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Arithmetical properties of double Möbius-Bernoulli numbers

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Abstract: Given positive integers n, n' and k, we investigate the Möbius-Bernoulli numbers $M_k(n)$, double Möbius-Bernoulli numbers $M_k(n, n')$, and Möbius-Bernoulli polynomials $M_k(n)(x)$. We find new identities involving double Möbius-Bernoulli, Barnes-Bernoulli numbers and Dedekind sums. In part of this paper, the Möbius-Bernoulli polynomials $M_k(n)(x)$, can be interpreted as critical values of the following Dirichlet type L-function

$$L_{HM}(s; n, x) := \sum_{d|n} \sum_{m=0}^{\infty} \frac{\mu(d)}{(md + x)^s} \quad (\text{for } \mathrm{Re}(s) > 1),$$

which has analytic continuation to the whole s-complex plane, where μ is the Möbius function.

Keywords: Möbius-Bernoulli numbers, Barnes-Bernoulli numbers, Dedekind sums

MSC: 11A05, 33E99

1 Introduction

Curiously, Möbius-Bernoulli numbers and polynomials are closely related to Dedekind Sums, critical values of certain Dirichlet series, Barnes-Bernoulli numbers and of course also to the Bernoulli numbers. In this paper, we will clarify all relationships between these arithmetical objects.

This paper consists of three parts. The first part treats Möbius-Bernoulli numbers and polynomials(cf. Section 2). In the second part, we consider new Dirichlet type series and show that their critical values are related to the Möbius-Bernoulli numbers and polynomials(cf. Section 3). In the third part, we study *double Möbius-Bernoulli* numbers and connect them to Barnes-Bernoulli numbers and Dedekind sums(cf. Sections 4 and 5). The three parts are more or less independent.

2 Möbius-Bernoulli numbers and polynomials

We fix some notations, definitions and preliminaries used in this paper. As to us we specify the motivations of our work.

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2.1 Identities on Möbius-Bernoulli numbers and polynomials

For n being a positive integer, we define the Möbius-Bernoulli polynomials by the generating function

$$\sum_{k=0}^{\infty} M_k(n)(x)\frac{t^k}{k!} = \sum_{d|n}\mu(d)\frac{te^{xt}}{e^{dt}-1}, \quad |t| < \frac{2\pi}{n}, \tag{2.1}$$

where as usual the Möbius function μ is given by

$$\mu(m) = \begin{cases} 1 & \text{if} \quad m = 1, \\ (-1)^k & \text{if} \quad m = p_1 \ldots p_k,\ p_1, \ldots, p_k \text{ are distinct primes}, \\ 0 & \text{otherwise}. \end{cases}$$

The Möbius-Bernoulli numbers $M_k(n)$ are given by $M_k(n) := M_k(n)(0)$. We recall the Bernoulli polynomial $B_k(x)$ is defined by the series:

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}, \quad |t| < 2\pi. \tag{2.2}$$

Note that $B_k(x)$ are monic polynomials with rational coefficients and $B_k := B_k(0)$ is the kth Bernoulli number. From the equations (2.1), (2.2) and Möbius inversion formula, we obtain

Theorem 1. *Let n and k be nonnegative integers. We have*

$$M_k(n)(x) = \sum_{d|n}\mu(d)d^{k-1}B_k(x/d), \tag{2.3}$$

$$B_k(nx) = \sum_{d|n}(n/d)^{k-1}M_k(d)(dx), \tag{2.4}$$

at $x = 0$, we get

$$M_k(n) = B_k\prod_{p|n}\left(1 - p^{k-1}\right) \tag{2.5}$$

and Gauss type formula

$$\sum_{d|n} d^k \prod_{p|d}\left(1 - 1/p^k\right) = n^k. \tag{2.6}$$

Let $n^* = p_1 \cdots p_r$ be the square free part of n. To get the equation (2.5), we use the relation (2.3) at $x = 0$. Since $\mu(d) = 0$ if d is not square-free, we have

$$\begin{aligned} M_k(n) &= B_k\sum_{d|n^*}\mu(d)d^{k-1} \\ &= B_k \sum_{\{i_1,\cdots,i_s\}\subset\{1,2,\cdots,r\}} \mu(p_{i_1}\cdots p_{i_s})(p_{i_1}\cdots p_{i_s})^{k-1} \\ &= B_k \sum_{\{i_1,\cdots,i_s\}\subset\{1,2,\cdots,r\}} (-1)^s(p_{i_1}\cdots p_{i_s})^{k-1} \\ &= B_k \prod_{p \text{ prime } |n} (1 - p^{k-1}). \end{aligned}$$

Among others, in this paper we will give an interpretation of the formula (2.5) in terms of critical values of Dirichlet L-series, and its generalization to *double Möbius-Bernoulli numbers*.

2.2 Generalized Bernoulli numbers

By Dirichlet character modulo a positive integer n, we mean as usual $\mathbb{C}$-valued function χ on $\mathbb{Z}$ such that $\chi(m) = 0$ if m is not coprime to n, and χ induces a character on $\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times}$. For a such χ, the generalized Bernoulli numbers

$$B_{k,\chi} \in \mathbb{Q}(\chi(1), \chi(2), \cdots, \chi(n-1))$$

are given through the generating function

$$\sum_{a=1}^{n} \chi(a) \frac{te^{at}}{e^{nt}-1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}. \tag{2.7}$$

From the equations (2.2) and (2.7) we have

$$B_{k,\chi} = n^{k-1} \sum_{a=1}^{n-1} \chi(a) B_k(a/n). \tag{2.8}$$

The numbers $B_{k,\chi}$ are very intriguing and still mysterious. If $k \geq 1$, then $B_{k,\chi} = 0$ if $\chi(-1) = (-1)^{k-1}$ (unless $n = k = 1$). In the simplest case where F is an imaginary quadratic field of discriminant $-D$, and χ is th unique quadratic character of conductor D such that $\chi(-1) = -1$, that is, $\chi(a) = \left(\frac{a}{-D}\right)$, and then we have the classical formula

$$B_{1,\chi} = -\frac{2h_F}{w_F},$$

where h_F is the class number of F and w_F is the number of roots of unity in F.
Recently, for such numbers $B_{k,\chi}$ many others interesting explicit formulae are obtained. In particular, when χ is a *quadratic character* see [1], for a *nontrivial primitive* Dirichlet character χ we refer to [2, 3].

2.3 Critical values of certain Dirichlet L-series

The Dirichlet L-series :

$$L(s, \chi) = \sum_{m \geq 1} \chi(m) m^{-s} \ (\operatorname{Re}(s) > 1)$$

is the L-series attached to any character χ.

The main interest of the numbers $B_{k,\chi}$ is that they give the value at non-positive integers of Dirichlet L-series. In fact, there is a well-known formula, proved by Hecke in [4]

$$L(1-k, \chi) = -\frac{B_{k,\chi}}{k} \ (k \geq 1). \tag{2.9}$$

In this paper, we are going to study *Möbius-Bernoulli and double Möbius-Bernoulli* numbers: $M_k(n)$, $M_k(n, n')$. We establish their relationship to Bernoulli, Apostol-Bernoulli and Barnes-Bernoulli numbers, and Dedekind sums. In part of this paper, the Möbius-Bernoulli, can be interpreted as critical values of the following Dirichlet type L-functions.

Lemma 2. *For k, n being positive integers, and χ_n the Dirichlet principal character modulo n, then we have*

$$M_k(n)/k = -L(1-k, \chi_n). \tag{2.10}$$

Proof of Lemma 2. We can get this result from relations (2.9) and Theorem 1. For Dirichlet character, the L-series has the Euler product

$$L(s, \chi) = \prod_{p \nmid n} \left(1 - \chi(p) p^{-s}\right)^{-1}.$$

While $\chi = \chi_n$, we have

$$L(s, \chi_n) = \zeta(s) \prod_{p \mid n} \left(1 - p^{-s}\right),$$

where $\zeta(s)$ is the Riemann zeta function. At $s = 1 - k$ we obtain

$$B_{k,\chi_n} = B_k \prod_{p|n} \left(1 - p^{k-1}\right). \tag{2.11}$$

The relation (2.5) and (2.9) completes the proof. □

From the equalities (2.5), (2.10) and Kummer's congruence [5, Theorem 5, p.239] for Bernoulli numbers B_k we obtain the following Kummer's type congruence formula.

Theorem 3 (Kummer's type Congruences). *Let p be a prime number, $p - 1 \nmid k$ and $k' \equiv k \pmod{(p-1)p^N}$ with N being a nonnegative integer. Then we have*

$$\frac{M_k(n)}{k} \equiv \frac{M_{k'}(n)}{k'} \pmod{p^{N+1}} \text{ if } p|n,$$

$$\left(1 - p^{k-1}\right)\frac{M_k(n)}{k} \equiv \left(1 - p^{k'-1}\right)\frac{M_{k'}(n)}{k'} \pmod{p^{N+1}} \text{ otherwise.}$$

Remark 4. One can use again the equalities (2.5), (2.10) and Von Satudt Clausen theorem [5, Theorem 3, p.233] to obtain von Staudt Clausen type result for $M_k(n)$.

3 Dirichlet type L-series and Möbius-Bernoulli polynomials

3.1 Möbius L-functions of Hurwitz type

For n being a positive integer and $x > 0$, let

$$L_{HM}(s; n, x) := \sum_{d|n} \sum_{m=0}^{\infty} \frac{\mu(d)}{(md + x)^s} \quad (\text{for } \mathrm{Re}(s) > 1). \tag{3.1}$$

We call $L_{HM}(s; n, x)$ *Hurwitz-Möbius L-functions*. Let

$$f(t; n, x) := \sum_{k=0}^{\infty} M_k(n)(x)\frac{t^k}{k!} = \sum_{d|n} \mu(d)\frac{te^{xt}}{e^{dt} - 1}$$

be the generating function of Möbius-Bernoulli polynomials, $M_k(n)(x)$ $(k \geq 0)$. Consider

$$\begin{aligned} \Gamma(s) L_{HM}(s; n, x) &= \int_0^{\infty} e^{-t} t^s \frac{dt}{t} \sum_{d|n} \sum_{m=0}^{\infty} \frac{\mu(d)}{(md + x)^s} \\ &= \sum_{d|n} \sum_{m=0}^{\infty} \int_0^{\infty} \frac{\mu(d)}{(md + x)^s} e^{-t} t^s \frac{dt}{t} \quad (\text{for } \mathrm{Re}(s) > 1). \end{aligned}$$

Substituting t by $(md+x)t$ in the last equality, we get

$$\begin{aligned}\Gamma(s)L_{HM}(s;n,x) &= \sum_{d|n}\sum_{m=0}^{\infty}\int_0^{\infty}\mu(d)e^{-(md+x)t}t^s\frac{dt}{t}\\ &= \int_0^{\infty}\sum_{d|n}\mu(d)\sum_{m=0}^{\infty}e^{-(md+x)t}t^s\frac{dt}{t}\\ &= \int_0^{\infty}\sum_{d|n}\mu(d)\frac{e^{-xt}}{1-e^{-dt}}t^s\frac{dt}{t}\\ &= \int_0^{\infty}\sum_{d|n}\mu(d)\frac{-te^{-xt}}{e^{-dt}-1}t^{s-1}\frac{dt}{t}\quad(\text{for }\operatorname{Re}(s)>1).\end{aligned}$$

The second equality is by Lebesgue's dominated convergence theorem. Therefore, we have

$$\begin{aligned}L_{HM}(s;n,x) &= \frac{1}{\Gamma(s)}\int_0^{\infty}\sum_{d|n}\mu(d)\frac{e^{-xt}}{1-e^{-dt}}t^s\frac{dt}{t}\\ &= \frac{1}{\Gamma(s)}\int_0^{\infty}f(-t;n,x)t^{s-1}\frac{dt}{t}\quad(\text{for }\operatorname{Re}(s)>1).\end{aligned}\tag{3.2}$$

This shows that the Hurwitz-Möbius L-function $L_{HM}(s;n,x)$ is almost the Mellin transform of $f(t;n,x)$, the generating function of Möbius-Bernoulli polynomials, $M_k(n)(x)$ $(k\geq 0)$.

Using Proposition 10.2.2 of [6] and formula (2.2) of [7], we can get the following theorem.

Theorem 5. *Notations as above. We have*

$$L_{HM}(s;n,x)=\sum_{d|n}\frac{\mu(d)}{d^s}\zeta(s;\frac{x}{d}),$$

where $\zeta(s;x)$ is the Hurwitz zeta function with parameter $x>0$; $L_{HM}(s;n,x)$ can be analytically continued to the whole complex plane, to a meromorphic function with a single pole at $s=1$, simple with residue $\frac{\phi(n)}{n}$ and

$$L_{HM}(-k;n,x)=-\frac{M_{k+1}(n)(x)}{k+1},$$

where $k\geq 0$ is an integer.

Proof. By definition (3.1), we have

$$\begin{aligned}L_{HM}(s;n,x) &= \sum_{d|n}\sum_{m=0}^{\infty}\frac{\mu(d)}{(md+x)^s}\\ &= \sum_{d|n}\frac{\mu(d)}{d^s}\sum_{m=0}^{\infty}\frac{1}{(m+x/d)^s}\\ &= \sum_{d|n}\frac{\mu(d)}{d^s}\zeta(s;\frac{x}{d})\quad(\text{ for }\operatorname{Re}(s)>1).\end{aligned}$$

This completes the proof of the first statement.

Now by equation (3.2), we get

$$\begin{aligned}&L_{HM}(s;n,x)-\frac{1}{\Gamma(s)}\int_0^{\infty}\left(\sum_{d|n}\frac{\mu(d)}{d}\frac{e^{-t}}{t}\right)t^s\frac{dt}{t}\\ &=\frac{1}{\Gamma(s)}\int_0^{\infty}\left(\sum_{d|n}\mu(d)\frac{e^{-xt}}{1-e^{-dt}}-\sum_{d|n}\frac{\mu(d)}{d}\frac{e^{-t}}{t}\right)t^s\frac{dt}{t}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{\Gamma(s)}\int_0^\infty \left(\sum_{d|n}\frac{\mu(d)}{d}\frac{e^{-t}}{t}\right)t^s\frac{dt}{t} &= \sum_{d|n}\frac{\mu(d)}{d}\frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{-t}}{t}t^s\frac{dt}{t}\\ &= \sum_{d|n}\frac{\mu(d)}{d}\frac{1}{\Gamma(s)}\int_0^\infty e^{-t}t^{s-1}\frac{dt}{t}\\ &= \sum_{d|n}\frac{\mu(d)}{d}\frac{\Gamma(s-1)}{\Gamma(s)}\\ &= \frac{1}{s-1}\sum_{d|n}\frac{\mu(d)}{d}\\ &= \frac{1}{s-1}\frac{\phi(n)}{n}.\end{aligned}$$

Let

$$\widetilde{f}(t;n,x) = \left(\sum_{d|n}\mu(d)\frac{e^{-xt}}{1-e^{-dt}} - \sum_{d|n}\frac{\mu(d)}{d}\frac{e^{-t}}{t}\right).$$

Then $\widetilde{f}(t;n,x)$ is C^∞ on $[0,\infty)$ and tending to zero rapidly at infinity. We define

$$L(s,\widetilde{f},x) := \frac{1}{\Gamma(s)}\int_0^\infty \widetilde{f}(t;n,x)t^s\frac{dt}{t}. \tag{3.3}$$

By Proposition 10.2.2 of [6], we know that $L(s,\widetilde{f},x)$ can be analytically continued to the whole complex plane, to a holomorphic function.

Obviously, the above computations show that

$$L_{HM}(s;n,x) = L(s,\widetilde{f},x) + \frac{1}{s-1}\prod_{p|n}(1-\frac{1}{p}). \tag{3.4}$$

So $L_{HM}(s;n,x)$ can be analytically continued to the whole complex plane, to a meromorphic function with a single pole at $s=1$, simple with residue $\frac{\phi(n)}{n}$. This completes the proof of the second statement.

Also by Proposition 10.2.2 of [6], we have

$$L(-k,\widetilde{f},x) = (-1)^k\frac{d^k\widetilde{f}}{dt^k}(0) \tag{3.5}$$

for $k \geq 0$ and $k \in \mathbb{Z}$.

Obviously, $\widetilde{f}(t;n,x)$ is analytic around zero. Computing its Taylor expansion at 0, we get

$$\begin{aligned}\widetilde{f}(t;n,x) &= \frac{1}{t}\left(\sum_{d|n}\mu(d)\frac{(-t)e^{-xt}}{e^{-dt}-1} - \sum_{d|n}\frac{\mu(d)}{d}e^{-t}\right)\\ &= \frac{1}{t}\left(\sum_{k=1}^\infty M_k(n)(x)\frac{(-t)^k}{k!} - \sum_{d|n}\frac{\mu(d)}{d}\sum_{k=1}^\infty\frac{(-t)^k}{k!}\right)\\ &= \sum_{k=1}^\infty(-1)^k\left(M_k(n)(x) - \sum_{d|n}\frac{\mu(d)}{d}\right)\frac{t^{k-1}}{k!}\\ &= \sum_{k=0}^\infty(-1)^{k+1}\left(M_{k+1}(n)(x) - \sum_{d|n}\frac{\mu(d)}{d}\right)\frac{1}{k+1}\frac{t^k}{k!}.\end{aligned}$$

Therefore,

$$L(-k,\widetilde{f},x)=-\frac{1}{k+1}\left(M_{k+1}(n)(x)-\sum_{d|n}\frac{\mu(d)}{d}\right). \tag{3.6}$$

From equations (3.4) and (3.6), we get

$$\begin{aligned}L_{HM}(-k;n,x)&=-\frac{1}{k+1}\prod_{p|n}\left(1-\frac{1}{p}\right)-\frac{1}{k+1}\left(M_{k+1}(n)(x)-\sum_{d|n}\frac{\mu(d)}{d}\right)\\&=-\frac{1}{k+1}M_{k+1}(n)(x).\end{aligned}$$

This completes the proof of the final statement. □

3.2 Modified Möbius L-functions

For n being a positive integer, let

$$g(t):=\sum_{k=0}^{\infty}\widetilde{M}_k(n)\frac{t^k}{k!}=\sum_{d|n}\mu(d)\frac{te^{dt}}{e^{nt}-1}.$$

We call $\widetilde{M}_k(n)$ *modified Möbius-Bernoulli numbers*.

In the following, we will first show their relations with Bernoulli polynomials. For $n=1$,

$$\sum_{k=0}^{\infty}\widetilde{M}_k(n)\frac{t^k}{k!}=t+\frac{t}{e^t-1}=t+\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}.$$

So $\widetilde{M}_k(1)$ is essentially Bernoulli number B_k.

Now we consider the general case. Then we have

$$\sum_{k=0}^{\infty}\widetilde{M}_k(n)\frac{t^k}{k!}=\sum_{d|n}\mu(d)\frac{te^{dt}}{e^{nt}-1}.$$

Changing variable t to t/n, we get,

$$\sum_{k=0}^{\infty}\widetilde{M}_k(n)\frac{1}{n^k}\frac{t^k}{k!}=\frac{1}{n}\sum_{d|n}\mu(d)\frac{te^{\frac{d}{n}t}}{e^t-1}=\frac{1}{n}\sum_{d|n}\mu(d)\sum_{k=0}^{\infty}B_k(\frac{d}{n})\frac{t^k}{k!}.$$

Thus we have the following relation

$$\widetilde{M}_k(n)=n^{k-1}\sum_{d|n}\mu(d)B_k(\frac{d}{n}). \tag{3.7}$$

For n being a positive integer, let

$$L_M(s;n):=\sum_{d|n}\sum_{m=0}^{\infty}\frac{\mu(d)}{(mn+d)^s}\ \ (\text{for } \mathrm{Re}(s)>1). \tag{3.8}$$

We call $L_M(s;n)$ the modified Möbius L-functions. Note that if $n=1$, then the modified Möbius L-function $L_M(s;n)$ is just the usual Riemann zeta function $\zeta(s)$. Similarly as in the previous subsection, we can prove that

$$\begin{aligned}L_M(s;n)&=\frac{1}{\Gamma(s)}\int_0^{\infty}\left(\sum_{d|n}\mu(d)\frac{e^{-dt}}{1-e^{-nt}}\right)t^s\frac{dt}{t}\\&=\frac{1}{\Gamma(s)}\int_0^{\infty}g(-t)t^{s-1}\frac{dt}{t}\ \ (\text{for } \mathrm{Re}(s)>1),\end{aligned} \tag{3.9}$$

which shows that the modified Möbius L-function $L_M(s;n)$ is almost the Mellin transform of $g(t)$, the generating function of modified Möbius-Bernoulli numbers, $\widetilde{M}_k(n)$ ($k \geq 0$).

Changing variable t to t/n in (3.9), we get

$$L_M(s;n) = \sum_{d|n} \frac{\mu(d)}{n^s} \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-\frac{d}{n}t}}{1-e^{-t}} t^s \frac{dt}{t} \quad \text{(for Re}(s) > 1). \tag{3.10}$$

Similarly as Theorem 5, we can prove the following theorem

Theorem 6. *Let $n \geq 1$ be an integer. Then we have*

$$L_M(s;n) = n^{-s} \sum_{d|n} \mu(d)\zeta(s;d/n).$$

We have $L_M(s;1) = \zeta(s)$. For $n \geq 2$, $L_M(s;n)$ has holomorphic continuation to the whole s-complex plane and

$$L_M(-k;n) = -\frac{\widetilde{M}_{k+1}(n)}{k+1}.$$

4 Double Möbius-Bernoulli and Barnes-Bernoulli numbers

In this section we introduce and give some properties of the *double Möbius-Bernoulli* numbers. We express these numbers in terms of Barnes-Bernoulli numbers. Using Barnes-Bernoulli numbers properties, explicit formulas will be given for *double Möbius-Bernoulli* in section 5.

Let a_1, a_2 be nonzero real numbers. The double Bernoulli-Barnes numbers $B_k((a_1,a_2))$ are defined through

$$\frac{z^2}{(e^{a_1 z}-1)(e^{a_2 z}-1)} = \sum_{k=0}^\infty B_k((a_1,a_2))\frac{z^k}{k!}, \ |t| < \min\left(\frac{2\pi}{|a_1|}, \frac{2\pi}{|a_2|}\right). \tag{4.1}$$

We investigate, for n, n' being positive integers, the double Möbius-Bernoulli numbers $M_k(n,n')$ given by

$$M_k(n,n') = \sum_{j=0}^k \binom{k}{j} M_j(n)M_{k-j}(n'). \tag{4.2}$$

Theorem 7. *Let n, n' be positive integers and $n^\star$, $n'^\star$ be their square free parts, respectively. We have the following results.*

$$M_k(n,n') = \sum_{d|n,\ d'|n'} \mu(d)\mu(d')B_k((d,d')). \tag{4.3}$$

$$M_k(n) = M_k(n^\star) \tag{4.4}$$

$$M_k(n,n') = M_k(n^\star, n'^\star). \tag{4.5}$$

Proof of Theorem 10 . Since $\mu(d) = 0$ if d is not square-free, we obtain the relations (4.4) and (4.5). On the other hand, it is easy to show that

$$\left(\sum_{d|n}\mu(d)\frac{t}{e^{dt}-1}\right)\left(\sum_{d'|n'}\mu(d')\frac{t}{e^{d't}-1}\right)$$
$$= \sum_{d|n,d'|n'} \mu(d)\mu(d')\frac{t^2}{(e^{dt}-1)(e^{d't}-1)}.$$

Expanding the Taylor series of both sides yields the following identity

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}M_j(n)M_{k-j}(n')\right)\frac{t^k}{k!}=\sum_{k=0}^{\infty}\left(\sum_{\substack{d|n,\\ d'|n'}}\mu(d)\mu(d')B_k((d,d'))\right)\frac{t^k}{k!}.$$

Thus we obtain identity (4.3). □

5 Double Möbius-Bernoulli numbers and Dedekind sums

In this section, we give an effective method to compute the Möbius-Bernoulli numbers, based on the Apostol-Dedekind reciprocity law for the generalized Dedekind sums.

5.1 Generalized Dedekind Sums $s_k(a, b)$

Let a and b be positive integers. The *Apostol–Dedekind sums* $s_k(a, b)$ are given by

$$s_k(a,b)=\sum_{r=0}^{b-1}\frac{r}{b}B_k\left(\left\{\frac{ar}{b}\right\}\right). \tag{5.1}$$

These sums are effectively computable through the Apostol-Dedekind reciprocity law [8] and its generalization in [9]. By use of the Apostol-Dedekind reciprocity we state the following results.

Theorem 8. *Let a, b be positive integers, k positive integer, and $d = \gcd(a, b)$. Then we have*

$$\begin{aligned}B_k((a,b)) =&k\left(a^{k-2}s_{k-1}(b,a)+b^{k-2}s_{k-1}(a,b)\right)-kd^{k-1}B_{k-1}\\ &-\frac{(k-1)d^{2(k-1)}}{ab}B_k.\end{aligned} \tag{5.2}$$

From the above theorem we get the following identities.

Corollary 9. *Let p, p_1, p_2 be prime numbers, with p_1, p_2 different. We have*

$$B_k((1,1))=-k\,B_{k-1}-(k-1)\,B_k, \tag{5.3}$$

$$B_k((p,1))=kp^{k-2}s_{k-1}(1,p)-k\,B_{k-1}-\frac{(k-1)}{p}B_k, \tag{5.4}$$

$$B_k((p,p))=-kp^{k-2}\,B_{k-1}-(k-1)p^{2(k-2)}\,B_k, \tag{5.5}$$

$$B_k((p_1,p_2))=k\left(p_2^{k-2}s_{k-1}(p_1,p_2)+p_1^{k-2}s_{k-1}(p_2,p_1)\right)-kB_{k-1}-\frac{(k-1)}{p_1p_2}B_k. \tag{5.6}$$

The equality (5.3) is well-known since Euler [10, p.32]. However, the equalities (5.4), (5.5), (5.6) seem new identities.

Theorem 10. *Let n, n' be positive coprime integers and $n^\star, n'^\star$ be their square free parts, respectively. We have*

$$\begin{aligned}M_k(n,n') =&k\sum_{d|n^\star,\ d'|n'^\star}\mu(d)\mu(d')\left[d^{k-2}s_{k-1}(d',d)+d'^{k-2}s_{k-1}(d,d')\right]\\ &-kB_{k-1}\delta_{1,n^\star n'^\star}-(k-1)B_k\frac{\varphi(n^\star n'^\star)}{n^\star n'^\star},\end{aligned} \tag{5.7}$$

where φ is the Euler's function and $\delta_{i,j}$ is the Kronecker's symbol.

We combine Corollary 9 with Theorem 10, and we obtain the following explicit formulas.

Corollary 11. *Let p_1, p_2, p be distinct primes and $\alpha \geq 1, \beta \geq 1$ are integers. We have*

$$\begin{aligned} M_k(p_1^\alpha, p_2^\beta) =& -(k-1)\left(1-1/p_1\right)\left(1-1/p_2\right)B_k + kp_2^{k-2}\left(s_{k-1}(p_1,p_2) - s_{k-1}(1,p_2)\right) \\ &+ kp_1^{k-2}\left(s_{k-1}(p_2,p_1) - s_{k-1}(1,p_1)\right), \\ M_k(p^\alpha, p^\beta) =& k\left(1-p^{k-2}\right)B_{k-1} - (k-1)\left(1+p^{2(k-2)} - 2/p\right)B_k \\ &- 2kp^{k-2}s_{k-1}(1,p). \end{aligned}$$

5.2 Proofs of the *Theorem 8 and Theorem 10*

Proof of Theorem 8. Combining (2.2) with (4.1), we get the relation between Bernoulli-Barnes numbers and Bernoulli numbers:

$$B_k((a_1,a_2)) = \sum_{m=0}^{k} \binom{k}{m} a_1^{m-1} a_2^{k-m-1} B_m B_{k-m}. \tag{5.8}$$

By use of the equation (5.8), Apostol results in [8, Theorem 1] , [11, Theorem 2] and Takacs generalization [9, Theorem 1], with $x = y = 0$, we achieve the proof of the Theorem 8. □

Proof of Theorem 10. Take n, n', a, b be positive integers such that:

$$(a,b) = (n,n') = 1.$$

Then we have the equalities

$$B_k((a,b)) = k\left(a^{k-2}s_{k-1}(b,a) + b^{k-2}s_{k-1}(a,b)\right) - k\,B_{k-1} - \frac{(k-1)}{ab}B_k, \tag{5.9}$$

$$M_k(n,n') = M_k(n^*, n'^*) = \sum_{d|n^*,\ d'|n'^*} \mu(d)\mu(d')B_k((d,d')), \tag{5.10}$$

$$\frac{\varphi(n)}{n} = \sum_{d|n} \mu(d)d^{-1}, \tag{5.11}$$

$$\delta_{1,n} = \sum_{d|n} \mu(d). \tag{5.12}$$

From these equalities we complete the proof of the Theorem 10. □

We conclude this paper by the following remark.

Remark 12. The generalized Dedekind sums $s_{k-1}(a,b)$ are very easy to evaluate for small a and b. For example, using (5.1), we get values of $s_{k-1}(a,b)$ in Table 1 with $1 \leq a, b \leq 5$ and $(a,b) = 1$.

Table 1: Examples for $s_{k-1}(a,b)$ with $1 \le a, b \le 5$ and $(a,b)=1$

(a,b)	$s_{k-1}(a,b)$
$(a,1)$	0
$(1,2),(3,2),(5,2)$	$\frac{1}{2}B_{k-1}(\frac{1}{2})$
$(1,3),(4,3)$	$\frac{1}{3}B_{k-1}(\frac{1}{3})+\frac{2}{3}B_{k-1}(\frac{2}{3})$
$(1,4),(5,4)$	$\frac{1}{4}B_{k-1}(\frac{1}{4})+\frac{1}{2}B_{k-1}(\frac{1}{2})+\frac{3}{4}B_{k-1}(\frac{3}{4})$
$(1,5)$	$\frac{1}{5}B_{k-1}(\frac{1}{5})+\frac{2}{5}B_{k-1}(\frac{2}{5})+\frac{3}{5}B_{k-1}(\frac{3}{5})+\frac{4}{5}B_{k-1}(\frac{4}{5})$
$(2,3),(5,3)$	$\frac{1}{3}B_{k-1}(\frac{2}{3})+\frac{2}{3}B_{k-1}(\frac{1}{3})$
$(2,5)$	$\frac{1}{5}B_{k-1}(\frac{2}{5})+\frac{2}{5}B_{k-1}(\frac{4}{5})+\frac{3}{5}B_{k-1}(\frac{1}{5})+\frac{4}{5}B_{k-1}(\frac{3}{5})$
$(3,4)$	$\frac{1}{4}B_{k-1}(\frac{3}{4})+\frac{1}{2}B_{k-1}(\frac{1}{2})+\frac{3}{4}B_{k-1}(\frac{1}{4})$
$(3,5)$	$\frac{1}{5}B_{k-1}(\frac{3}{5})+\frac{2}{5}B_{k-1}(\frac{1}{5})+\frac{3}{5}B_{k-1}(\frac{4}{5})+\frac{4}{5}B_{k-1}(\frac{2}{5})$
$(4,5)$	$\frac{1}{5}B_{k-1}(\frac{4}{5})+\frac{2}{5}B_{k-1}(\frac{3}{5})+\frac{3}{5}B_{k-1}(\frac{2}{5})+\frac{4}{5}B_{k-1}(\frac{1}{5})$

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