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Research Article

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Centralizers of automorphisms permuting free generators

Abstract: By $\sigma \in S_{km}$ we denote a permutation of the cycle-type k^m and also the induced automorphism permuting subscripts of free generators in the free group F_{km}. It is known that the centralizer of the permutation σ in S_{km} is isomorphic to a wreath product $Z_k \wr S_m$ and is generated by its two subgroups: the first one is isomorphic to Z_k^m, the direct product of m cyclic groups of order k, and the second one is S_m. We show that the centralizer of the automorphism $\sigma \in \mathrm{Aut}(F_{km})$ is generated by its subgroups isomorphic to Z_k^m and $\mathrm{Aut}(F_m)$.

Dedicated to the memory of Vitaly Sushchanskyy (1946-2016)

1 Introduction

This paper was inspired by a question of Vitaly Sushchanskyy who asked about the structure of centralizers of automorphisms permuting free generators in a free group.

Another motivation of the present paper are numerous papers in which the authors investigate centralizers of finite subgroups of both $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$. K. Vogtmann's paper [1] is the vast survey on this and related topics with many references. We present here a few results. If G is a finite subgroup of $\mathrm{Aut}(F_n)$, then $E = F_n \rtimes G$ is a finite extension of F_n. In general, if E is given by the short exact sequence $1 \to F_n \to E \to K \to 1$, then conjugation action of E on F_n induces a homomorphism $\theta : K \to \mathrm{Out}(F_n)$. There exists a connection between the group of outer automorphisms of E and the centralizer of $\theta(K)$. For example, it is known (see in [2, 3]) that $\mathrm{Out}(E)$ is finite if and only if the centralizer of $\theta(K)$ is finite. D. Boutin in [2] and M. Pettet in [3] describe finite extensions E of a free group F_n for which the group $\mathrm{Out}(E)$ is finite. In the present paper, if an automorphism of F_n is induced by a permutation σ of the cycle-type k^m (m cycles of length k), then the centralizer of σ in $\mathrm{Aut}(F_n)$ is finite if and only if σ is a long cycle (e.g. it is the cycle without fixed points). Since the automorphism induced by the permutation σ of the cycle-type k^m has no fixed points, there are no non-trivial inner automorphisms in the centralizer $C(\sigma)$. Hence the subgroup $C(\sigma) < \mathrm{Aut}(F_n)$ is isomorphic to its image in $\mathrm{Out}(F_n)$. Y. Algom-Kfir and C. Pfaff study in [4] a centralizer of a subgroup of $\mathrm{Out}(F_n)$ cyclically generated by lone axis fully irreducible outer automorphisms. The centralizer of such group is infinite cyclic group. It follows from the main theorem of the present paper that the centralizer of

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an automorphism induced by a permutation of a cycle-type k^m is either finite cyclic or infinite non-abelian. S. Krstič proves in [5] (Theorem 2) that the centralizer of a finite subgroup of $Aut(F_n)$ is finitely presentable.

Let σ be a permutation of the cycle-type k^m (m cycles of length k). We use the same letter σ for the induced automorphism of the free group F_{km} with standard basis $X = \{x_1, \ldots, x_{km}\}$. The automorphism σ acts on the basis X by permuting subscripts of generators, which we write as:

$$\sigma : x_i \to x_{\sigma(i)} \ \ or \ \ x_i^\sigma = x_{\sigma(i)}. \tag{1}$$

We describe here the structure of the centralizer $C(\sigma) \subseteq Aut(F_{km})$ of the automorphism σ induced by the permutation σ.

It is known that the centralizer of σ in S_{km} is isomorphic to the wreath product $Z_k \wr S_m$ of a cyclic group of order k and a symmetric group S_m (see for example [6, 7] for details) and it is generated by subgroups isomorphic to Z_k^m and S_m (in fact $C(\sigma) \cong Z_k^m \rtimes S_m$). We show that the centralizer of the automorphism induced by σ in the free group F_{km} has a similar structure and is generated by subgroups isomorphic to Z_k^m and $Aut(F_m)$ but unfortunately in this case Z_k^m is not a normal subgroup of $Aut(F_m)$.

2 Notation

Since the structure of the centralizer $C(\sigma)$ does not depend on the contents of cycles of the permutation σ, we shall assume:

$$\sigma = (1, \ldots, k)(k+1, \ldots, 2k) \ldots ((m-1)k+1, \ldots, mk). \tag{2}$$

The elements of the basis X of the free group F_{km} can be written as the entries of the $k{\times}m$-matrix with columns denoted by $\mathfrak{X}_i$ so that the columns correspond to the σ-orbits.

$$X = \begin{pmatrix} x_1 & x_{k+1} & \cdots & x_{(m-1)k+1} \\ x_1^\sigma & x_{k+1}^\sigma & \cdots & x_{(m-1)k+1}^\sigma \\ x_1^{\sigma^2} & x_{k+1}^{\sigma^2} & \cdots & x_{(m-1)k+1}^{\sigma^2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\sigma^{k-1}} & x_2^{\sigma^{k-1}} & \cdots & x_m^{\sigma^{k-1}} \end{pmatrix} = $$

$$\begin{pmatrix} x_1 & x_{k+1} & \cdots & x_{(m-1)k+1} \\ x_2 & x_{k+2} & \cdots & x_{(m-1)k+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_k & x_{2k} & \cdots & x_{mk} \end{pmatrix} = (\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_m). \tag{3}$$

Note that the subscripts of the generators in columns are the elements of orbits of the permutation σ. Note also, that the first row of this matrix defines the other rows by applying the automorphism σ.

3 Properties of automorphisms commuting with σ.

If $A := \{a_1, a_2, ..., a_{km}\}$ is another basis in F_{km}, then the action of σ on a_i is defined by its action on X. We are interested in the case, when the basis A is also a union of σ-orbits.

Definition 1. *The basis $A := \{a_1, a_2, ..., a_{km}\}$ in F_{km} is called a σ-**basis** if it can be ordered so that*

$$\sigma : a_i \to a_{\sigma(i)}, \quad (a_i^\sigma = a_{\sigma(i)}), \tag{4}$$

that is for σ of the form (2) the σ-basis A can be written as

$$
A = \begin{pmatrix}
a_1 & a_{k+1} & \dots & a_{(m-1)k+1} \\
a_1^\sigma & a_{k+1}^\sigma & \dots & a_{(m-1)k+1}^\sigma \\
a_1^{\sigma^2} & a_{k+1}^{\sigma^2} & \dots & a_{(m-1)k+1}^{\sigma^2} \\
\vdots & \vdots & \ddots & \vdots \\
a_1^{\sigma^{k-1}} & a_2^{\sigma^{k-1}} & \dots & a_{(m-1)k+1}^{\sigma^{k-1}}
\end{pmatrix} =
$$

$$
\begin{pmatrix}
a_1 & a_{k+1} & \dots & a_{(m-1)k+1} \\
a_2 & a_{k+2} & \dots & a_{(m-1)k+2} \\
\vdots & \vdots & \ddots & \vdots \\
a_k & a_{2k} & \dots & a_{mk}
\end{pmatrix} = (\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_m).
\tag{5}
$$

Example 1. Let $\sigma = (1, 2, 3)(4, 5, 6)$, $n = 6$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Take another basis in F_6: $A = \{x_1 x_4, x_2 x_5, x_3 x_6, x_4, x_5, x_6\}$.

$$
X = \begin{pmatrix} x_1 & x_4 \\ x_2 & x_5 \\ x_3 & x_6 \end{pmatrix}, \quad
A = \begin{pmatrix} x_1 x_4 & x_4 \\ x_2 x_5 & x_5 \\ x_3 x_6 & x_6 \end{pmatrix}
$$

We formulate now a criterion for an automorphism $\alpha \in Aut(F_n)$ to centralize the automorphism $\sigma \in Aut(F_n)$.

Proposition 1. *An automorphism $\alpha \in Aut(F_n)$, $\alpha : x_i \to a_i$, $X \to A$ commutes with the automorphism σ if and only if A is a σ-basis.*

Proof. Let $\alpha : x_i \to a_i$. Then $x_i^{\alpha\sigma} = a_i^\sigma$ and $x_i^{\sigma\alpha} = (x_{\sigma(i)})^\alpha = a_{\sigma(i)}$. So the equality $\alpha\sigma = \sigma\alpha$ holds if and only if $(a_i)^\sigma = a_{\sigma(i)}$ as required. $\square$

Corollary 1. *An automorphisms α is in $C(\sigma)$ if and only if it maps each orbit $\{x, x^\sigma, \dots, x^{\sigma^{k-1}}\}$ onto some orbit $\{a, a^\sigma, \dots, a^{\sigma^{k-1}}\}$.*

Example 2. An automorphism δ, acting as σ in only one σ-orbit commutes with σ, hence it is the σ-automorphism and is in $C(\sigma)$.

4 m-tuples and Nielsen transformations

We remind now some useful notions from [8, Chapter 3].
By a **basic m-tuple** we mean any ordered set of free generators in a free group F_m with the basis $(w_1, \dots w_i, \dots w_j, \dots w_m)$. The following m-tuples are called **elementary m-tuples**:

$$(i) \ (w_1, \dots w_i^{-1}, \dots w_j, \dots w_m), \quad \text{replaced } w_i \text{ with } w_i^{-1},$$

$$(ii) \ (w_1, \dots w_i w_j, \dots w_j, \dots w_m), \text{replaced } w_i \text{ with } w_i w_j \text{ (or } w_j w_i),$$

$$(iii) \ (w_1, \dots w_j, \dots, w_i, \dots w_m), \quad \text{switched } w_i \text{ and } w_j.$$

$$\tag{6}$$

The basic m-tuples in F_m form a group with the unity element $X = (x_1, x_2, \dots, x_m)$ and with the following multiplication:

$$(a_1, a_2, \dots, a_m) \cdot (b_1, b_2, \dots, b_m) = (a_1(b_v), a_2(b_v), \dots, a_m(b_v)).
\tag{7}$$

Example 3. $(x_1x_3,\ x_2x_3,\ x_3^{-1}) \cdot (b_2b_3^{-1},\ b_3b_1,\ b_1) = (b_2b_3^{-1}b_1,\ b_3b_1^2,\ b_1^{-1})$.

$(x_1x_3,\ x_2x_3,\ x_3^{-1}) \cdot (x_2x_3^{-1},\ x_3x_1,\ x_1) = (x_2x_3^{-1}x_1,\ x_3x_1^2,\ x_1^{-1})$.

Each m-tuple $(b_1, b_2, \ldots, b_m)$ defines **an automorphism** $\beta : x_i \to b_i$ which acts on m-tuple $(a_1, a_2, \ldots, a_m)$ by multiplication **from the right**, as in (7),

$$(a_1, a_2, \ldots, a_m)^\beta = (a_1(b_v), a_2(b_v), \ldots, a_m(b_v)) = X^{\alpha\beta}.$$

To define the notion of a Nielsen transformation, we recall that in matrix theory, to switch i-th and j-th rows in a matrix M, we multiply M from the left by the identity matrix with switched i-th and j-th rows. The action of the Nielsen transformations is similar, namely:

Each m-tuple $(a_1, a_2, \ldots, a_m)$ defines **a Nielsen transformation** N_α acting on another m-tuple $(b_1, b_2, \ldots, b_m)$ by multiplication **from the left**, as in (7),

$$N_\alpha(b_1, b_2, \ldots, b_m) = (a_1(b_v), a_2(b_v), \ldots, a_m(b_v)).$$

For example if $n = 2$, the Nielsen transformation $(x_1x_2,\ x_1)$ changes the basis (a, b) into $(x_1x_2,\ x_1) \cdot (a, b) = (ab, a)$.

Since $N_\alpha N_\beta X = N_\alpha(b_1, b_2, \ldots, b_n) = (a_1(b_v), a_2(b_v), \ldots, a_n(b_v)) = X^{\alpha\beta}$,

the group of Nielsen transformations and the group of automorphisms $Aut(F_n)$ are anti-isomorphic [8] (sec 3.2, (11)) and the following equality holds

$$N_{\alpha_1}N_{\alpha_2}\ldots N_{\alpha_k}X = X^{\alpha_1\alpha_2\ldots\alpha_k}.$$

$$N_{\alpha_k}\ldots N_{\alpha_2}N_{\alpha_1}X = X^{\alpha_k\ldots\alpha_2\alpha_1}. \tag{8}$$

The elementary m-tuples (6) define the so called elementary automorphisms and elementary Nielsen transformations.

5 SE-automorphisms

Let $\widehat{F}_m, \widehat{F}_m^\sigma, \cdots, \widehat{F}_m^{\sigma^{k-1}}$ be the free subgroups generated by the elements of the rows in the matrix (3), then

$$F_{km} = \widehat{F}_m \star \widehat{F}_m^\sigma \star \cdots \star \widehat{F}_m^{\sigma^{k-1}},$$

$$\begin{pmatrix} \langle x_1 & x_{k+1} & \cdots & x_{(m-1)k+1} \rangle \\ \langle x_1^\sigma & x_{k+1}^\sigma & \cdots & x_{(m-1)k+1}^\sigma \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1^{\sigma^{k-1}} & x_{k+1}^{\sigma^{k-1}} & \cdots & x_{(m-1)k+1}^{\sigma^{k-1}} \rangle \end{pmatrix} = \begin{pmatrix} \widehat{F}_m \\ \widehat{F}_m^\sigma \\ \vdots \\ \widehat{F}_m^{\sigma^{k-1}} \end{pmatrix}.$$

If the same elementary automorphism τ acts on every row (in every $\widehat{F}_m^{\sigma^i}$), then it defines a so-called **simultaneous elementary** automorphism which we address as a SE-automorphism in F_{km} [9].

Note that if the same τ acts on every row, then it acts on the row of columns, which generate the free group $\widetilde{F}_m$ of rank m

$$\widetilde{F}_m := \langle \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m \rangle.$$

So the elementary automorphism τ in $\widetilde{F}_m$ induces the above SE-automorphism in F_{km}. We can write this as

Corollary 2. *An elementary automorphism τ in the free m-generator group $\widetilde{F_m} = \langle \mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_m \rangle$ defines the SE-automorphism in $F_{km} = \widehat{F}_m \star \cdots \star \widehat{F}_m^{\sigma^{k-1}}$ acting as τ in each of k factors $\widehat{F}_m^{\sigma^i}$.*

There are three types (a), (b), (c) of the elementary automorphisms τ in $\widetilde{F_m}$, defining the SE-automorphisms in $F_{km} = \widehat{F}_m \star \widehat{F}_m^{\sigma} \star \cdots \star \widehat{F}_m^{\sigma^{k-1}}$:

(a) for a fixed i and all $x \in \mathfrak{X}_i$, $x^\tau = x^{-1}$. while the elements in other orbits are fixed, $X^\tau = \left(\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_i^{-1}, \ldots, \mathfrak{X}_j, \ldots, \mathfrak{X}_m \right)$,

(b) for fixed i, j on each level in (3), for all $x \in \mathfrak{X}_i$ $y \in \mathfrak{X}_j$: $x^\tau = xy$ (or yx), while $\mathfrak{X}_j$ and other orbits are fixed, so
$X^\tau = \left(\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_i \mathfrak{X}_j, \ldots, \mathfrak{X}_j, \ldots, \mathfrak{X}_m \right)$ or
$X^\tau = \left(\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_j \mathfrak{X}_i, \ldots, \mathfrak{X}_j, \ldots, \mathfrak{X}_m \right)$,

(c) for fixed i, j, and all $x \in \mathfrak{X}_i$ $y \in \mathfrak{X}_j$ on each level: $x^\tau = y$, $y^\tau = x$. Other orbits are fixed, so $X^\tau = \left(\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_j, \ldots, \mathfrak{X}_i, \ldots, \mathfrak{X}_m \right)$.

Proposition 2. *The SE-automorphisms (a), (b), (c) in $Aut(F_{km})$ generate a subgroup in the centralizer $C(\sigma)$ isomorphic to $Aut(F_m)$.*

Proof. Since by [8] (Theorem 3.2 on page 131) the elementary automorphisms generate the full automorphism group, it follows by Corollary 2 that the group generated by all SE-automorphisms in $F_{km} = \widehat{F}_m \star \widehat{F}_m^{\sigma} \star \cdots \star \widehat{F}_m^{\sigma^{k-1}}$ is isomorphic to the group $Aut(\widetilde{F_m})$ and hence to $Aut(F_m)$.

Each elementary automorphism in $\widetilde{F_m}$ has one of the forms (a), (b), (c), changes the basis X into a σ-basis X^τ, and by Proposition 1 is in the centralizer of σ, which finishes the proof. □

The subgroup generated by the SE-automorphisms is the proper subgroup in the centralizer $C(\sigma)$, since it does not contain automorphisms δ_i, which maps each generator a in the $i-th$ column to a^σ (vertical permutations in $i-th$ columns). So we have to consider also this type of σ-automorphisms mentioned in Example 4.

(d) *for a fixed i and all $y \in \mathfrak{X}_i$, $y^{\delta_i} := y^\sigma$, that is δ_i acts on $\mathfrak{X}_i$ as a cyclic permutation of order k, while the other orbits are fixed,*
$$X^{\delta_i} := \left(\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_i^\sigma, \ldots, \mathfrak{X}_j, \ldots, \mathfrak{X}_m \right).$$

Proposition 3. *The automorphisms of the type (d) form a subgroup isomorphic to the direct product Z_k^m of m cyclic groups of order k.*

Note The σ-automorphism (a), (b), (c) act in rows of the basis-matrix, while (d) permutes vertically elements from different rows.

Our goal now is to show that the SE-automorphisms (a), (b), (c), $and (d)$ generate the centralizer of σ in $Aut(F_{km})$. To proceed we shall use the Nielsen transformations on the m-tuples of orbits.

6 Transformations for m-tuples of σ-orbits

Let F_{km} be a free group with the σ-basis A which is the union of m σ-orbits.

$$A = X^\alpha = \left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_i, \ldots, \mathcal{A}_m \right).$$

Note that the Nielsen transformations of this m-tuples into

(a) $\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_i^{-1}, \ldots, \mathcal{A}_j, \ldots, \mathcal{A}_m \right)$,

(b) $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_i \mathcal{A}_j, \ldots, \mathcal{A}_j, \ldots, \mathcal{A}_m)$ or
$(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_j \mathcal{A}_i, \ldots, \mathcal{A}_j, \ldots, \mathcal{A}_m)$,

(c) $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_j, \ldots, \mathcal{A}_i, \ldots, \mathcal{A}_m)$,

(d) $(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_i^\sigma, \ldots, \mathcal{A}_j, \ldots, \mathcal{A}_m)$,

define the ES-automorphisms in F_{km}. (For convenience we shall call the automorphisms and transformations (*d*) also elementary)

Corollary 3. *Since the set of σ-bases in F_n is closed under the above Nielsen transformations on m-tuples of σ-orbits the equality* (8) *holds:*

$$N_{\tau_k} \ldots N_{\tau_2} N_{\tau_1} A = A^{\tau_k \ldots \tau_2 \tau_1},$$

where the elementary automorphisms τ correspond to the elementary Nielsen transformations.

7 Transformation of a *σ*-basis *A* to the standard basis *X*

We have to show that each σ-basis A written as (5) can be transformed to the σ-basis X written as (3) by the sequence $N_{\tau_1}, \ldots, N_{\tau_k}$ of elementary Nielsen transformations, applied to the *m*-tuples of orbits, which are the columns in the matrices. The proof is similar to that for Theorem 3.1 in [8] and uses the following terminology for the words in the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$, $n = km$:

– A word is freely reduced if no cancellations in it is possible.
– An *x*-length $|a|$ of a word a is a number of $x_i^{\pm 1}$ in its freely reduced form.
– The *x*-length of a basis A is the sum of *x*-lengths of all its elements.
– A major initial (major terminal) segment of a word has a minimally bigger length then half of the word's length.
– A major initial (major terminal) segment of a word $a \in A$ is **isolated** if no word $b \in A$ has such an initial (or terminal) segment. Note that otherwise $|a^{-1}b| < |b|$ (or $|ba^{-1}| < |b|$).
– A subset S in the free group F_n is **Nielsen-reduced** if the major initial and major terminal segments of each $a \in S$ are isolated and for an element a of even length either its left or its right half is isolated.
– The basis X in F_{km} is Nielsen-reduced and has the minimal *x*-length, which is equal to $n = km$.

Since by Definition 1, a σ-basis A in general case has a form

$$A = \begin{pmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \ldots & \mathbf{w}_m \\ \mathbf{w}_1^\sigma & \mathbf{w}_2^\sigma & \ldots & \mathbf{w}_m^\sigma \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{w}_1^{\sigma^{k-1}} & \mathbf{w}_2^{\sigma^{k-1}} & \ldots & \mathbf{w}_m^{\sigma^{k-1}} \end{pmatrix} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \ldots, \mathcal{A}_m), \tag{9}$$

and the permutation σ has no fixed points, it follows that every σ-orbit $\{\mathbf{w}_i, \mathbf{w}_i^\sigma, \ldots, \mathbf{w}_i^{\sigma^{k-1}}\}$ is a Nielsen reduced set.

Theorem 1. *For each σ-basis A in F_{km} there is a sequence of Nielsen transformations $N_{\tau_1}, N_{\tau_2}, \ldots, N_{\tau_r}$ of types in $\{(a), (b), (c), (d)\}$, such that*

$$N_{\tau_r} \ldots N_{\tau_1} A = X.$$

Proof. Let $A = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ be the freely reduced σ-basis in F_{km} written as the matrix (9). Since elements in columns have different first (last) letters x_i, we have that the subsets in each column are Nielsen-reduced, while the whole basis A need not be Nielsen-reduced.

1. Assume first, that the *x*-length of the σ-basis A is equal km. Then its entries are of the form x_i^ε, $\varepsilon = \pm 1$. Since $(x_i^\varepsilon)^\sigma = (x_i^\sigma)^\varepsilon$, all elements in a σ-orbit have the same value of ε. Then by the Nielsen-transformations of the type (a) we can eliminate all $\varepsilon = -1$. Now by permuting (cyclically) elements in the columns by

transformations (d), we get the proper order inside each column. Then the permutations of the columns (transformations (c)) lead to the basis X, as required.

2. Let now the x-length of the σ-basis A be greater than km. Then by Lemma 3.1 in [8] the basis A is not Nielsen-reduced. It follows that there is a word $a = uv$ in some column $\mathcal{A}_i$ in (9) which has a major (say initial) segment u ($|u| > |v|$), equal to the initial segment of some word $b = uw$ in some other column $\mathcal{A}_j$ in (9). We apply the Nielsen transformations of the type (d) to these columns to get the top elements in i-th and j-th columns respectively:

$$uv \text{ and } uw$$

Then the i-th and j-th columns consist of elements $uv, (uv)^\sigma, \dots$ and $uw, (uw)^\sigma, \dots$ respectively. Now we change $\mathcal{A}_i$ to $\mathcal{A}_i^{-1}$ by transformation (a) and apply transformation (b) to change $\mathcal{A}_j$ to $\mathcal{A}_i^{-1}\mathcal{A}_j$. We get the top elements in i-th and j-th columns

$$(uv)^{-1} \text{ and } v^{-1}w.$$

Since $|u| > |v|$, we get $|uw| > |vw| \geq |v^{-1}w|$. It diminishes the length of the j-th column at least for m and hence diminishes the x-length of the σ-basis A. Similarly we can isolate the halves of the entries. By repeating these steps we get the Nielsen reduced σ-basis of minimal length equal to km, which was considered in the part 1 of the proof. So the proof is complete.

$\square$

Example 4. Let $\sigma = (1, 2, 5)(3, 4, 6)$. We transform

the σ-basis $\quad A = X^\alpha = \{x_1 x_6,\ x_2 x_3,\ x_5 x_4\}\{x_4 x_2 x_3,\ x_6 x_5 x_4,\ x_3 x_1 x_6\}$

to the σ-basis $X = \{x_1, x_2, x_5\}\{x_3, x_4, x_6\}$.

$$A = \begin{pmatrix} x_1 x_6 & x_4 x_2 x_3 \\ x_2 x_3 & x_6 x_5 x_4 \\ x_5 x_4 & x_3 x_1 x_6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} x_1 x_6 & x_3 x_1 x_6 \\ x_2 x_3 & x_4 x_2 x_3 \\ x_5 x_4 & x_6 x_5 x_4 \end{pmatrix} \xrightarrow{(a)} \begin{pmatrix} (x_1 x_6)^{-1} & x_3 x_1 x_6 \\ (x_2 x_3)^{-1} & x_4 x_2 x_3 \\ (x_5 x_4)^{-1} & x_6 x_5 x_4 \end{pmatrix}$$

$$\xrightarrow{(b)} \begin{pmatrix} (x_1 x_6)^{-1} & x_3 \\ (x_2 x_3)^{-1} & x_4 \\ (x_5 x_4)^{-1} & x_6 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} (x_1 x_6)^{-1} & x_6 \\ (x_2 x_3)^{-1} & x_3 \\ (x_5 x_4)^{-1} & x_4 \end{pmatrix} \xrightarrow{(b)} \begin{pmatrix} x_1^{-1} & x_6 \\ x_2^{-1} & x_3 \\ x_5^{-1} & x_4 \end{pmatrix} \xrightarrow{(a)}$$

$$\begin{pmatrix} x_1 & x_6 \\ x_2 & x_3 \\ x_5 & x_4 \end{pmatrix} \xrightarrow{(d)} \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \\ x_5 & x_6 \end{pmatrix} = X.$$

8 The Main Theorem

Theorem 2. *The centralizer $C(\sigma)$ in $Aut(F_{km})$ for the automorphism σ with m orbits of order k is generated by the SE-automorphisms (a), (b), (c) and (d).*

Proof. Let an automorphism $\alpha \in C(\sigma)$ map $X \to A$,

$$A = X^\alpha. \tag{10}$$

By Theorem 1 the basis A can be changed into X by a sequence of transformations N_τ of the types (a),(b),(c),(d), then by (8) and (10)

$$X = N_{\tau_k} \dots N_{\tau_2} N_{\tau_1} A = A^{\tau_k \dots \tau_2 \tau_1} = X^{\alpha \tau_k \dots \tau_2 \tau_1},$$

and hence, for the corresponding SE-automorphisms τ we get

$$\alpha = \tau_1^{-1} \tau_2^{-1} \dots \tau_k^{-1}$$

is a product of SE-automorphisms of types (a), (b), (c) and (d) as required.

$\square$

Corollary 4. *By Propositions 2 and 3, for C(σ) in Aut(F_{km}) we have*

$$C(\sigma) \simeq \left\langle Z_k^m, \ Aut(F_m) \right\rangle,$$

which shows the similarity with the results from [6, 7] for permutation σ ∈ S_{km} and C(σ) ⊆ S_{km} but in the case of automorphism permuting generators Z_k^m is not a normal subgroup of C(σ).

Theorem 3. *The centralizer C(σ) in Aut(F_{km}) for the automorphism σ with m cycles of order k can be generated by the automorphism cyclically permuting elements in the first σ-orbit (the type (d)), and in addition, by two automorphisms if m ≥ 4 and by three automorphisms for m = 2, 3. Moreover, C(σ) is finitely presented.*

Proof. By Proposition 2, the SE-automorphisms in F_{km}, being the elementary automorphisms in the free group $\langle \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m \rangle$, generate the group

$$Aut(\langle \mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m \rangle) \cong Aut(F_m).$$

It is shown in [10] (see [8], p.165), that the group $Aut(F_m)$ can be generated by two automorphisms which map (for m = 4, 6, 8, . . .) the free generators as:

$$(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m) \to (\mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_m, \mathcal{X}_1),$$

$$(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m) \to \left(\mathcal{X}_2^{-1}, \mathcal{X}_1, \mathcal{X}_3, \ldots, \mathcal{X}_m \mathcal{X}_{m-1}^{-1}, \mathcal{X}_{m-1}^{-1} \right).$$

and for odd m ≥ 5 by two automorphisms:

$$(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m) \to \left(\mathcal{X}_2^{-1}, \mathcal{X}_3^{-1}, \ldots, \mathcal{X}_m^{-1}, \mathcal{X}_1^{-1} \right),$$

$$(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_m) \to \left(\mathcal{X}_2^{-1}, \mathcal{X}_1, \mathcal{X}_3, \ldots, \mathcal{X}_m \mathcal{X}_{m-1}^{-1}, \mathcal{X}_{m-1}^{-1} \right).$$

For m = 2, 3 we need three additional automorphisms.

The centralizer C(σ) is finitely presented by Krstič's Theorem (see [5], Theorem 2). □

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