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Characterizations of the Solution Sets of Generalized Convex Fuzzy Optimization Problem

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Abstract: This paper provides some new characterizations of the solution sets for non-differentiable generalized convex fuzzy optimization problem. Firstly, we introduce some new generalized convex fuzzy functions and discuss the relationships among them. Secondly, some properties of these new generalized convex fuzzy functions are given. Finally, as applications, some characterizations of the solution sets for non-differentiable generalized convex fuzzy optimization problem are obtained.

Keywords: Generalized convex fuzzy mapping, Solution sets, Optimization problem

MSC: 90C26; 90C29; 90C30; 90C46

1 Introduction

As we all know, convexity and generalized convexity play a crucial role in many aspects of mathematical programming including, for example, optimality conditions, duality theorems, saddle points, variational inequalities and characterizations of the solution sets, one can refer to [1–18]. In terms of some properties and characterizations of the solution sets, they are very useful for understanding the behavior of solution methods for optimization problems that have multiple optimal solutions. Mangasarian, [8], initially gave the characterizations of the solution sets for differentiable convex programming problems and obtained some equivalent representations of the solution sets. From then on, the characterizations of the solution sets have been widely studied under convexity or generalized convexity by many scholars. For instance, Jeyakumar and Yang [9] gave the solution sets of multi-objective optimization problems with convexity and extended the results in [8] to pseudolinear programming problems. By means of sub-differentiable and Gateaux differentiable, Wu and Wu, [12], characterized the solution sets for a general convex optimization problem in normed vector spaces. Yang [13] investigated the solution sets for differentiable extremum problems under the assumption of pseudoinvexity. Liu et al., [15], extended the results in [13] to the non-differentiable pseudoinvex programs and gave the characterizations of the solution sets involving the non-differentiable pseudoinvex functions.

On the other hand, in [19], Chang and Zadeh introduced the concept of fuzzy mapping. Since then, many researchers conducted in-depth investigations into fuzzy mapping and obtained a series of important conclusions in the problems of integrability, differentiability and measurability of fuzzy mappings [20–

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25]. At the same time, convexity and generalized convexity of fuzzy mappings and their applications have been deeply and widely studied. For instance, Nanda and Kar, [26], proposed a concept of convex fuzzy mapping and verified that a fuzzy mapping is convex if and only if its epigraph is a convex set. Yan and Xu, [20], introduced a new class of convexity and quasiconvexity of fuzzy mapping by considering the order relation proposed by Goestschel and Voxman [27]. Panigrahi et al., [28], solved the minimization of fuzzy mapping and modified the definition of quasiinvex fuzzy mapping, which is different from the one proposed by Nanda and Kar [26]. Syau, [29], introduced B-preinvexity, pseudo-B-vexity, B-invexity and pseudo-B-invexity of fuzzy mappings and obtained sufficient optimality conditions for B-invex and B-preinvex fuzzy mappings. In [30], Wu and Xu defined the concepts of some generalized convex fuzzy mappings including fuzzy pseudoconvexity, fuzzy invexity, fuzzy concavity, fuzzy preinvexity, fuzzy prequasiinvexity and fuzzy pseudoinvexity. Moreover, under the assumptions of these generalized convex fuzzy mappings, Wu and Xu in [30] established the relations between the fuzzy variational-like inequality and the fuzzy optimization problems. By using parameterized representation of fuzzy numbers, Syau and Lee in [31] gave the criteria for a lower semicontinuous fuzzy mapping defined on a non-empty convex subset of $\mathbb{R}^n$ to be a convex fuzzy mapping. Li and Noor, [32], generalized the work of Syau and Lee in [31] and proved that a fuzzy mapping is preinvex if and only if the endpoint functions are preinvex. Li et al., [33], introduced a kind of fuzzy weakly uninvex mapping and obtained the optimality conditions and duality results for constrained fuzzy minimization problem under the assumption of weak fuzzy uninvexity. Based on Wu and Xu in [30], Rufián-Lizanaa et al. in [34] presented more general notions of invex and concave fuzzy mappings involving strongly generalized differentiable fuzzy mapping. In [35], Osuna-Gómez et al. discussed necessary and sufficient conditions for fuzzy optimality problems under the assumption of the new fuzzy generalized convexity. Arana-Jiménez et al. in [25] studied efficiency and weak efficiency in fuzzy vector optimization through a linear ordering. Motivated by these works in [21, 26, 28, 30–34, 36–38], our first idea is to introduce some new classes of generalized fuzzy convex mappings, namely, α-preinvexity, α-prequasiinvexity and α-pseudoinvexity of fuzzy mappings and discuss the relationships and several basic properties among them.

Convexity and generalized convexity of fuzzy functions play an important role in optimization theory including the fuzzy variational inequality [30, 34, 38–40], the sufficient and necessary optimality conditions [28, 33, 35, 37, 41], the saddle point [42, 43] and duality [33]. Recently, some scholars have begun to study the characterization of the solution sets for fuzzy programming. Yang and Wu in [41] used the concept of sub-differential to define the quasiconvex fuzzy mapping and obtained some properties of the solution sets in the fuzzy settings. Mishra et al., [44], introduced the pseudolinear and η-pseudolinear fuzzy mappings by relaxing the definitions of the pseudoconvex and pseudoinvex fuzzy mappings and derived the characterizations of the solution sets of the pseudolinear and η-pseudolinear fuzzy programs. However, to the best of our knowledge, the property and characterization of the solution sets, as an important part of fuzzy optimization theory, have not been extensively studied by scholars. Therefore, it is very significant to explore the theory in this respect. The second aim of this paper is to obtain some characterizations of the solution sets for fuzzy optimization problem by virtue of fuzzy α-preinvexity, fuzzy α-prequasiinvexity and fuzzy α-pseudoinvexity.

This paper is organized as follows. In Section 2, some definitions and basic results about fuzzy numbers are recalled. In Section 3, we introduce some new classes of generalized fuzzy convex mappings, namely, α-preinvexity, α-prequasiinvexity and α-pseudoinvexity of fuzzy mappings and define the $\alpha\eta$-directional differentiability of a fuzzy mapping before discussing the several relationships. In Section 4, we obtain some properties with respect to α-preinvexity, α-prequasiinvexity and $\alpha\eta$-directional differentiability of a fuzzy mapping under a series of assumptions. In Section 5, by applying some results obtained in Section 4, we present some equivalent characterizations of the solution sets of the fuzzy optimization problem involving the fuzzy α-pseudoinvexity.

2 Preliminaries

Throughout the paper, $\mathbb{R}^n$, $\mathbb{R}$ and $\mathbb{R}_+$ represent the n-dimensional Euclidean space, the set of real numbers and the set of nonnegative real numbers, respectively. Now, we recall some definitions and known results about fuzzy numbers which will be used throughout the paper.

A fuzzy set on $\mathbb{R}^n$ is a mapping $\tilde{u} : \mathbb{R}^n \to [0,1]$. The r-level set of a fuzzy set $\tilde{u}$ on $\mathbb{R}^n$ is denoted by $[\tilde{u}]^r = \{x \in \mathbb{R}^n | \tilde{u}(x) \geqslant r\}$ for any $r \in (0,1]$. We use $supp\,\tilde{u} = \{x \in \mathbb{R}^n | \tilde{u}(x) > 0\}$ to represent the support of $\tilde{u}$. The closure of $supp\,\tilde{u}$ is written as $[\tilde{u}]^0$.

Definition 2.1. [21] *A fuzzy number $\tilde{u}$ is a fuzzy set with the following properties:*
(i) $\tilde{u}$ is normal, i.e., there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{u}(x_0) = 1$;
(ii) $\tilde{u}$ is upper semi-continuous (u.s.c);
(iii) $\tilde{u}(\lambda x + (1-\lambda)y) \geqslant \min\{\tilde{u}(x), \tilde{u}(y)\}$ for all $x, y \in \mathbb{R}^n$, $\lambda \in [0,1]$;
(iv) $[\tilde{u}]^0$ is compact.

The family of fuzzy numbers is denoted by $\mathbb{E}$. Clearly, $\tilde{u} \in \mathbb{E}$ is a fuzzy number if and only if $[\tilde{u}]^r$ can be represented by $[\tilde{u}_\star(r), \tilde{u}^\star(r)]$ which is a nonempty compact convex subset for any $r \in [0,1]$, where $\tilde{u}_\star(r)$ denotes the left-hand endpoint of $[\tilde{u}]^r$ and $\tilde{u}^\star(r)$ denotes the right-hand endpoint of $[\tilde{u}]^r$. Thus, a fuzzy number is determined by the endpoints of the interval $[\tilde{u}_\star(r), \tilde{u}^\star(r)]$. A precise number $a \in \mathbb{R}$ is a special case of fuzzy number encoded as

$$\tilde{a}(t) = \begin{cases} 1, & \text{if} \quad t = a, \\ 0, & \text{if} \quad t \neq a. \end{cases}$$

In particular, the fuzzy number $\tilde{0}$ is defined as $\tilde{0}(t) = 1$ if $t = 0$, and $\tilde{0}(t) = 0$ if $t \neq 0$. Thus, a fuzzy number $\tilde{u}$ can be identified by a parameterized triple $\{(\tilde{u}_\star(r), \tilde{u}^\star(r), r) | r \in [0,1]\}$.

The following lemma makes the connection between fuzzy numbers and their endpoint functions.

Lemma 2.1. [27] *Assume that $I = [0,1]$, $\tilde{u}_\star : I \to \mathbb{R}$ and $\tilde{u}^\star : I \to \mathbb{R}$ satisfy the following conditions:*
(i) $\tilde{u}_\star : I \to \mathbb{R}$ is a bounded increasing function;
(ii) $\tilde{u}^\star : I \to \mathbb{R}$ is a bounded decreasing function;
(iii) $\tilde{u}_\star(1) \leqslant \tilde{u}^\star(1)$;
(iv) $\lim_{r\to k^-} \tilde{u}_\star(r) = \tilde{u}_\star(k)$ and $\lim_{r\to k^-} \tilde{u}^\star(r) = \tilde{u}^\star(k)$ for $0 < k \leqslant 1$;
(v) $\lim_{r\to 0^+} \tilde{u}_\star(r) = \tilde{u}_\star(0)$ and $\lim_{r\to 0^+} \tilde{u}^\star(r) = \tilde{u}^\star(0)$.
Then, $\tilde{u} : \mathbb{R} \to I$ defined by $\tilde{u}(x) = \sup\{r | \tilde{u}_\star(r) \leqslant x \leqslant \tilde{u}^\star(r)\}$ is a fuzzy number with parameterization given by $\{(\tilde{u}_\star(r), \tilde{u}^\star(r), r) | r \in [0,1]\}$. Moreover, if $\tilde{u} : \mathbb{R} \to I$ is a fuzzy number with parametrization given by $\{(\tilde{u}_\star(r), \tilde{u}^\star(r), r) | r \in [0,1]\}$, then the functions $\tilde{u}_\star(r)$ and $\tilde{u}^\star(r)$ satisfy the above conditions (i)-(v).

Triangular fuzzy numbers are a special type of fuzzy numbers which are defined by three real numbers a, b and c with $a \leqslant b \leqslant c$ and we write $\tilde{u} := \langle a, b, c \rangle$ and $[\tilde{u}]^r = [a + (b-a)r, c - (c-b)r]$, $\forall r \in [0,1]$. For given fuzzy numbers $\tilde{u}, \tilde{v} \in \mathbb{E}$ denoted by intervals $[\tilde{u}_\star(r), \tilde{u}^\star(r)]$ and $[\tilde{v}_\star(r), \tilde{v}^\star(r)]$ for any $r \in [0,1]$, respectively, and for any real number $\lambda \in \mathbb{R}$, we define the fuzzy addition $\tilde{u} \tilde{+} \tilde{v}$ and scalar multiplication $\lambda \tilde{u}$ as follows:

$$(\tilde{u} \tilde{+} \tilde{v})(x) = \sup_{y+z=x} \min[\tilde{u}(y), \tilde{v}(z)], \quad (\lambda \tilde{u})(x) = \begin{cases} \tilde{u}(\frac{x}{\lambda}), & \text{if} \quad \lambda \neq 0, \\ 0, & \text{if} \quad \lambda = 0. \end{cases}$$

For $\tilde{u}, \tilde{v} \in \mathbb{E}$, $\lambda \in \mathbb{R}$, $r \in [0,1]$, we have $[\tilde{u} \tilde{+} \tilde{v}]^r = [\tilde{u}]^r \tilde{+} [\tilde{v}]^r$ and $[\lambda \tilde{u}]^r = \lambda [\tilde{u}]^r$, i.e.,

$$(\tilde{u} \tilde{+} \tilde{v})_\star(r) = \tilde{u}_\star(r) + \tilde{v}_\star(r), \quad (\tilde{u} \tilde{+} \tilde{v})^\star(r) = \tilde{u}^\star(r) + \tilde{v}^\star(r),$$

$$(\lambda \tilde{u})_\star(r) = \begin{cases} \lambda \tilde{u}_\star(r), & \lambda \geqslant 0, \\ \lambda \tilde{u}^\star(r), & \lambda < 0, \end{cases} \quad (\lambda \tilde{u})^\star(r) = \begin{cases} \lambda \tilde{u}^\star(r), & \lambda \geqslant 0, \\ \lambda \tilde{u}^\star(r), & \lambda < 0. \end{cases}$$

Definition 2.2. [45] *Let $\tilde{u}, \tilde{v} \in \mathbb{E}$. If there exists $\tilde{\omega} \in \mathbb{E}$ such that $\tilde{u} = \tilde{v}\tilde{+}\tilde{\omega}$, then $\tilde{\omega}$ is called the Hukuhara difference (H-difference, for short and denoted by $\tilde{u} \ominus_H \tilde{v}$) of $\tilde{u}$ and $\tilde{v}$.*

Remark 2.1. *Clearly, if the H-difference $\tilde{\omega} = \tilde{u} \ominus_H \tilde{v}$ exists, then for any $r \in [0,1]$, $\tilde{\omega}_\star(r) = (\tilde{u} \ominus_H \tilde{v})_\star(r) = \tilde{u}_\star(r) - \tilde{v}_\star(r)$ and $\tilde{\omega}^\star(r) = (\tilde{u} \ominus_H \tilde{v})^\star(r) = \tilde{u}^\star(r) - \tilde{v}^\star(r)$. Moreover, we also have $k(\tilde{u} \ominus_H \tilde{v}) = k\tilde{u} \ominus_H k\tilde{v}$ for any $k \in \mathbb{R}_+$.*

Definition 2.3. [21] *Let $\tilde{u}, \tilde{v} \in \mathbb{E}$.*
(i) $\tilde{u} \preccurlyeq \tilde{v}$ iff $\tilde{u}_\star(r) \leqslant \tilde{v}_\star(r)$ and $\tilde{u}^\star(r) \leqslant \tilde{v}^\star(r)$ for each $r \in [0,1]$;
(ii) If $\tilde{u} \preccurlyeq \tilde{v}$ and $\tilde{v} \preccurlyeq \tilde{u}$, then $\tilde{u} = \tilde{v}$;
(iii) $\tilde{u} \prec v$ iff $\tilde{u} \preccurlyeq \tilde{v}$ and there exists $r_0 \in [0,1]$ such that $\tilde{u}_\star(r_0) < \tilde{v}_\star(r_0)$ or $\tilde{u}^\star(r_0) < \tilde{v}^\star(r_0)$;
(iv) $\tilde{u}$ and $\tilde{v}$ are comparable iff either $\tilde{u} \preccurlyeq \tilde{v}$ or $\tilde{v} \preccurlyeq \tilde{u}$; otherwise they are non-comparable.

Note that $\preccurlyeq$ is a partial order relation on $\mathbb{E}$. It is sometimes convenient to write $\tilde{v} \succcurlyeq \tilde{u}$ (respectively, $\tilde{v} \succ \tilde{u}$) in place of $\tilde{u} \preccurlyeq \tilde{v}$ (respectively, $\tilde{u} \prec \tilde{v}$).

Remark 2.2. *(i) Let $[\tilde{u}]^r = [\tilde{u}_\star(r), \tilde{u}^\star(r)]$ for any $r \in [0,1]$. Then, $k[\tilde{u}]^r \succcurlyeq \tilde{0}$ iff $k\tilde{u}_\star(r) \geqslant 0$ and $k\tilde{u}^\star(r) \geqslant 0$ for any $r \in [0,1]$, where $k \in \mathbb{R}_+$ and $\tilde{0} = [0,0]$. (ii) $\tilde{u} \preccurlyeq \tilde{v}$ iff $\tilde{v}_\star(r) - \tilde{u}_\star(r) \geqslant 0$ and $\tilde{v}^\star(r) - \tilde{u}^\star(r) \geqslant 0$ for any $r \in [0,1]$. (iii) Let $\tilde{u} \preccurlyeq \tilde{v}$. If $\lambda \geqslant 0$, then $\lambda\tilde{u} \preccurlyeq \lambda\tilde{v}$. Accordingly, if $\lambda < 0$, then $\lambda\tilde{u} \succcurlyeq \lambda\tilde{v}$.*

Let $\tilde{f} : K(\subseteq \mathbb{R}^n) \to \mathbb{E}$ be a fuzzy mapping. The r-cut of $\tilde{f}$ at $x \in K$, which is a closed and bounded interval, can be denoted by $[\tilde{f}(x)]^r = \tilde{f}(x,r) := [\tilde{f}_\star(x,r), \tilde{f}^\star(x,r)]$ for any $r \in [0,1]$, where $\tilde{f}_\star(x,r)$ and $\tilde{f}^\star(x,r)$ are functions from $K \times [0,1]$ to the set of real numbers $\mathbb{R}$. $\tilde{f}_\star(x,r)$ is a bounded increasing function of r and $\tilde{f}^\star(x,r)$ is a bounded decreasing function of r. Moreover, $\tilde{f}_\star(x,r) \leqslant \tilde{f}^\star(x,r)$.

Definition 2.4. [36] *Let $\tilde{f} : K \to \mathbb{E}$ be a fuzzy mapping.*
(i) $\tilde{f}$ is called u.s.c at $x_0 \in K$ iff both $\tilde{f}_\star(x,r)$ and $\tilde{f}^\star(x,r)$ are u.s.c at x_0 uniformly in $r \in [0,1]$. $\tilde{f}$ is called u.s.c. on K iff it is u.s.c. at each point of K.
(ii) $\tilde{f}$ is called lower semicontinuous (l.s.c) at $x_0 \in K$ iff both $\tilde{f}_\star(x,r)$ and $\tilde{f}^\star(x,r)$ are l.s.c at x_0 uniformly in $r \in [0,1]$. $\tilde{f}$ is called l.s.c on K iff it is l.s.c at each point of K.

Lemma 2.2. [46] *Let K be a nonempty closed and bounded subset of $\mathbb{R}^n$. A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ which is u.s.c (respectively, l.s.c) on K attains its maximum (respectively, minimum) on K.*

3 Generalized convexity of fuzzy mappings

Let K be a nonempty subset of $\mathbb{R}^n$. Let $\alpha(\cdot,\cdot) : K \times K \to \mathbb{R}\backslash\{0\}$ be a real-valued function and $\eta(\cdot,\cdot) : K \times K \to \mathbb{R}^n$ be a vector-valued mapping.

Definition 3.1. [47] *Let $y \in K$. Then the set K is called an α-invex at y with respect to (w.r.t., shortly) α and η iff for each $x \in K$, $\lambda \in [0,1]$, $y + \lambda\alpha(x,y)\eta(x,y) \in K$. K is called an α-invex set w.r.t. α and η iff K is α-invex at each $y \in K$.*

Remark 3.1. *Obviously, K is a convex set with $\alpha(x,y) = 1$, $\eta(x,y) = x - y$ and is an invex set in [48] with $\alpha(x,y) = 1$ for all $x, y \in K$. However, the following example shows that the converse is not true.*

Example 3.1. *Let $K = [-3,-2] \cup [2,5]$,*

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x>0,\ y>0, \\ 1, & \text{if } x<0,\ y<0, \\ -1, & \text{if } x>0,\ y<0, \\ -1, & \text{if } x<0,\ y>0, \end{cases} \quad \eta(x,y) = \begin{cases} x-y, & \text{if } x>0,\ y>0, \\ x-y, & \text{if } x<0,\ y<0, \\ y+3, & \text{if } x>0,\ y<0, \\ y-2, & \text{if } x<0,\ y>0, \end{cases}$$

where $x, y \in K$. Obviously, K is an α-invex set w.r.t. α and η for each $x, y \in K$ and $\lambda \in [0,1]$. Indeed,

$$y + \lambda\alpha(x,y)\eta(x,y) = \begin{cases} \lambda x + (1-\lambda)y \in K, & \text{if } x>0,\ y>0, \\ \lambda x + (1-\lambda)y \in K, & \text{if } x<0,\ y<0, \\ -3\lambda + (1-\lambda)y \in K, & \text{if } x>0,\ y<0, \\ 2\lambda + (1-\lambda)y \in K, & \text{if } x<0,\ y>0. \end{cases}$$

However, let $\bar{x} = -2, \bar{y} = 5, \bar{\lambda} = \frac{1}{2}$. We have $\bar{y} + \bar{\lambda}\eta(\bar{x},\bar{y}) = \bar{y} + \bar{\lambda}(\bar{y}-2) = \frac{13}{2} \notin K$, which implies that K is not an invex set w.r.t. the same η.

Definition 3.2. [26] *Let K be a nonempty convex subset of $\mathbb{R}^n$. A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called convex on K iff*

$$\tilde{f}(\lambda x + (1-\lambda)y) \preccurlyeq \lambda\tilde{f}(x)\tilde{+}(1-\lambda)\tilde{f}(y),\ \forall x, y \in K, \lambda \in [0,1].$$

Lemma 3.1. [31] *Let K be a nonempty convex subset of $\mathbb{R}^n$. $\tilde{f}$ is convex on K iff the endpoint functions $\tilde{f}_\star(x,r)$ and $\tilde{f}^\star(x,r)$ are convex on K, i.e.,*

$$\tilde{f}_\star(\lambda x + (1-\lambda)y, r) \leqslant \lambda\tilde{f}_\star(x,r) + (1-\lambda)\tilde{f}_\star(y,r),\ \forall x, y \in K, \lambda, r \in [0,1],$$

$$\tilde{f}^\star(\lambda x + (1-\lambda)y, r) \leqslant \lambda\tilde{f}^\star(x,r) + (1-\lambda)\tilde{f}^\star(y,r),\ \forall x, y \in K, \lambda, r \in [0,1].$$

Based on Definition 3.2, Noor [49] introduced the preinvex fuzzy mapping which is a proper generalization of the convex fuzzy mapping.

Definition 3.3. [49] *Let K be a nonempty invex set of $\mathbb{R}^n$ w.r.t. η. A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called preinvex on K w.r.t. η iff*

$$\tilde{f}(y + \lambda\eta(x,y)) \preccurlyeq \lambda\tilde{f}(x)\tilde{+}(1-\lambda)\tilde{f}(y),\ \forall x, y \in K, \lambda \in [0,1].$$

Lemma 3.2. [33] *Let K be a nonempty invex set of $\mathbb{R}^n$ w.r.t. η. $\tilde{f}$ is preinvex on K w.r.t. η iff the endpoint functions $\tilde{f}_\star(x,r)$ and $\tilde{f}^\star(x,r)$ are preinvex on K w.r.t. η, i.e.,*

$$\tilde{f}_\star(y + \lambda\eta(x,y), r) \leqslant \lambda\tilde{f}_\star(x,r) + (1-\lambda)\tilde{f}_\star(y,r),\ \forall x, y \in K, \lambda, r \in [0,1],$$

$$\tilde{f}^\star(y + \lambda\eta(x,y), r) \leqslant \lambda\tilde{f}^\star(x,r) + (1-\lambda)\tilde{f}^\star(y,r),\ \forall x, y \in K, \lambda, r \in [0,1].$$

Remark 3.2. *When $\eta(x,y) = x - y$, Lemma 3.2 reduces to Lemma 3.1. It is clear that if the invex set K is not a convex set, then the fuzzy preinvexity of $\tilde{f}$ does not imply the fuzzy convexity of $\tilde{f}$. The following example shows that, even if K is convex, the fuzzy preinvexity of $\tilde{f}$ cannot imply the fuzzy convexity of $\tilde{f}$.*

Example 3.2. *Let $K = \mathbb{R}$ and $\tilde{f} : K \to \mathbb{E}$ be a fuzzy mapping defined by $\tilde{f}(x) = \langle 0,1,2\rangle(-|x|)$. Let*

$$\eta(x,y) = \begin{cases} x-y, & \text{if } x \geqslant 0,\ y \geqslant 0 \text{ or } x \leqslant 0,\ y \leqslant 0, \\ y-x, & \text{if } x<0,\ y>0 \text{ or } x>0,\ y<0. \end{cases}$$

It is easy to check that $\tilde{f}$ is preinvex on K w.r.t. η. However, there exist $\bar{x} = 2, \bar{y} = -1, \bar{\lambda} = \frac{1}{2}$ and $r_0 \in (0,1)$ such that $\tilde{f}_\star(\bar{\lambda}\bar{x} + (1-\bar{\lambda})\bar{y}), r_0) = -\frac{1}{2}r_0 \nleqslant -\frac{3}{2}r_0 = \lambda\tilde{f}_\star(x,r_0) + (1-\lambda)\tilde{f}^\star(y,r_0)$. It follows from Lemma 3.1 that $\tilde{f}$ is not convex on K.

Definition 3.4. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called α-preinvex on K w.r.t. α and η iff*

$$\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \preccurlyeq \lambda\tilde{f}(x)\tilde{+}(1-\lambda)\tilde{f}(y), \ \forall x, y \in K, \lambda \in [0, 1].$$

Similar to Lemma 3.1 and 3.2, we have the following property.

Proposition 3.1. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. $\tilde{f}$ is α-preinvex on K w.r.t. α and η iff the endpoint functions $\tilde{f}_\star(x, r)$ and $\tilde{f}^\star(x, r)$ are α-preinvex on K w.r.t. α and η, i.e.,*

$$\tilde{f}_\star(y + \lambda\alpha(x, y)\eta(x, y), r) \leqslant \lambda\tilde{f}_\star(x, r) + (1-\lambda)\tilde{f}_\star(y, r), \ \forall x, y \in K, \lambda, r \in [0, 1],$$

$$\tilde{f}^\star(y + \lambda\alpha(x, y)\eta(x, y), r) \leqslant \lambda\tilde{f}^\star(x, r) + (1-\lambda)\tilde{f}^\star(y, r), \ \forall x, y \in K, \lambda, r \in [0, 1].$$

Remark 3.3. *When $\alpha(x, y) = 1$, Proposition 3.1 becomes Lemma 3.2. Furthermore, Proposition 3.1 becomes Lemma 3.1 when $\alpha(x, y) = 1$ and $\eta(x, y) = x - y$. However, The following example shows that α-preinvexity of $\tilde{f}$ does not imply preinvexity of $\tilde{f}$. Thus, Definition 3.4 is a proper generalization of Definition 3.3.*

Example 3.3. *Let $\tilde{f} : K \to \mathbb{E}$ be a fuzzy mapping defined by $\tilde{f}(x) = \langle 0, 1, 2\rangle x$, $x \in K = [0, +\infty)$. For any $x, y \in K$, let*

$$\alpha(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x \geqslant y, \\ 2, & \text{if } x < y, \end{cases} \qquad \eta(x, y) = \frac{1}{3}x - \frac{1}{2}y.$$

(i) Clearly, K is an α-invex set w.r.t. α and η. Moreover, K is an invex set w.r.t. the same η since $y + \lambda\eta(x, y) = \frac{1}{3}\lambda x + (1 - \frac{1}{2}\lambda)y \in K$.
(ii) It is easy to verify that $\tilde{f}$ is α-preinvex on K w.r.t. α and η. However, there exist $r_0 \in (0, 1)$, $\bar{x} = 3$, $\bar{y} = 5$ and $\bar{\lambda} = \frac{1}{2}$ such that $\tilde{f}_\star(\bar{y} + \bar{\lambda}\bar{\eta}(\bar{x}, \bar{y}), r_0) = \frac{17}{4}r_0 > 4r_0 = \bar{\lambda}\tilde{f}_\star(\bar{x}, r_0) + (1-\bar{\lambda})\tilde{f}_\star(\bar{y}, r_0)$. It follows from Lemma 3.2 that $\tilde{f}$ is not a preinvex fuzzy function on K w.r.t. the same η.

Nanda and Kar, [26], introduced the notion of a quasiconvex fuzzy mapping. Panigrahi in [28] pointed out that, however, the notion for finding the maximum of two fuzzy numbers has not been discussed in their paper. It may happen that two fuzzy numbers are not comparable. Therefore, Panigrahi defined a class of comparable and non-comparable fuzzy functions (see the following Definition 3.6) and furnished some reasonable examples (see Examples 3.9 and 3.10 in [28]). Furthermore, Panigrahi modified the definition of quasiconvex fuzzy mappings (see Definition 4.8 in [28]). Motivated by Panigrahi, [28], and Noor, [38], we now define the α-prequasiinvex fuzzy mapping.

Definition 3.5. [28] *A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called comparable iff $\tilde{f}(x_1)$ and $\tilde{f}(x_2)$ are comparable for every pair $x_1 \neq x_2 \in K$. Otherwise, $\tilde{f}$ is called non-comparable. Let $\mathcal{F}$ denote the set of all comparable fuzzy functions.*

Definition 3.6. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called α-prequasiinvex on K w.r.t. α and η iff*

$$\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \preccurlyeq \max\{\tilde{f}(x), \tilde{f}(y)\}, \forall x, y \in K, \lambda \in [0, 1],$$

where $\tilde{f}(x)$ and $\tilde{f}(y)$ are comparable.

Remark 3.4. *If $\alpha(x, y) = 1$, then Definition 3.6 reduces to the prequasiinvex fuzzy mapping introduced by Wu and Xu in [30]. Obviously, if the α-invex set K is not an invex set, then the fuzzy α-prequasiinvexity of $\tilde{f}$ cannot imply the fuzzy prequasiinvexity of $\tilde{f}$. The following example shows that, even if K is an invex set w.r.t. the same η, the α-prequasiinvexity of $\tilde{f}$ does not imply the prequasiinvexity of $\tilde{f}$, either. Therefore, the fuzzy α-prequasiinvexity is a proper generalization of the fuzzy prequasiinvexity.*

Example 3.4. *Let $K = [0, +\infty)$. For any $x, y \in K$, let*

$$\eta(x, y) = \begin{cases} 2(x - y), & \text{if } x \geqslant y, \\ -\frac{y}{2}, & \text{if } x < y, \end{cases} \quad \alpha(x, y) = \begin{cases} \frac{1}{2}, & \text{if } x \geqslant y, \\ 2, & \text{if } x < y \end{cases}$$

and $\tilde{f}(x) = \langle 0, 1, 2\rangle x$. Indeed, it follows from Definition 3.5 that $\tilde{f}$ is a comparable fuzzy function. Moreover, it is easy to verify that K is an α-invex set and $\tilde{f}$ is α-prequasiinvex fuzzy function on K w.r.t. α and η. On the other hand, K is also an invex set w.r.t. the same η. But there exist $r_0 \in (0, 1)$, $\bar{x} = 2$, $\bar{y} = 1$ and $\bar{\lambda} = \frac{2}{3}$ such that $\tilde{f}_\star(\bar{y} + \bar{\lambda}\eta(\bar{x}, \bar{y}), r_0) = \frac{7}{3}r_0 > 2r_0 = xr_0 = \max\{\tilde{f}_\star(\bar{x}, r_0), \tilde{f}_\star(\bar{y}, r_0)\}$. Hence, $\tilde{f}$ is not prequasiinvex on K w.r.t. the same η.

Remark 3.5. *It is not hard to see that the fuzzy α-preinvexity of $\tilde{f}$ implies that the fuzzy α-prequasiinvexity of $\tilde{f}$. However, the example shows that the fuzzy α-prequasiinvexity of $\tilde{f}$ does not imply the fuzzy α-preinvexity of $\tilde{f}$. Hence, the fuzzy α-prequasiinvexity is also a proper generalization of the fuzzy α-preinvexity.*

Example 3.5. *Let $K = \mathbb{R}$. For any $x, y \in K$, let*

$$\tilde{f}(x) = \begin{cases} \langle 0, 1, 2\rangle(-x), & \text{if } x \geqslant 0, \\ \tilde{0}, & \text{if } x < 0, \end{cases} \quad \alpha(x, y) = \begin{cases} 1, & \text{if } x \geqslant 0, y \geqslant 0 \text{ or } x \leqslant 0, y \leqslant 0, \\ -1, & \text{if } x > 0, y < 0 \text{ or } x < 0, y > 0 \end{cases}$$

and $\eta(x, y) = x - y$. Then, we obtain, for all $r \in [0, 1]$,

$$\tilde{f}(x, r) = \begin{cases} [-(2 - r)x, -rx], & \text{if } x \geqslant 0, \\ [0, 0], & \text{if } x < 0. \end{cases}$$

Clearly, K is an α-invex set w.r.t. α and η. Moreover, it is easy to check that $\tilde{f}$ is α-prequasiinvex on K w.r.t. α and η. However, there exist $r_0 \in (0, 1)$, $\bar{x} = 2$, $\bar{y} = -1$ and $\bar{\lambda} = \frac{1}{2}$ such that

$$\tilde{f}^\star(\bar{y} + \bar{\lambda}\alpha(\bar{x}, \bar{y})\eta(\bar{x}, \bar{y}), r_0) = \frac{5}{2}r_0 > -\frac{1}{2}r_0 = \bar{\lambda}\tilde{f}^\star(\bar{x}, r_0) + (1 - \bar{\lambda})\tilde{f}^\star(\bar{y}, r_0),$$

It follows from Proposition 3.1 that $\tilde{f}$ is not α-prequasiinvex on K w.r.t. the same α and η.

Motivated by the fuzzy directional derivative in [30], we introduce the concept of the fuzzy $\alpha\eta$-directional derivative by virtue of H-difference.

Definition 3.7. *Let $\tilde{f} : K \to \mathbb{E}$ be a fuzzy mapping, where K is an α-invex set w.r.t. α and η. If for any $x, y \in K$, there exists $\delta > 0$ such that the H-difference $\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \ominus_H \tilde{f}(y)$ exists for any real number $\lambda \in (0, \delta)$, and there exists $h \in \mathbb{E}$ such that $\lim_{\lambda \to 0^+}((\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \ominus_H \tilde{f}(y))\backslash\lambda) = h$, then $\tilde{f}$ is called fuzzy αη-directionally differentiable at y and h (denote $\tilde{f}'(y; \alpha(x, y)\eta(x, y))$) is called fuzzy αη-directional derivative at y in the direction $\alpha(x, y)\eta(x, y)$.*

Remark 3.6. It follows from Proposition 3.1 in [21] that if $\tilde{f}$ is fuzzy $\alpha\eta$-directionally differentiable at y in the direction $\alpha(x, y)\eta(x, y)$, then for any fixed $r \in [0, 1]$, $\tilde{f}_\star(x, r)$ and $\tilde{f}^\star(x, r)$ are $\alpha\eta$-directionally differentiable at y in the direction $\alpha(x, y)\eta(x, y)$, and

$$\tilde{f}'((y; \alpha(x, y)\eta(x, y)), r) = [\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r), \tilde{f}^{\star\prime}((y; \alpha(x, y)\eta(x, y)), r)],$$

where $\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r)$ and $\tilde{f}^{\star\prime}((y; \alpha(x, y)\eta(x, y)), r)$ are respectively, the $\alpha\eta$-directional derivatives of $\tilde{f}_\star(x, r)$ and $\tilde{f}^\star(x, r)$ at y in the direction $\alpha(x, y)\eta(x, y)$. That is,

$$\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r) = \lim_{\lambda \to 0^+} \frac{\tilde{f}_\star(y + \lambda\alpha(x, y)\eta(x, y), r) - \tilde{f}_\star(y, r)}{\lambda},$$

$$\tilde{f}^{\star\prime}((y; \alpha(x, y)\eta(x, y)), r) = \lim_{\lambda \to 0^+} \frac{\tilde{f}^\star(y + \lambda\alpha(x, y)\eta(x, y), r) - \tilde{f}^\star(y, r)}{\lambda}.$$

The following example is given to illustrate Definition 3.7.

Example 3.6. *Let $K = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > x_1 > 0\}$. Let $\alpha(x, y) = p \in (0, 2)$, $\eta(x, y) = x - \frac{1}{2}y$ for any x and $y \in K$. Define $\tilde{f}(x_1, x_2) = \langle 0, 1, 2\rangle x_1^2 \tilde{+} \langle 0, 1, 2\rangle x_2^2 \tilde{+} \langle 1, 3, 5\rangle$ for any $(x_1, x_2) \in K$. Obviously, K is an α-invex set w.r.t. α and η. For any $r \in [0, 1]$, we have*

$$\tilde{f}((x_1, x_2), r) = [r, 2 - r]x_1^2 \tilde{+} [r, 2 - r]x_2^2 \tilde{+} [1 + 2r, 5 - 2r].$$

Thus, we obtain $\tilde{f}_\star((x_1, x_2), r) = rx_1^2 + rx_2^2 + (1+2r)$ and $\tilde{f}^\star((x_1, x_2), r) = (2-r)x_1^2 + (2-r)x_2^2 + (5-2r)$. Take $x = (\frac{3}{2}, 2)$ and $y = (1, 2)$. By a direct calculation, we get $\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r) = 6pr$ and $\tilde{f}^{\star'}((y; \alpha(x, y)\eta(x, y)), r) = -6pr + 12p$. Hence, $\tilde{f}'((y; \alpha(x, y)\eta(x, y)), r) = [6pr, -6pr + 12p]$.

By applying the fuzzy $\alpha\eta$-directional derivative, the concepts of the fuzzy α-pseudoinvexity and the fuzzy α-pseudomonotonicity are defined.

Definition 3.8. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called α-pseudoinvex on K w.r.t. α and η iff, for any $x, y \in K$,*

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \succcurlyeq \tilde{0} \Rightarrow \tilde{f}(x) \succcurlyeq \tilde{f}(y).$$

The above implication is equivalent to the following implication:

$$\tilde{f}(x) \prec \tilde{f}(y) \Rightarrow \tilde{f}'(y; \alpha(x, y)\eta(x, y)) \prec \tilde{0}.$$

The following example is used to illustrate Definition 3.8.

Example 3.7. *Let $K = \{(x_1, x_2) \in \mathbb{R}^2 | x_2 > x_1 > 0\}$. The fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is defined by $\tilde{f}(x_1, x_2) = \langle 0, 1, 2\rangle \cdot \frac{x_2}{x_1} \tilde{+} \langle 1, 3, 5\rangle$, $\forall (x_1, x_2) \in K$. Let $\alpha(x, y) = 2$ and $\eta(x, y) = x + y$ for any $x, y \in K$. Clearly, K is an α-invex set w.r.t. α and η. On the other hand, the endpoint functions of the fuzzy mapping $\tilde{f}$ are*

$$\tilde{f}_\star((x_1, x_2), r) = r \cdot \frac{x_2}{x_1} + (1 + 2r), \forall r \in [0, 1],$$

$$\tilde{f}^\star((x_1, x_2), r) = (2 - r) \cdot \frac{x_2}{x_1} + (5 - 2r), \forall r \in [0, 1].$$

It follows from a direct computation that

$$\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r) = r \cdot \frac{2(x_2y_1 - x_1y_2)}{y_1^2}, \forall (x_1, x_2) \in K, r \in [0, 1], \tag{3.1}$$

$$\tilde{f}^{\star'}((y; \alpha(x, y)\eta(x, y)), r) = (2 - r) \cdot \frac{2(x_2y_1 - x_1y_2)}{y_1^2}, \forall (x_1, x_2) \in K, r \in [0, 1], \tag{3.2}$$

$$\tilde{f}_\star(x, r) - \tilde{f}_\star(y, r) = r \cdot \frac{x_2y_1 - x_1y_2}{x_1y_1}, \forall (x_1, x_2) \in K, r \in [0, 1], \tag{3.3}$$

$$\tilde{f}^\star(x, r) - \tilde{f}^\star(y, r) = (2 - r) \cdot \frac{x_2y_1 - x_1y_2}{x_1y_1}, \forall (x_1, x_2) \in K, r \in [0, 1]. \tag{3.4}$$

By Eqs. (3.1)-(3.4), $\tilde{f}$ is a fuzzy α-pseudoinvex mapping.

Definition 3.9. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. A fuzzy mapping $\tilde{f} : K \to \mathbb{E}$ is called $\alpha\eta$-pseudomonotone on K w.r.t. α and η iff, for any $x, y \in K$,*

$$\tilde{f}'(x; \alpha(y, x)\eta(y, x)) \succcurlyeq \tilde{0} \Rightarrow \tilde{f}'(y; \alpha(x, y)\eta(x, y)) \preccurlyeq \tilde{0}.$$

The above implication is equivalent to the following implication:

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \succ \tilde{0} \Rightarrow \tilde{f}'(x; \alpha(y, x)\eta(y, x)) \prec \tilde{0}.$$

The following example will be used to illustrate Definition 3.9.

Example 3.8. *In Example 3.7, we have*

$$\tilde{f}_\star'((x;\alpha(y,x)\eta(y,x)),r)=r\cdot\frac{2(x_1y_2-x_2y_1)}{x_1^2},\tag{3.5}$$

$$\tilde{f}^{\star\prime}((x;\alpha(y,x)\eta(y,x)),r)=(2-r)\cdot\frac{2(x_1y_2-x_2y_1)}{x_1^2}.\tag{3.6}$$

By Eqs. (3.1), (3.2), (3.5) and (3.6), $\tilde{f}$ is an $\alpha\eta$-pseudomonotone mapping.

Definition 3.10. *Let $\tilde{f}:\mathbb{R}^n\to\mathbb{E}$ be a fuzzy mapping.*
(i) $\tilde{f}$ is called positive homogeneous iff $\tilde{f}(\theta x)=\theta\tilde{f}(x)$ for any $x\in\mathbb{R}^n$ and $\theta>0$.
(ii) $\tilde{f}$ is called subodd iff $\tilde{f}(x)\tilde{+}\tilde{f}(-x)\succcurlyeq\tilde{0}$ for any $x\in\mathbb{R}^n\backslash\{0\}$.

Example 3.9. *Let $\tilde{f}:X\to\mathbb{E}$ be defined by $\tilde{f}(x)=\langle 0,1,2\rangle|x|$. Then, $\tilde{f}(x,r)=[rx,(2-r)x]$ for any $r\in[0,1]$ and $x\in X$. If $X:=\mathbb{R}^n$, then $\tilde{f}(\theta x)=[r(\theta|x|),(2-r)(\theta|x|)]=\theta[r|x|,(2-r)|x|]=\theta\tilde{f}(x)$ for $\theta>0$. If $X:=\mathbb{R}^n\backslash\{0\}$, then $\tilde{f}(x)\tilde{+}\tilde{f}(-x)=[2r|x|,2(2-r)|x|]\succcurlyeq\tilde{0}$ for any $r\in[0,1]$ and $x\in\mathbb{R}^n\backslash\{0\}$.*

Remark 3.7. *The fuzzy $\alpha\eta$-directional derivative is positive homogeneous w.r.t. the direction $\alpha(x,y)\eta(x,y)$.*

Definition 3.11. *Let $\tilde{f}:K\to\mathbb{E}$ be a fuzzy mapping. $\tilde{f}$ is called fuzzy radially upper semicontinuous (r.u.s.c) iff the function $\tilde{\varphi}(\lambda):=\tilde{f}(y+\lambda\alpha(x,y)\eta(x,y))$ is u.s.c for any $x,y\in K$ and $\lambda\in[0,1]$.*

To obtain some results in Section 4 and 5, we will give the following assumption regarding the fuzzy function $\tilde{f}$. This assumption plays an important part in studying the properties of generalized convex fuzzy functions.
Assumption A. Let $\tilde{f}:K\subseteq\mathbb{R}^n\to\mathbb{E}$ satisfy the assumption

$$\tilde{f}(y+\alpha(x,y)\eta(x,y))\preccurlyeq\tilde{f}(x),\forall x,y\in K.$$

To understand the Assumption A, we consider the following example.

Example 3.10. *In Example 3.3. Let $x\geqslant y$, in this case, we have $\tilde{f}(y+\alpha(x,y)\eta(x,y),r)=[r(\frac{1}{6}x+\frac{3}{4}y),(2-r)(\frac{1}{6}x+\frac{3}{4}y)]\preccurlyeq\tilde{f}(x,r)$, since*

$$\tilde{f}_\star(y+\alpha(x,y)\eta(x,y),r)=r\left(\frac{1}{6}x+\frac{3}{4}y\right)\leqslant r\left(\frac{1}{4}x+\frac{3}{4}x\right)=rx=\tilde{f}_\star(x,r),$$

$$\tilde{f}^\star(y+\alpha(x,y)\eta(x,y),r)=(2-r)\left(\frac{1}{6}x+\frac{3}{4}y\right)\leqslant(2-r)\left(\frac{1}{4}x+\frac{3}{4}x\right)=(2-r)x=\tilde{f}^\star(x,r),$$

for any $r\in[0,1]$. A similar result holds, if $x<y$.

We also need the following assumption regarding the functions $\alpha(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$.
Assumption B. [50] Let $\alpha(\cdot,\cdot):K\times K\to\mathbb{R}\setminus\{0\}$ and $\eta(\cdot,\cdot):K\times K\to\mathbb{R}^n$ satisfy the assumptions:

$$\eta(y,y+\lambda\alpha(x,y)\eta(x,y))=-\lambda\eta(x,y),$$
$$\eta(x,y+\lambda\alpha(x,y)\eta(x,y))=(1-\lambda)\eta(x,y),\forall x,y\in K,\lambda\in[0,1].$$

Remark 3.8. *Clearly, $\eta(y,y)=0$. If Assumption B holds and $\alpha(x,y)=\alpha(y,y+\lambda\alpha(x,y)\eta(x,y))$ for any $x,y\in K$, then we have $\eta(y+\lambda\alpha(x,y)\eta(x,y),y)=\lambda\eta(x,y)$. Indeed,*

$$\begin{aligned}&\eta(y+\lambda\alpha(x,y)\eta(x,y),y)\\&=\eta(y+\lambda\alpha(x,y)\eta(x,y),y+\lambda\alpha(x,y)\eta(x,y)+\alpha(x,y)\eta(y,y+\lambda\alpha(x,y)\eta(x,y)))\\&=\eta(y+\lambda\alpha(x,y)\eta(x,y),y+\lambda\alpha(x,y)\eta(x,y)+\alpha(y,y+\lambda\alpha(x,y)\eta(x,y))\eta(y,y+\lambda\alpha(x,y)\eta(x,y)))\\&=-\eta(y,y+\lambda\alpha(x,y)\eta(x,y))\\&=\lambda\eta(x,y).\end{aligned}$$

Lemma 3.3. [51] *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. For any $x, y \in K$ and $\lambda \in [0,1]$, if $\eta(y, y+\lambda\alpha(x,y)\eta(x,y)) = -\lambda\eta(x,y)$ and $\alpha(x,y) = \alpha(y, y+\lambda\alpha(x,y)\eta(x,y))$, then for any $\lambda_1, \lambda_2 \in [0,1]$ and $\lambda_1 > \lambda_2$, the following equalities hold,*

$$\alpha(x,y) = \alpha(y+\lambda_1\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y)),$$
$$\eta(y+\lambda_1\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y)) = (\lambda_1-\lambda_2)\eta(x,y).$$

4 Some properties of generalized convex fuzzy mapping

In this section, we turn our attention to investigating some basic properties of the generalized convex fuzzy mapping.

Theorem 4.1. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α satisfies

$$\alpha(x,y) = \alpha(y, y+\lambda\alpha(x,y)\eta(x,y)),\ \forall x, y \in K,\ \lambda \in [0,1].$$

Then, the fuzzy mapping $\tilde{f}$ is α-preinvex on K w.r.t. α and η iff the fuzzy mapping $\tilde{\varphi}(\lambda) := \tilde{f}(y+\lambda\alpha(x,y)\eta(x,y))$ is convex on $[0,1]$.

Proof. Necessity. Suppose that the fuzzy mapping $\tilde{f}$ is α-preinvex on K w.r.t. α and η. In order to verify that the fuzzy mapping $\tilde{\varphi}(\lambda)$ is convex on $[0,1]$, we need to show

$$\tilde{\varphi}(\lambda_2 + k(\lambda_1-\lambda_2)) \preccurlyeq k\tilde{\varphi}(\lambda_1)\tilde{+}(1-k)\tilde{\varphi}(\lambda_2),\ \forall \lambda_1, \lambda_2, k \in [0,1]. \tag{4.1}$$

From Eq. (4.1) and Lemma 3.1, we only need to prove that

$$\tilde{\varphi}_\star(\lambda_2 + k(\lambda_1-\lambda_2), r) \leqslant k\tilde{\varphi}_\star(\lambda_1, r) + (1-k)\tilde{\varphi}_\star(\lambda_2, r),\ \forall \lambda_1, \lambda_2, k, r \in [0,1], \tag{4.2}$$

$$\tilde{\varphi}^\star(\lambda_2 + k(\lambda_1-\lambda_2), r) \leqslant k\tilde{\varphi}^\star(\lambda_1, r) + (1-k)\tilde{\varphi}^\star(\lambda_2, r),\ \forall \lambda_1, \lambda_2, k, r \in [0,1] \tag{4.3}$$

By Lemma 3.3 and the α-preinvexity of $\tilde{f}$, we get

$$\begin{aligned}
&\tilde{\varphi}_\star(\lambda_2 + k(\lambda_1-\lambda_2), r)\\
&\quad = \tilde{f}_\star(y+\lambda_2\alpha(x,y)\eta(x,y) + k(\lambda_1-\lambda_2)\alpha(x,y)\eta(x,y), r)\\
&\quad = \tilde{f}_\star(y+\lambda_2\alpha(x,y)\eta(x,y) + k\alpha(x,y)\eta(y+\lambda_1\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y)), r)\\
&\quad = \tilde{f}_\star(y+\lambda_2\alpha(x,y)\eta(x,y) + k\alpha(y+\lambda_1\alpha(x,y)\eta(x,y), y+\lambda_2\alpha(x,y)\eta(x,y))\eta(y+\lambda_1\alpha(x,y)\eta(x,y),\\
&\qquad y+\lambda_2\alpha(x,y)\eta(x,y)), r)\\
&\quad \leqslant k\tilde{f}_\star(y+\lambda_1\alpha(x,y)\eta(x,y), r) + (1-k)\tilde{f}_\star(y+\lambda_2\alpha(x,y)\eta(x,y), r)\\
&\quad = k\tilde{\varphi}_\star(\lambda_1, r) + (1-k)\tilde{\varphi}_\star(\lambda_2, r), \forall \lambda_1, \lambda_2, k, r \in [0,1],
\end{aligned}$$

which means Eq. (4.2) holds. Similarly, Eq. (4.3) holds. Hence, $\tilde{\varphi}(\lambda)$ is convex on $[0,1]$.

Sufficiency. Since Assumptions A and B hold, we have

$$\begin{aligned}
\tilde{f}_\star(y+\lambda\alpha(x,y)\eta(x,y), r) &= \tilde{\varphi}_\star(\lambda, r)\\
&= \tilde{\varphi}_\star(\lambda\cdot 1 + (1-\lambda)\cdot 0, r)\\
&\leqslant \lambda\tilde{\varphi}_\star(1, r) + (1-\lambda)\tilde{\varphi}_\star(0, r)\\
&= \lambda\tilde{f}_\star(y+\alpha(x,y)\eta(x,y), r) + (1-\lambda)\tilde{f}_\star(y, r)\\
&\leqslant \lambda\tilde{f}_\star(x, r) + (1-\lambda)\tilde{f}_\star(y, r), \forall x, y \in K, \lambda, r \in [0,1].
\end{aligned} \tag{4.4}$$

Similarly, we have

$$\tilde{f}^{\star}(y+\lambda\alpha(x,y)\eta(x,y),r)\leqslant\lambda\tilde{f}^{\star}(x,r)+(1-\lambda)\tilde{f}^{\star}(y,r),\forall x,y\in K,\lambda,r\in[0,1]. \tag{4.5}$$

It follows from Eqs. (4.4) and (4.5) that $\tilde{f}$ is α-preinvex on K w.r.t. α and η. □

Remark 4.1. *When $\alpha(x,y)=1$, Theorem 4.1 reduces to Theorem 3.1 in [32]. When $\alpha(x,y)=1$ and $\eta(x,y)=x-y$, Theorem 4.1 reduces to Theorem 4.6 in [31].*

Theorem 4.2. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that the fuzzy mapping $\tilde{f}$ is r.u.s.c on K and the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α is a symmetric function such that

$$\alpha(x,y)=\alpha(y,y+\lambda\alpha(x,y)\eta(x,y)),\ \forall x,y\in K,\ \lambda\in[0,1].$$

Then, for any $x,y\in K$ with $x\neq y$, there exists a point $z\in\{y+\lambda\alpha(x,y)\eta(x,y)|\lambda\in[0,1)\}$ such that

$$\tilde{f}(x)\ominus_H\tilde{f}(y)\succcurlyeq\tilde{f}'(z;\alpha(x,y)\eta(x,y)).$$

Proof. By contradiction. Suppose that there exist $x,y\in K$ with $x\neq y$, for any $z\in\{y+\lambda\alpha(x,y)\eta(x,y)|\lambda\in[0,1)\}$ such that $\tilde{f}'(z;\alpha(x,y)\eta(x,y))\succ\tilde{f}(x)\ominus_H\tilde{f}(y)$, which implies there exists $r_0\in[0,1]$ such that

$$\tilde{f}_{\star}'((z;\alpha(x,y)\eta(x,y)),r_0)>\tilde{f}_{\star}(x,r_0)-\tilde{f}_{\star}(y,r_0) \tag{4.6}$$

or

$$\tilde{f}^{\star\prime}((z;\alpha(x,y)\eta(x,y)),r_0)>\tilde{f}^{\star}(x,r_0)-\tilde{f}^{\star}(y,r_0). \tag{4.7}$$

Without loss of generality, we suppose that Eq. (4.6) holds. Let $\tilde{\tau}:[0,1]\to\mathbb{E}$ be a fuzzy mapping defined by

$$\tilde{\tau}(\lambda):=\tilde{f}(y+\lambda\alpha(x,y)\eta(x,y)),\ \forall\lambda\in[0,1]. \tag{4.8}$$

Thus, we have

$$\begin{aligned}\tilde{\tau}_{\star}'((\lambda;1),r)&=\lim_{t\to0^+}\frac{\tilde{\tau}_{\star}(\lambda+t,r)-\tilde{\tau}_{\star}(\lambda,r)}{t}\\&=\lim_{t\to0^+}\frac{\tilde{f}_{\star}(y+(\lambda+t)\alpha(x,y)\eta(x,y),r)-\tilde{f}_{\star}(y+\lambda\alpha(x,y)\eta(x,y),r)}{t}\\&=\tilde{f}_{\star}'((y+\lambda\alpha(x,y)\eta(x,y);\alpha(x,y)\eta(x,y)),r),\ \forall\lambda,r\in[0,1].\end{aligned} \tag{4.9}$$

Next, let $\tilde{\pi}:[0,1]\to\mathbb{E}$ be a fuzzy mapping defined by

$$\tilde{\pi}(\lambda):=\tilde{\tau}(\lambda)\tilde{+}\lambda(\tilde{f}(y)\ominus_H\tilde{f}(y+\alpha(x,y)\eta(x,y))),\ \forall\lambda\in[0,1]. \tag{4.10}$$

According to Eq. (4.10), we have

$$\tilde{\pi}_{\star}(\lambda,r)=\tilde{\tau}_{\star}(\lambda,r)+\lambda(\tilde{f}_{\star}(y,r)-\tilde{f}_{\star}(y+\alpha(x,y)\eta(x,y),r)),\ \forall\lambda,r\in[0,1]. \tag{4.11}$$

Since $\tilde{f}$ is r.u.s.c on K, it follows from Definitions 2.4 and 3.11 that $\tilde{\pi}_{\star}(\lambda,r)$ is u.s.c on interval $[0,1]$. By Lemma 2.1, there exists $\bar{\lambda}\in[0,1]$ such that $\tilde{\pi}_{\star}(\lambda,r)$ attains its maximum at $\bar{\lambda}$. Write $M:=\tilde{\pi}_{\star}(\bar{\lambda},r)$. It follows from Eq. (4.8) and (4.11) that $\tilde{\pi}_{\star}(0,r)=\tilde{\pi}_{\star}(1,r)=\tilde{f}_{\star}(y,r)$. Hence, $\bar{\lambda}\in[0,1)$. Thus, there exists $\delta>0$ such that

$$\tilde{\pi}_{\star}((\bar{\lambda}+t),r)-\tilde{\pi}_{\star}(\bar{\lambda},r)\leqslant0,\ \forall t\in(0,\delta),r\in[0,1]. \tag{4.12}$$

Dividing Eq. (4.12) by t and taking limits for $t\to0^+$, we get

$$\tilde{\pi}_{\star}'((\bar{\lambda};1),r)=\lim_{t\to0^+}\frac{\tilde{\pi}_{\star}((\bar{\lambda}+t),r)-\tilde{\pi}_{\star}(\bar{\lambda},r)}{t}\leqslant0,\ \forall r\in[0,1]. \tag{4.13}$$

By Eq. (4.11), we obtain

$$\tilde{\pi}_\star'((\bar{\lambda}; 1), r) = \tilde{\tau}_\star'((\bar{\lambda}; 1), r) + \tilde{f}_\star(y, r) - \tilde{f}_\star(y + \alpha(x, y)\eta(x, y), r), \ \forall r \in [0, 1]. \tag{4.14}$$

Combining Eqs. (4.9) and (4.14), we get

$$\begin{aligned}\tilde{\pi}_\star'((\bar{\lambda}; 1), r) =& \tilde{f}_\star'((y + \bar{\lambda}\alpha(x, y)\eta(x, y); \alpha(x, y)\eta(x, y)), r) \\ &+ \tilde{f}_\star(y, r) - \tilde{f}_\star(y + \alpha(x, y)\eta(x, y), r), \ \forall r \in [0, 1].\end{aligned} \tag{4.15}$$

It follows from Eqs. (4.13) and (4.15) that

$$\tilde{f}_\star'((y + \bar{\lambda}\alpha(x, y)\eta(x, y); \alpha(x, y)\eta(x, y)), r) \leqslant \tilde{f}_\star(y + \alpha(x, y)\eta(x, y), r) - \tilde{f}_\star(y, r), \forall r \in [0, 1]. \tag{4.16}$$

Write $\bar{z} := y + \bar{\lambda}\alpha(x, y)\eta(x, y) \in \{y + \lambda\alpha(x, y)\eta(x, y) | \lambda \in [0, 1)\}$. By Eq. (4.16) and Assumption A, we have

$$\tilde{f}_\star'((\bar{z}; \alpha(x, y)\eta(x, y)), r) \leq \tilde{f}_\star(x, r) - \tilde{f}_\star(y, r), \forall r \in [0, 1]. \tag{4.17}$$

Taking $r = r_0$ in Eq. (4.17), we obtain $\tilde{f}_\star'((\bar{z}; \alpha(x, y)\eta(x, y)), r_0) \leq \tilde{f}_\star(x, r_0) - \tilde{f}_\star(y, r_0)$, which contradicts Eq. (4.6). □

Remark 4.2. *If $\alpha(x, y) = 1$ and the fuzzy mapping $\tilde{f}$ is replaced by a r.u.s.c real-valued function f, then Theorem 4.2 becomes Theorem 2.1 in [15].*

Theorem 4.3. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that the fuzzy mapping $\tilde{f} \in \mathcal{F}$ is r.u.s.c on K and the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α is a symmetric function such that

$$\alpha(x, y) = \alpha(y, y + \lambda\alpha(x, y)\eta(x, y)), \ \forall x, y \in K, \ \lambda \in [0, 1];$$

(iii) $\tilde{f}$ is an α-pseudoinvex fuzzy mapping on K w.r.t. α and η;
(iv) For any $x \in K$, $\tilde{f}'(x; \cdot)$ is subodd in the second argument.
Then, $\tilde{f}$ is an α-prequasiinvex fuzzy mapping on K w.r.t. the same α and η.

Proof. By contradiction. Suppose that $\tilde{f}$ is not α-prequasiinvex fuzzy mapping on K w.r.t. the same α and η. Then, there exist $x, y \in K$ and $\hat{\lambda} \in [0, 1]$ such that

$$\tilde{f}(y + \hat{\lambda}\alpha(x, y)\eta(x, y)) \succ \max\{\tilde{f}(x), \tilde{f}(y)\}. \tag{4.18}$$

Without loss of generality, let

$$\tilde{f}(y) \succcurlyeq \tilde{f}(x). \tag{4.19}$$

By (4.18) and (4.19), we have $\tilde{f}(y + \hat{\lambda}\alpha(x, y)\eta(x, y)) \succ \tilde{f}(y)$. Therefore, there exists $r_0 \in [0, 1]$ such that

$$\tilde{f}_\star(y + \hat{\lambda}\alpha(x, y)\eta(x, y), r_0) > \tilde{f}_\star(y, r_0) \tag{4.20}$$

or

$$\tilde{f}^\star(y + \hat{\lambda}\alpha(x, y)\eta(x, y), r_0) > \tilde{f}^\star(y, r_0). \tag{4.21}$$

Without loss of generality, suppose that Eq. (4.20) holds. Let

$$\phi(\lambda) := \tilde{f}_\star(y + \lambda\alpha(x, y)\eta(x, y), r_0) - \tilde{f}_\star(y, r_0), \ \forall \lambda \in [0, 1]. \tag{4.22}$$

Since the fuzzy mapping $\tilde{f}$ is r.u.s.c on K, it follows from Definition 3.11 that $\phi(\lambda)$ is u.s.c on $[0, 1]$. By Lemma 2.2, there exists $\lambda^\star \in [0, 1]$ such that

$$\phi(\lambda) - \phi(\lambda^\star) \leqslant 0, \ \forall \lambda \in [0, 1]. \tag{4.23}$$

By Eq. (4.22) and Assumption A, we have $\phi(0) = 0$ and $\phi(1) = \tilde{f}_\star(y + \alpha(x, y)\eta(x, y), r_0) - \tilde{f}_\star(y, r_0) \leqslant 0$. Hence, $\lambda^* \in [0, 1)$. Choose a $\delta > 0$ such that $\lambda^* + t \in [0, 1]$ for any $t \in (0, \delta)$. It follows from Eqs. (4.22) and (4.23) that

$$\tilde{f}_\star(y + (\lambda^* + t)\alpha(x, y)\eta(x, y), r_0) - \tilde{f}_\star(y + \lambda^*\alpha(x, y)\eta(x, y), r_0) \leqslant 0, \ \forall t \in (0, \delta). \tag{4.24}$$

Dividing Eq. (4.24) by t and taking limit for $t \to 0^+$, we get

$$\begin{aligned} &\tilde{f}_\star'((y + \lambda^*\alpha(x, y)\eta(x, y); \alpha(x, y)\eta(x, y)), r_0) \\ &= \lim_{t\to 0^+} \frac{\tilde{f}_\star(y + (\lambda^* + t)\alpha(x, y)\eta(x, y), r_0) - \tilde{f}_\star(y + \lambda^*\alpha(x, y)\eta(x, y), r_0)}{t} \leqslant 0. \end{aligned} \tag{4.25}$$

Multiplying Eq. (4.25) by $-\lambda^*$, it follows from Condition (iv) that

$$\tilde{f}_\star'((y + \lambda^*\alpha(x, y)\eta(x, y); -\lambda^*\alpha(x, y)\eta(x, y)), r_0) \geqslant 0. \tag{4.26}$$

From Assumption B, we have

$$\eta(y, y + \lambda^*\alpha(x, y)\eta(x, y)) = -\lambda^*\eta(x, y). \tag{4.27}$$

By Condition (ii), we obtain

$$\alpha(x, y) = \alpha(y, y + \lambda^*\alpha(x, y)\eta(x, y)). \tag{4.28}$$

It follows from Eqs. (4.26)-(4.28) that

$$\tilde{f}_\star'((y + \lambda^*\alpha(x, y)\eta(x, y); \alpha(y, y + \lambda^*\alpha(x, y)\eta(x, y))\eta(y, y + \lambda^*\alpha(x, y)\eta(x, y))), r_0) \geqslant 0. \tag{4.29}$$

By the α-pseudoinvexity of $\tilde{f}$, Eq. (4.29) implies

$$\tilde{f}_\star(y, r_0) \geqslant \tilde{f}_\star(y + \lambda^*\alpha(x, y)\eta(x, y), r_0). \tag{4.30}$$

Since $\hat{\lambda} \in [0, 1]$, it follows from (4.30) that $\tilde{f}_\star(y, r_0) \geqslant \tilde{f}_\star(y + \hat{\lambda}\alpha(x, y)\eta(x, y), r_0)$, which contradicts Eq. (4.20). Therefore, $\tilde{f}$ is an α-prequasiinvex fuzzy mapping on K w.r.t. the same α and η. □

Theorem 4.4. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that $\tilde{f}$ is a fuzzy r.u.s.c function on K and the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α is a symmetric function such that

$$\alpha(x, y) = \alpha(y, y + \lambda\alpha(x, y)\eta(x, y)), \ \forall x, y \in K, \ \lambda \in [0, 1];$$

(iii) For any $x \in K$, $\tilde{f}'(x; \cdot)$ is subodd in the second argument.
Then, the following statements are equivalent:
(a) $\tilde{f}$ is an α-pseudoinvex mapping on K w.r.t. α and η;
(b) $\tilde{f}$ is an $\alpha\eta$-pseudomonotone function on K w.r.t. α and η.

Proof. $(a) \Rightarrow (b)$. Suppose that $\tilde{f}$ is an α-pseudoinvex mapping on K w.r.t. α and η. For any $x, y \in K$, let

$$\tilde{f}'(x; \alpha(y, x)\eta(y, x)) \succcurlyeq \tilde{0}. \tag{4.31}$$

We need to verify that

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \preccurlyeq \tilde{0}. \tag{4.32}$$

Suppose that Eq. (4.32) does not hold. We have

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \succ \tilde{0}. \tag{4.33}$$

It follows from Eq. (4.33) and the α-pseudoinvexity of $\tilde{f}$ that $\tilde{f}(x) \succcurlyeq \tilde{f}(y)$. By the α-pseudoinvexity of $\tilde{f}$, Eq. (4.31) implies $\tilde{f}(y) \succcurlyeq \tilde{f}(x)$. Therefore, we have

$$\tilde{f}(x) = \tilde{f}(y). \tag{4.34}$$

By Theorem 4.3, $\tilde{f}$ is α-prequasiinvex mapping on K w.r.t. the same α and η, i.e.,

$$\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \preccurlyeq \max\{\tilde{f}(x), \tilde{f}(y)\}, \ \forall\lambda \in [0, 1]. \tag{4.35}$$

From Eqs. (4.34) and (4.35), we have

$$\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \preccurlyeq \tilde{f}(y), \ \forall\lambda \in [0, 1].$$

Hence,

$$\tilde{f}_\star(y + \lambda\alpha(x, y)\eta(x, y), r) - \tilde{f}_\star(y, r) \le 0, \forall\lambda, r \in [0, 1], \tag{4.36}$$

$$\tilde{f}^\star(y + \lambda\alpha(x, y)\eta(x, y), r) - \tilde{f}^\star(y, r) \le 0, \forall\lambda, r \in [0, 1]. \tag{4.37}$$

Dividing Eq. (4.36) and Eq. (4.37) by λ and taking limits $\lambda \to 0^+$, respectively, we get

$$\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r) \le 0, \forall r \in [0, 1], \tag{4.38}$$

$$\tilde{f}^{\star\prime}((y; \alpha(x, y)\eta(x, y)), r) \le 0, \forall r \in [0, 1]. \tag{4.39}$$

It follows from Eqs. (4.38) and (4.39) that $\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \preccurlyeq \tilde{0}$, which contradicts Eq. (4.33). Hence, $\tilde{f}$ is an $\alpha\eta$-pseudomonotone function on K w.r.t. α and η.

$(b) \Rightarrow (a)$. Suppose that $\tilde{f}$ is an $\alpha\eta$-pseudomonotone function on K w.r.t. α and η. Let $x, y \in K$ such that

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \succcurlyeq \tilde{0}. \tag{4.40}$$

We assert that

$$\tilde{f}(x) \succcurlyeq \tilde{f}(y). \tag{4.41}$$

Case 1. If $x = y$, then Eq. (4.41) holds.

Case 2. If $x \neq y$, then Eq. (4.41) holds, too. Otherwise,

$$\tilde{f}(x) \prec \tilde{f}(y). \tag{4.42}$$

Using Theorem 4.2, we can find $\lambda^* \in [0, 1)$ such that $z := y + \lambda^*\alpha(x, y)\eta(x, y)$ and

$$\tilde{0} \succ \tilde{f}(x) \ominus_H \tilde{f}(y) \succcurlyeq \tilde{f}'(z; \alpha(x, y)\eta(x, y)). \tag{4.43}$$

When $\lambda^* = 0$, Eq. (4.43) is equivalent to $\tilde{0} \succ \tilde{f}'(y; \alpha(x, y)\eta(x, y))$, which contradicts Eq. (4.40). Hence, $\lambda^* \in (0, 1)$. By positive homogeneity and suboddness of $\tilde{f}'(x, \cdot)$, multiplied (4.43) by $-\lambda^*$, we get

$$\tilde{f}'(z; -\lambda^*\alpha(x, y)\eta(x, y)) \succ \tilde{0}. \tag{4.44}$$

From Assumption B, we have

$$\eta(y, y + \lambda^*\alpha(x, y)\eta(x, y)) = -\lambda^*\eta(x, y). \tag{4.45}$$

By Condition (ii), we get

$$\alpha(x, y) = \alpha(y, y + \lambda^*\alpha(x, y)\eta(x, y)). \tag{4.46}$$

It follows from Eqs. (4.44)-(4.46) that

$$\tilde{f}'(z; \alpha(y, y + \lambda^*\alpha(x, y)\eta(x, y))\eta(y, y + \lambda^*\alpha(x, y)\eta(x, y))) \succ \tilde{0}. \tag{4.47}$$

From Eq. (4.47) and the $\alpha\eta$-pseudomonotonicity of $\tilde{f}$, we obtain

$$\tilde{f}'(y; \alpha(y + \lambda^*\alpha(x, y)\eta(x, y), y)\eta(y + \lambda^*\alpha(x, y)\eta(x, y), y)) \prec \tilde{0}. \tag{4.48}$$

By Remark 3.8, Condition (ii), Assumption B and Eq. (4.48), we obtian $\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \prec \tilde{0}$, which contradicts Eq. (4.40). Therefore, $\tilde{f}$ is a fuzzy α-pseudoinvex function w.r.t. α and η on K. □

Remark 4.3. *In the non-fuzzy settings, when $\tilde{f}$ is a r.u.s.c real-valued function and $\alpha(x, y) = 1$, Theorem 4.4 becomes Theorem 3.2 in [15], and furthermore, if $\eta(x, y) = x - y$, then Theorem 4.4 becomes Theorem 5.2 in [52].*

5 Characterization of the solution sets

Let K be a nonempty α-invex subset of $\mathbb{R}^n$ w.r.t. α and η, and let $\tilde{f} : K \to \mathbb{E}$ be a non-differentiable fuzzy α-pseudoinvex mapping. We consider the following fuzzy optimization problem (FOP):

$$\text{(FOP)} \qquad \min \tilde{f}(x) \quad \text{s.t. } x \in K.$$

Now, we assume from here onwards that the solution set of (FOP) denoted by $S := \operatorname{argmin}\{\tilde{f}(x)|x \in K\}$ is nonempty.

Theorem 5.1. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that $\tilde{f}$ is a fuzzy r.u.s.c function on K and the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α is a symmetric function such that

$$\alpha(x, y) = \alpha(y, y + \lambda\alpha(x, y)\eta(x, y)), \ \forall x, y \in K, \ \lambda \in [0, 1];$$

(iii) $\tilde{f}$ is an α-pseudoinvex mapping on K w.r.t. α and η;
(iv) For any $x \in K$, $\tilde{f}'(x; \cdot)$ is subodd in the second argument.
Then, the solution set S of (FOP) is an α-invex set.

Proof. Let $x, y \in S$. Then, for any $z \in K$, we have

$$\tilde{f}(x) = \tilde{f}(y) \preccurlyeq \tilde{f}(z). \tag{5.1}$$

It follows from Theorem 4.3 that $\tilde{f}$ is an α-prequasiinvex mapping on K w.r.t. the same α and η. Thus, by Eq. (5.1), we have

$$\tilde{f}(y + \lambda\alpha(x, y)\eta(x, y)) \preccurlyeq \tilde{f}(z), \forall\lambda \in [0, 1],$$

which implies $y + \lambda\alpha(x, y)\eta(x, y) \in S$. Hence, the solution set S of (FOP) is an α-invex set. □

Theorem 5.2. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that $\tilde{f}$ is a r.u.s.c function on K and the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α is a symmetric function such that

$$\alpha(x, y) = \alpha(y, y + \lambda\alpha(x, y)\eta(x, y)), \ \forall x, y \in K, \ \lambda \in [0, 1];$$

(iii) $\tilde{f}$ is an α-pseudoinvex fuzzy mapping on K w.r.t. α and η;
(iv) For any $x \in K$, $\tilde{f}'(x; \cdot)$ is subodd in the second argument.
Then, for any $x, y \in S$, $\tilde{f}'(y; \alpha(x, y)\eta(x, y)) = \tilde{f}'(x; \alpha(y, x)\eta(y, x))$.

Proof. Since $x, y \in S$, we have $\tilde{f}(y) \preccurlyeq \tilde{f}(y + \lambda\alpha(x, y)\eta(x, y))$ for any $\lambda \in [0, 1]$, which implies

$$\tilde{f}_\star(y + \lambda\alpha(x, y)\eta(x, y), r) - \tilde{f}_\star(y, r) \geqslant 0, \ \forall\lambda, r \in [0, 1], \tag{5.2}$$

$$\tilde{f}^\star(y + \lambda\alpha(x, y)\eta(x, y), r) - \tilde{f}^\star(y, r) \geqslant 0, \ \forall\lambda, r \in [0, 1]. \tag{5.3}$$

Dividing Eq. (5.2) and Eq. (5.3) by λ and taking limits $\lambda \to 0^+$, respectively, we get

$$\tilde{f}_\star'((y; \alpha(x, y)\eta(x, y)), r) \geqslant 0, \ \forall r \in [0, 1],$$

$$\tilde{f}^{\star\prime}((y; \alpha(x, y)\eta(x, y)), r) \geqslant 0, \ \forall r \in [0, 1].$$

Therefore,

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \succcurlyeq \tilde{0}. \tag{5.4}$$

Similarly, we have

$$\tilde{f}'(x; \alpha(y, x)\eta(y, x)) \succcurlyeq \tilde{0}. \tag{5.5}$$

By Theorem 4.4, $\tilde{f}'$ is $\alpha\eta$-pseudomonotone. Hence, Eqs. (5.4) and (5.5) imply, respectively,

$$\tilde{f}'(x; \alpha(y, x)\eta(y, x)) \preccurlyeq \tilde{0}, \tag{5.6}$$

$$\tilde{f}'(y; \alpha(x, y)\eta(x, y)) \preccurlyeq \tilde{0}. \tag{5.7}$$

It follows from Eqs. (5.4)-(5.7) that $\tilde{f}'(y; \alpha(x, y)\eta(x, y)) = \tilde{f}(x; \alpha(y, x)\eta(y, x))$. □

Theorem 5.3. *Let K be a nonempty α-invex set of $\mathbb{R}^n$ w.r.t. α and η. Suppose that $\tilde{f}$ is a r.u.s.c. function on K and the following conditions hold:*
(i) assumptions A and B are satisfied;
(ii) α is a symmetric function such that

$$\alpha(x, y) = \alpha(y, y + \lambda\alpha(x, y)\eta(x, y)), \ \forall x, y \in K, \ \lambda \in [0, 1];$$

(iii) $\tilde{f}$ is an α-pseudoinvex mapping on K w.r.t. α and η;
(iv) For each $x \in K$, $\tilde{f}'(x; \cdot)$ is subodd in the second argument.
For any given $x^\star \in S$, $S = S_1 = S_2 = S_3 = S_4 = S_5$, where

$$\begin{aligned}
S_1 &= \{x \in K | \tilde{f}'(x; \alpha(x^\star, x)\eta(x^\star, x)) = \tilde{0}\},\\
S_2 &= \{x \in K | \tilde{f}'(x; \alpha(x^\star, x)\eta(x^\star, x)) \succcurlyeq \tilde{0}\},\\
S_3 &= \{x \in K | \tilde{f}'(x^\star; \alpha(x, x^\star)\eta(x, x^\star)) = \tilde{f}'(x; \alpha(x^\star, x)\eta(x^\star, x))\},\\
S_4 &= \{x \in K | \tilde{f}'(x^\star; \alpha(x, x^\star)\eta(x, x^\star)) \preccurlyeq \tilde{f}'(x; \alpha(x^\star, x)\eta(x^\star, x))\},\\
S_5 &= \{x \in K | \tilde{f}'(x^\star; \alpha(x, x^\star)\eta(x, x^\star)) = \tilde{0}\}.
\end{aligned}$$

Proof. Step 1. $S \subseteq S_1$. Let $x \in S$, we have $\tilde{f}(x) = \tilde{f}(x^\star)$. It follows from Theorem 5.1 that S is an α-invex set. Therefore, $x + \lambda\alpha(x^\star, x)\eta(x^\star, x) \in S$ for any $\lambda \in [0, 1]$. Then, $\tilde{f}(x + \lambda\alpha(x^\star, x)\eta(x^\star, x)) = \tilde{f}(x)$, i.e., for every $r \in [0, 1]$, we have

$$\tilde{f}_\star'((x; \alpha(x^\star, x)\eta(x^\star, x)), r) = \lim_{\lambda \to 0^+} \frac{\tilde{f}_\star(x + \lambda\alpha(x^\star, x)\eta(x^\star, x), r) - \tilde{f}_\star(x, r)}{\lambda} = 0,$$

$$\tilde{f}^{\star\prime}((x; \alpha(x^\star, x)\eta(x^\star, x)), r) = \lim_{\lambda \to 0^+} \frac{\tilde{f}^\star(x + \lambda\alpha(x^\star, x)\eta(x^\star, x), r) - \tilde{f}^\star(x, r)}{\lambda} = 0,$$

i.e., $\tilde{f}'(x; \alpha(x^\star, x)\eta(x^\star, x)) = \tilde{0}$. Thus, $x \in S_1$ and hence $S \subseteq S_1$.

Step 2. $S_1 \subseteq S_2$ is clear.

Step 3. $S_2 \subseteq S$. Assume that $x \in S_2$. Then $\tilde{f}'(x; \alpha(x^\star, x)\eta(x^\star, x)) \succcurlyeq \tilde{0}$. By the α-pseudoinvexity of $\tilde{f}$, $\tilde{f}(x^\star) \succcurlyeq \tilde{f}(x)$. Since $x^\star \in S$, $\tilde{f}(x^\star) \preccurlyeq \tilde{f}(x)$. Therefore, $\tilde{f}(x^\star) = \tilde{f}(x)$, which shows that $x \in S$. Hence, $S_2 \subseteq S$.

Step 4. $S \subseteq S_3$. Let $x \in S$. Then $\tilde{f}(x) = \tilde{f}(x^\star)$. It follows from Theorem 5.1 that S is an α-invex set. For any $\lambda \in [0, 1]$, $x^\star + \lambda\alpha(x, x^\star)\eta(x, x^\star) \in S$ and $x + \lambda\alpha(x^\star, x)\eta(x^\star, x) \in S$. Then, $\tilde{f}(x^\star + \lambda\alpha(x, x^\star)\eta(x, x^\star)) = \tilde{f}(x^\star)$, i.e., for every $r \in [0, 1]$, we have

$$\tilde{f}_\star'((x^\star; \alpha(x, x^\star)\eta(x, x^\star)), r) = \lim_{\lambda \to 0^+} \frac{\tilde{f}_\star(x^\star + \lambda\alpha(x, x^\star)\eta(x, x^\star), r) - \tilde{f}_\star(x^\star, r)}{\lambda} = 0,$$

$$\tilde{f}^{\star\prime}((x^\star; \alpha(x, x^\star)\eta(x, x^\star)), r) = \lim_{\lambda \to 0^+} \frac{\tilde{f}^\star(x^\star + \lambda\alpha(x, x^\star)\eta(x, x^\star), r) - \tilde{f}^\star(x^\star, r)}{\lambda} = 0.$$

Therefore,

$$\tilde{f}'(x^\star; \alpha(x, x^\star)\eta(x, x^\star)) = \tilde{0}. \tag{5.8}$$

Similarly, we have

$$\tilde{f}'(x;\alpha(x^\star,x)\eta(x^\star,x))=\tilde{0}. \tag{5.9}$$

It follows from Eqs. (5.8) and (5.9) that $\tilde{f}^{\star'}(x^\star;\alpha(x,x^\star)\eta(x,x^\star)) = \tilde{f}^{\star'}(x;\alpha(x^\star,x)\eta(x^\star,x))$. Thus, $x\in S_3$, i.e., $S\subseteq S_3$.

Step 5. It is clear that $S_3\subseteq S_4$.

Step 6. $S_4\subseteq S$. To show that $S_4\subseteq S$, we only need to verify that $S_4\subseteq S_2$. Let $x\in S_4$. Since $x^\star\in S$, we get $\tilde{f}(x^\star)\preccurlyeq\tilde{f}(x^\star+\lambda\alpha(x,x^\star)\eta(x,x^\star))$ for any $\lambda\in[0,1]$, i.e., for each $r\in[0,1]$, we have

$$\tilde{f}_\star'((x^\star;\alpha(x,x^\star)\eta(x,x^\star)),r)=\lim_{\lambda\to0^+}\frac{\tilde{f}_\star(x^\star+\lambda\alpha(x,x^\star)\eta(x,x^\star),r)-\tilde{f}_\star(x^\star,r)}{\lambda}\geqslant0, \tag{5.10}$$

$$\tilde{f}^{\star'}((x^\star;\alpha(x,x^\star)\eta(x,x^\star)),r)=\lim_{\lambda\to0^+}\frac{\tilde{f}^\star(x^\star+\lambda\alpha(x,x^\star)\eta(x,x^\star),r)-\tilde{f}^\star(x^\star,r)}{\lambda}\geqslant0. \tag{5.11}$$

By Eqs. (5.10) and (5.11), we get $\tilde{f}'(x^\star;\alpha(x,x^\star)\eta(x,x^\star))\succcurlyeq\tilde{0}$. Since $x\in S_4$, $\tilde{f}'(x;\alpha(x^\star,x)\eta(x^\star,x))\succcurlyeq\tilde{0}$. Thus, $x\in S_2$.

Step 7. $S=S_5$. By Steps 1-6, $S=S_1=S_3$. So $S_5=S_1\cap S_3=S$. □

Remark 5.1. *In the non-fuzzy settings, when $\tilde{f}$ is a r.u.s.c differentiable real-valued function and $\alpha(x,y)=1$, Theorem 5.3 becomes Theorem 3.1 in [13]. In addition, when $\tilde{f}$ is a r.u.s.c non-differentiable real-valued function and $\alpha(x,y)=1$, Theorem 5.3 is Theorem 4.2 in [15].*

6 Conclusions

In this paper, we introduced some new classes of generalized convex fuzzy mappings called α-preinvex fuzzy mapping, α-prequasiinvex fuzzy mapping, fuzzy $\alpha\eta$ directional derivative, α-pseudoinvex fuzzy mapping and fuzzy $\alpha\eta$-pseudomonotone function, respectively. Moreover, some relationships among several kinds of generalized convex fuzzy mapping are given. Under some suitable assumptions, some properties of α-pseudoinvex fuzzy mapping, which play an important role in characterizations of solution sets of a non-differentiable α-pseudoinvex fuzzy optimization problem, are obtained. As applications of these properties, five equivalent characterizations of the solution sets are obtained for a class of the generalized convex fuzzy optimization problem. It is interesting to discuss the relationships between fuzzy variational-like inequalities and fuzzy optimization problems and give the characterizations of the solution sets of the fuzzy variational-like inequalities under the assumption of the α-pseudoinvex fuzzy mapping.

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