varphi_{u'\vdash'\xi'}((\varphi^{-1}_{u'}(u))^{-1}_{l_\xi}),(u'\vdash'\xi')\dashv'(u')^{-1}_{l_{\xi'}}\big).\end{aligned}$$

Let

$$w(u,u',v)=\big((u\vdash\varphi_{u'}(v))\dashv\varphi_{u'\vdash'\xi'}((\varphi^{-1}_{u'}(u))^{-1}_{l_\xi}),(u'\vdash'\xi')\dashv'(u')^{-1}_{l_{\xi'}}\big).$$

It is clear that $w(u,u',v)\in D$. On the other hand, since

$$(u'\vdash'\xi')\dashv'(u')^{-1}_{l_{\xi'}}=u'\vdash'(\xi'\dashv'(u')^{-1}_{l_{\xi'}})=u'\vdash'(u')^{-1}_{l_{\xi'}}=\xi',$$

then $(u,u')\rhd(v,\xi')\lhd(u,u)^{-1}_{l_{(\xi,\xi')}}\in D^{\xi'}$.

For the last part we consider Proposition 9. Let $\xi,\eta\in E$, due to the fact that

$$(\xi,u')\rhd(\xi,v')=(\xi\vdash\varphi_{u'}(\eta),u'\vdash' v')=(\varphi_{u'}(\eta),u'\vdash' v')\in E\times D'$$

and

$$(\xi,u')\lhd(\eta,v')=(\xi\dashv\varphi_{u'}(\eta),u'\dashv' v')=(\xi,u'\dashv' v')\in E\times D',$$

we conclude that $E\times D'$ is a generalized subdigroup of $D\rtimes_\varphi D'$. □

Definition 10. *The digroup defined is the* ***generalized semidirect product*** *of D and D' with respect to φ.*

We end this section with the construction of $D \otimes D$, for the case in which $(D, \vdash, \dashv)$ is a generalized digroup. We just need to consider the binary operations

$$u_1 \otimes u_2 \vdash v_1 \otimes v_2 = (u_1 \vdash u_2 \vdash v_1) \otimes v_2 \tag{10}$$

and

$$u_1 \otimes u_2 \dashv v_1 \otimes v_2 = u_1 \otimes (u_2 \dashv v_1 \dashv v_2). \tag{11}$$

It is not hard to prove that the halo of $D \otimes D$ is the set

$$E(D \otimes D) = E(D) \otimes E(D).$$

Besides, if $\xi \otimes \widehat{\xi} \in E(D \otimes D)$ and $v_1 \otimes v_1 \in D \otimes D$, then

$$(v_1 \otimes v_2)^{-1}_{r_{\xi \otimes \widehat{\xi}}} = (v_1 \vdash v_2)^{-1}_{r_\xi}) \otimes \widehat{\xi}$$

and

$$(v_1 \otimes v_2)^{-1}_{l_{\xi \otimes \widehat{\xi}}} = \xi \otimes (v_1 \dashv v_2)^{-1}_{l_{\widehat{\xi}}}.$$

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