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The product of quasi-ideal refined generalised quasi-adequate transversals

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Abstract: As the real common generalisations of both orthodox transversals and adequate transversals in the abundant case, the concept of *refined generalised quasi-adequate transversal*, for short, *RGQA transversal* was introduced by Kong and Wang. In this paper, an interesting characterization for a generalised quasi-adequate transversal to be refined is acquired. It is shown that the product of every two quasi-ideal RGQA transversals of the abundant semigroup S satisfying the regularity condition is a quasi-ideal RGQA transversal of S and that all quasi-ideal RGQA transversals of S compose a rectangular band. The related results concerning adequate transversals are generalised and enriched.

Keywords: abundant semigroup, product, RGQA transversal, quasi-ideal

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1 Introduction

Suppose that S is a regular semigroup with a subsemigroup S^o. We denote the intersection of $V(a)$ and S^o by $V_{S^o}(a)$ and that $I = \{aa^o : a \in S, a^o \in V_{S^o}(a)\}$ and $\Lambda = \{a^o a : a \in S, a^o \in V_{S^o}(a)\}$. An *inverse transversal* of the semigroup S is a subsemigroup S^o that contains exactly one inverse of every element of S, that is, S is regular and S^o inverse with $|V_{S^o}(a)| = 1$. This important concept was introduced by Blyth and McFadden [1]. Thereafter, this class of regular semigroups excited many semigroup researchers' attention and a good many important results were obtained (see [1-4] and their references). Tang [4] showed that for S a regular semigroup with an inverse transversal S^o, then I and Λ are both bands with I left regular and Λ right regular. These two bands play an important role in the study of regular semigroups with inverse transversals. Other important subsets of S are $R = \{x \in S : x^o x = x^o x^{oo}\}$ and $L = \{x \in S : xx^o = x^{oo} x^o\}$. They are subsemigroups with R and L left and right inverse respectively. The concept of an orthodox transversal was introduced by Chen [5] as an interesting generalisation of inverse transversals, and a structure for a regular semigroup with a quasi-ideal orthodox transversal was established. Chen and Guo [6] further considered the general case of an orthodox transversal and acquired many properties focused on the sets I and Λ. In [7,8], Kong and Zhao introduced two interesting sets R and L and established the structure for a regular semigroup with quasi-ideal orthodox transversals. In 2014, Kong [9] introduced the concept of *a generalised orthodox transversal* and Kong and Meng [10] acquired the characterization for a generalised orthodox transversal to be an orthodox transversal and obtain a concrete characterization of the maximum idempotent-separating congruence on a regular semigroup with an orthodox transversal. If the concept of transversals could be introduced in the

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range of E-inversive semigroups, then the congruences [11,12] on them will be characterized more neatly. More recently, Kong [13] investigated the weakly simplistic orthodox transversal and obtained the sufficient and necessary condition for the orthodox transversal S^o to be weakly simplistic.

The concept of an *adequate transversal*, was introduced in the range of abundant semigroups by El-Qallali [14] as the generalisation of the concept of an *inverse transversal*. Chen, Guo and Shum [15,16] obtained some important results about a quasi-ideal adequate transversal. Kong [17] explored some properties about adequate transversals. Kong and Wang [18] considered the product of quasi-ideal adequate transversals and proposed the open problem of the isomorphism of adequate transversals. The concept of a *quasi-adequate transversal* was introduced by Ni [19] and followed by Luo, Kong and Wang [20,21], their work mainly focused on the properties and the structure of multiplicative quasi-adequate transversals. Unfortunately, quasi-adequate transversals are neither the generalisation of an orthodox transversal nor adequate transversals. Inspired by the characterization of orthodox transversals [10], the concept of *a RGQA transversal* was introduced by Kong and Wang [22]. It was demonstrated that RGQA transversals are the real common generalisations of both orthodox transversals and adequate transversals in the abundant case.

In the present paper, we continue along the lines of [3, 18, 22] by exploring the relationship between the quasi-ideal RGQA transversals of the abundant semigroup. The main result of this paper is that the product of two quasi-ideal RGQA transversals of an abundant semigroup S satisfying the regularity condition is a quasi-ideal RGQA transversal of S and that all quasi-ideal RGQA transversals of S compose a rectangular band. The related results concerning adequate transversals are generalised and enriched.

2 Preliminaries

The Miller-Clifford theorem is crucial in the study of semigroups.

Lemma 2.1. [23] (1) *Suppose that e and f are idempotents of a semigroup S. For every a in $R_e \cap L_f$, there exists a unique b in $R_f \cap L_e$ such that $ab = e$ and $ba = f$;*
(2) *Suppose that a, b ars elements of a semigroup S. Then $ab \in R_a \cap L_b$ if and only if there exists an idempotent in $L_a \cap R_b$.*

Definition 2.2. [5] Let S^o be an orthodox subsemigroup of the regular semigroup S. S^o is called be an *orthodox transversal* of S, if the following two conditions are satisfied:
(1) $(\forall\, a \in S)\quad V_{S^o}(a) \neq \emptyset$;
(2) For any $a, b \in S$, if $\{a, b\} \cap S^o \neq \emptyset$, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$.

Lemma 2.3. [10] *Let S^o be an orthodox subsemigroup of the regular semigroup S with $V_{S^o}(a) \neq \emptyset$ for every $a \in S$. Then the sufficient and necessary condition for S^o to be an orthodox transversal of S is*

$$(\forall a, b \in S)\ \ [V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset \Rightarrow V_{S^o}(a) = V_{S^o}(b)].$$

A subsemigroup T of a semigroup S is called a *quasi-ideal* of S, if it satisfies $TST \subseteq T$.

Let S and S^o be semigroups. In this paper, the set of idempotents of S and S^o are denoted by E and E^o respectively to avoid confusion. If the product of any two elements of E is regular, then S is called *satisfying the regularity condition*.

On any semigroup S the relation $\mathcal{L}^*$ is defined by $a\mathcal{L}^*b$ if and only if $\{\forall x, y \in S^1, ax = ay \Leftrightarrow bx = by\}$. The relation $\mathcal{R}^*$ is dually defined. Certainly, $\mathcal{L}^*$ and $\mathcal{R}^*$ are right and left congruences respectively with $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. If a, b are regular in S, then $a\mathcal{L}^*b$ ($a\mathcal{R}^*b$) if and only if $a\mathcal{L}b$ ($a\mathcal{R}b$). A semigroup is said to be *abundant* [24] if every $\mathcal{L}^*$- class and each $\mathcal{R}^*$- class contains an idempotent. An abundant semigroup S is said to be *quasi-adequate* [25] (*adequate semigroup*) if its idempotents constituate a band (semilattice). A band is said to be a *rectangular band* if it satisfies $abc = ac$. Suppose that S is an abundant semigroup with U an abundant subsemigroup, U is a $\star$-subsemigroup of S iff $\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)$ and $\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$.

Let S^o be a $\star$-adequate subsemigroup of the abundant semigroup S. S^o is said to be an *adequate transversal* of S, if for every $x \in S$ there exist two idempotents e, f in S and a unique element $\overline{x}$ in S^o such that $x = e\overline{x}f$, with $e\mathcal{L}\overline{x}^+$ and $f\mathcal{R}\overline{x}^\star$. (see [14] for detail).

Suppose that S is an abundant semigroup having the set of idempotents E and S^o a quasi-adequate $\star$–subsemigroup of S having the set of idempotents E^o. S^o is said to be a *generalised quasi-adequate transversal* of S if $C_{S^o}(x) = \{x^o \in S^o \mid x = ex^o f, e\mathcal{L}x^{o+}, f\mathcal{R}x^{o^\star} \text{ for some } x^{o+}, x^{o^\star} \in E^o\} \neq \emptyset$. Let

$$I_x = \{e \in E \mid (\exists x^o \in C_{S^o}(x))\ x = ex^o f, e\mathcal{L}x^{o+}, f\mathcal{R}x^{o^\star} \text{ for some } x^{o+}, x^{o^\star} \in E^o\},$$

$$\Lambda_x = \{f \in E \mid (\exists x^o \in C_{S^o}(x))\ x = ex^o f, e\mathcal{L}x^{o+}, f\mathcal{R}x^{o^\star} \text{ for some } x^{o+}, x^{o^\star} \in E^o\},$$

$$I = \bigcup_{x\in S} I_x, \quad \Lambda = \bigcup_{x\in S} \Lambda_x.$$

By Ni[19], the generalised quasi-adequate transversal S^o is said to be a *quasi-adequate transversal* of S if it satisfies $(\forall e \in E)\ (\forall g \in E^o)$, $C_{S^o}(e)C_{S^o}(g) \subseteq C_{S^o}(ge)$ and $C_{S^o}(g)C_{S^o}(e) \subseteq C_{S^o}(eg)$.

Definition 2.4. [22] Suppose that S^o is a generalised quasi-adequate transversal of the abundant semigroup S. If for all $a, b \in RegS$, $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$ implies that $V_{S^o}(a) = V_{S^o}(b)$, then S^o is said to be a *a RGQA transversal* of S.

Lemma 2.5. [22] *Suppose that S is an abundant semigroup having a generalised quasi-adequate transversal S^o. Then S^o is refined if and only if $IE^o, E^o\Lambda \subseteq E$ and for all $e \in I, \lambda \in \Lambda, f \in E^o$, if $fe, \lambda f$ are regular, then they are idempotent.*

Lemma 2.6. [22] *Let S^o be a refined generalised quasi-adequate transversal of the abundant semigroup S. Then*
(i) S^o is an orthodox transversal of S if and only if S is a regular semigroup.
(ii) S^o is an adequate transversal of S if and only if S^o is an adequate semigroup.

Therefore, by Lemma 2.6 we can say that RGQA transversals are the real common generalisation of an orthodox transversal and adequate transversals in the abundant case.

3 A characterization for a generalised quasi-adequate transversal to be refined

Lemma 2.5 gives the important equivalent conditions for a generalised quasi-adequate transversal to be refined. Now we supplement Lemma 2.5 with another characterization of RGQA transversals within the class of abundant semigroups. It is analogous to the definition of an orthodox transversal in an interesting manner.

Theorem 3.1. *Suppose that S is an abundant semigroup having a generalised quasi-adequate transversal S^o. Then S^o is refined if and only if for any regular elements $a \in S$, $b \in S^o$, if ba is regular, then $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$; and if ab is regular, then $V_{S^o}(b)V_{S^o}(a) \subseteq V_{S^o}(ab)$.*

Proof. (Necessity) For any regular elements $a \in S$, $b \in S^o$, we may take $a^o \in V_{S^o}(a)$, $b^o \in V_{S^o}(b)$, if the generalised quasi-adequate transversal S^o is refined, then by Lemma 2.5, $aa^o b^o b \in IE^o \subseteq E$. If ba is regular, take $(ba)^o \in V_{S^o}(ba)$, then

$$(b^o baa^o)(a(ba)^o b)(b^o baa^o) = b^o(baa^o a)(ba)^o(bb^o ba)a^o = b^o(ba)(ba)^o(ba)a^o = b^o(ba)a^o = b^o baa^o.$$

Thus $b^o baa^o$ is regular and so $b^o baa^o \in E^o I \subseteq E$. Therefore

$$a^o b^o \cdot ba \cdot a^o b^o = a^o(aa^o b^o b)(aa^o b^o b)b^o = a^o \cdot aa^o b^o b \cdot b^o = a^o b^o$$

$$ba \cdot a^o b^o \cdot ba = b(b^o baa^o)(b^o baa^o)a = b \cdot b^o baa^o \cdot a = ba$$

and so $V_{S^o}(a)V_{S^o}(b) \subseteq V_{S^o}(ba)$. Similarly, if ab is regular, then $V_{S^o}(b)V_{S^o}(a) \subseteq V_{S^o}(ab)$.

(Sufficiency) For any regular elements $s_1, s_2 \in S^o$, if $V(s_1) \cap V(s_2) \neq \emptyset$, take $s \in V(s_1) \cap V(s_2)$ and $s_1^o \in V_{S^o}(s_1)$. From $s_2 s \mathcal{L} s_1 s \mathcal{R} s_1 s_1^o$, by Lemma 2.1, $s_2 \mathcal{R} s_2 s s_1 s_1^o \mathcal{L} s_1^o$ and $(s_2 s s_1 s_1^o)^2 = s_2(s s_1 s_1^o s_2 s s_1)s_1^o = s_2 s s_1 s_1^o$ since by the assumption $s_1^o s_2 \in V_{S^o}(s s_1)$. Similarly, $s_2 \mathcal{L} s_1^o s_1 s s_2 \mathcal{R} s_1^o$ with $s_1^o s_1 s s_2 \in E$. Thus

$$s_1^o s_2 s_1^o = s_1^o s_2 s s_1 s s_2 s_1^o = s_1^o(s_2 s s_1 s_1^o)s_1 s s_2 s_1^o = (s_1^o s_1 s s_2)s_1^o = s_1^o$$

and

$$s_2 s_1^o s_2 = s_2(s s_2 s_1^o s_1)s_1^o(s_1 s_1^o s_2 s)s_2 = s_2 s s_1 s_1^o s_1 s s_2 = s_2 s s_1 s s_2 = s_2 s s_2 = s_2.$$

Hence $s_1^o \in V_{S^o}(s_2)$, that is, $V_{S^o}(s_1) \cap V_{S^o}(s_2) \neq \emptyset$. Therefore $V_{S^o}(s_1) = V_{S^o}(s_2)$ since the regular elements of S^o form an orthodox subsemigroup of S.

For any $e \in S$, if $V_{S^o}(e) \cap E^o \neq \emptyset$, take $f \in V_{S^o}(e) \cap E^o$. Then for any $e^o \in V_{S^o}(e)$, we have $e \in V(f) \cap V(e^o)$ and so by the above result, $V_{S^o}(f) = V_{S^o}(e^o)$. Consequently, e^o is an inverse of f in S^o and $e^o \in E^o$ since S^o is quasi-adequate. That is, if $V_{S^o}(e) \cap E^o \neq \emptyset$, then $V_{S^o}(e) \subseteq E^o$.

Let $e, f \in I$ with $e \mathcal{L} f$. Let $h \in E^o$ be such that $h \mathcal{L} e \mathcal{L} f$, then $h \in V_{S^o}(e) \cap V_{S^o}(f)$. For any $g \in V_{S^o}(e)$, by the result above we have $g \in E^o$. It is clear that $ghg \in V_{S^o}(gfg)$ and $ghg \in V_{S^o}(geg) = V_{S^o}(g)$. Then gfg and g have a common inverse ghg and so $ghg \cdot gfg \cdot ghg = ghg$. Thus $gfg = g$. Since $ge \mathcal{L} e \mathcal{L} f$, by Lemma 2.1, $fg \mathcal{R} f$ and so $fgf = f$. Thus $g \in V_{S^o}(f)$ and so $V_{S^o}(e) \subseteq V_{S^o}(f)$. Similarly, the reverse inclusion is hold and hence $V_{S^o}(e) = V_{S^o}(f)$. Dually, if $e, f \in \Lambda$ with $e \mathcal{R} f$, then $V_{S^o}(e) = V_{S^o}(f)$.

It is a routine matter to show that for any regular element $a \in S$ with $a^o \in V_{S^o}(a)$, then $V_{S^o}(a) = V_{S^o}(a^o a)a^o V_{S^o}(aa^o)$.

If regular elements $a, b \in S$ with $V_{S^o}(a) \cap V_{S^o}(b) \neq \emptyset$, we can take $x^o \in V_{S^o}(a) \cap V_{S^o}(b)$. It follows that

$$V_{S^o}(a) = V_{S^o}(x^o a)x^o V_{S^o}(ax^o) \text{ and } V_{S^o}(b) = V_{S^o}(x^o b)x^o V_{S^o}(bx^o).$$

Since $ax^o, bx^o \in I$ with $ax^o \mathcal{L} bx^o$, we have $V_{S^o}(ax^o) = V_{S^o}(bx^o)$. Dually, $V_{S^o}(x^o a) = V_{S^o}(x^o b)$. Therefore $V_{S^o}(a) = V_{S^o}(b)$ and so S^o is refined. □

4 The main result

In 1986, Saito [3] had acquired the result that the product of two quasi-ideal inverse transversals of the regular semigroup S is again a quasi-ideal inverse transversal of S. This important result was generalised to adequate transversals by Kong and Wang [18]. In this section we obtain the main result that the product of any two quasi-ideal RGQA transversals of the abundant semigroup S satisfying the regularity condition is a quasi-ideal RGQA transversal of S. And that, if S has quasi-ideal RGQA transversals, then all quasi-ideal RGQA transversals of S compose a rectangular band.

Let A and B be subsets of a semigroup S and AB for $\{ab : a \in A, b \in B\}$. It is obvious that $(\forall A, B, C \subseteq S)\ (AB)C = A(BC)$, and we denote it by ABC.

Lemma 4.1. Suppose that S^o is a quasi-ideal RGQA transversal of the abundant semigroup S and A a subset of S. Then

(1) $ASS^o = AS^o$ and $S^o SA = S^o A$;

(2) AS^o and $S^o A$ are both subsemigroups and quasi-ideals of S;

(3) For any regular element $x \in S$, if $|V(x) \bigcap A| \geq 1$, then $|V(x) \bigcap AS^o| \geq 1$ and $|V(x) \bigcap S^o A| \geq 1$.

Proof. (1) Suppose that $a \in A$, $x \in S$ and $s \in S^o$. Then $a = e_a \overline{a} f_a$ with $f_a \mathcal{R} \overline{a}^\star$ and so $axs = a\overline{a}^\star f_a xs \in AS^o SS^o \subseteq AS^o$. It is obvious that $as = af_a s \in ASS^o$ and thus $ASS^o = AS^o$. Similarly, $S^o SA = S^o A$.

(2) Obviously, $AS^o \cdot AS^o \subseteq A \cdot S^o SS^o \subseteq AS^o$, thus AS^o is closed and a subsemigroup of S. Similarly,

$AS^o \cdot S \cdot AS^o \subseteq A \cdot S^oSS^o \subseteq AS^o$ and so AS^o is a quasi-ideal of S. There is a dual result for S^oA

(3) For any regular element $x \in S$, take $x' \in V(x) \bigcap A$, then for any $x^o \in V_{S^o}(x)$, $x'xx^o \in V(x) \bigcap ASS^o = V(x) \bigcap AS^o$, that is $|V(x) \bigcap AS^o| \geq 1$. Similarly, $|V(x) \bigcap S^oA| \geq 1$. □

Lemma 4.2. Let S^o, $S^\square$ be quasi-ideal RGQA transversals of an abundant semigroup S. For every regular element $a \in S$, $V_{S^\square S^o}(a) = V_{S^\square}(a) \cdot a \cdot V_{S^o}(a)$.

Proof. Take $a^\square \in V_{S^\square}(a)$, $a^o \in V_{S^o}(a)$, then $a^\square aa^o \in S^\square SS^o = S^\square S^o$ and $a^\square aa^o \in V(a)$, thus $V_{S^\square}(a) \cdot a \cdot V_{S^o}(a) \subseteq V_{S^\square S^o}(a)$. For any $x^\square y^o \in V_{S^\square S^o}(a)$, we have

$$a = ax^\square y^o a, \quad x^\square y^o = x^\square y^o \cdot a \cdot x^\square y^o.$$

Thus

$$x^\square y^o = x^\square y^o \cdot aa^\square aa^o a \cdot x^\square y^o = x^\square y^o aa^\square \cdot a \cdot a^o ax^\square y^o.$$

and

$$x^\square y^o aa^\square \in S^\square SS^\square \subseteq S^\square, \quad a^o ax^\square y^o \in S^oSS^o \subseteq S^o,$$

Meanwhile

$$a \cdot x^\square y^o aa^\square \cdot a = a \cdot x^\square y^o \cdot a = a,$$

$$x^\square y^o aa^\square \cdot a \cdot x^\square y^o aa^\square = x^\square y^o ax^\square y^o aa^\square = x^\square y^o aa^\square.$$

Hence $x^\square y^o aa^\square \in V_{S^\square}(a)$, Similarly, $a^o ax^\square y^o \in V_{S^o}(a)$. Therefore $V_{S^\square S^o}(a) \subseteq V_{S^\square}(a) \cdot a \cdot V_{S^o}(a)$ and hence $V_{S^\square S^o}(a) = V_{S^\square}(a) \cdot a \cdot V_{S^o}(a)$. □

Lemma 4.3. Suppose that S^o is a quasi-ideal RGQA transversal of the abundant semigroup S. For any $x, y \in S$, there exist $\overline{x} \in C_{S^o}(x), \overline{y} \in C_{S^o}(y)$ such that $x = e_x\overline{x}f_x, e_x\mathcal{L}\overline{x}^+, f_x\mathcal{R}\overline{x}^\star$ for some $\overline{x}^+, \overline{x}^\star \in E^o$ and $y = e_y\overline{y}f_y, e_y\mathcal{L}\overline{y}^+, f_y\mathcal{R}\overline{y}^\star$ for some $\overline{y}^+, \overline{y}^\star \in E^o$. Then

(1) $\overline{x}f_xe_y\overline{y} \in C_{S^o}(xy)$;

(2) $e_x(\overline{x}f_xe_y)^+ \in I_{xy}$;

(3) $(f_xe_y\overline{y})^\star f_y \in \Lambda_{xy}$.

Proof. Certainly

$$xy = e_x\overline{x}f_xe_y\overline{y}f_y = e_x(\overline{x}f_xe_y)^+(\overline{x}f_xe_y\overline{y})(f_xe_y\overline{y})^\star f_y,$$

where $e_x(\overline{x}f_xe_y)^+ \in IE^o \subseteq E$, $(f_xe_y\overline{y})^\star f_y \in E^o\Lambda \subseteq E$ and $\overline{x}f_xe_y\overline{y} \in S^o$ since S^o is a quasi-ideal. Since $\mathcal{R}^\star, \mathcal{R}$ are left congruences and $\mathcal{L}^\star, \mathcal{L}$ are right congruences, it is easy to see

$$e_x(\overline{x}f_xe_y)^+ \;\mathcal{L}\; \overline{x}^+(\overline{x}f_xe_y)^+ \;\mathcal{R}^\star\; \overline{x}^+(\overline{x}f_xe_y) = \overline{x}f_xe_y\overline{y}^+ \;\mathcal{R}^\star\; \overline{x}f_xe_yy^o,$$

$$(f_xe_y\overline{y})^\star f_y \;\mathcal{R}\; (f_xe_y\overline{y})^\star\overline{y}^\star \;\mathcal{L}^\star\; (f_xe_y\overline{y})\overline{y}^\star = \overline{x}^\star f_xe_y\overline{y} \;\mathcal{L}^\star\; \overline{x}f_xe_y\overline{y}.$$

Thus the needed results are proved. □

Theorem 4.1. *Let S^o be a quasi-ideal RGQA transversal of the abundant semigroup S satisfying the regularity condition. Then both I and Λ are bands.*

Proof. Since S^o is a RGQA transversal of S and S satisfying the regularity condition, then by Lemma 2.5 $IE^o \subseteq E$ and $E^oI \subseteq E$. Let $e, f \in I$ with $e\mathcal{L}\, e^\star \in E^o$. Obviously, $e^\star f = e^\star ff^\star \in S^oSS^o \bigcap E^oI \subseteq S^o \bigcap I = E^o$ since S^o is a quasi-ideal. Thus $ef = ee^\star f \in IE^o \subseteq E$ and $ef\,\mathcal{L}e^\star f \in E^o$ and so $ef \in I$. Therefore I is a band and there is a dual result for Λ. □

In the following S^o and $S^{\square}$ denote a pair of RGQA transversals of the abundant semigroup S and E_{S^o} and $E_{S^{\square}}$ denote the idempotents of them respectively to avoid confusion, and similar as E_S, $E_{S^{\square}S^o}$, I_o, $I_{\square}$, Λ_o and $\Lambda_{\square}$ speaking for themselves. For simplicity, in $S^{\square}$, a typical idempotent which $\mathcal{L}^*$-related and $\mathcal{R}^*$-related to $a \in S^{\square}$ are denoted by a^* and a^+ respectively. For every $x \in S$, we denote $x = e_x\overline{x}f_x$ in S^o and $x = i_x\widetilde{x}\lambda_x$ in $S^{\square}$ as the decompositions and of x respectively. Then $\widetilde{x} \in S^{\square}$ has the same sense as in the definition of generalised quasi-adequate transversals, that is $i_x, \lambda_x \in E_S$ and $\widetilde{x}^*, \widetilde{x}^+ \in E_{S^{\square}}$ with $\widetilde{x}^*\mathcal{L}^*\widetilde{x}\mathcal{R}^*\widetilde{x}^+$ and $i_x\mathcal{L}\widetilde{x}^+$, $\lambda_x\mathcal{R}\widetilde{x}^*$, and also $i_x\mathcal{R}^*x\mathcal{L}^*\lambda_x$.

Let $S^{\square}$ and S^o be RGQA transversals of the abundant semigroup S. Write

$$I(S^{\square}, S^o) = \{aa^o : a \in Reg(S^{\square}), a^o \in V_{S^o}(a)\},$$

$$\Lambda(S^o, S^{\square}) = \{a^{\square}a : a \in Reg(S^o), a^{\square} \in V_{S^{\square}}(a)\}.$$

Theorem 4.2. *Let $S^{\square}$ and S^o be a pair of quasi-ideal RGQA transversals of the abundant semigroup S satisfying the regularity condition. Then $I(S^{\square}, S^o) = \Lambda(S^o, S^{\square}) = I_o \bigcap \Lambda_{\square}$.*

Proof. For any $aa^o \in I(S^{\square}, S^o)$, where $a \in Reg(S^{\square})$, $a^o \in V_{S^o}(a)$, certainly, $a \in V_{S^{\square}}(a^o)$ and so $aa^o = a^{o\square}a^o \in \Lambda(S^o, S^{\square})$. Thus $I(S^{\square}, S^o) \subseteq \Lambda(S^o, S^{\square})$ and dually $\Lambda(S^o, S^{\square}) \subseteq I(S^{\square}, S^o)$. Therefore,

$$I(S^{\square}, S^o) = \Lambda(S^o, S^{\square}) = \Sigma, \quad say$$

It is clear from the above definitions that $\Sigma \subseteq I_o \bigcap \Lambda_{\square}$.

For the reverse part, let $x \in I_o \bigcap \Lambda_{\square}$. Since $x \in \Lambda_{\square}$, we have $x = x^{\square}x$ for some $x^{\square} \in V_{S^{\square}}(x)$ with $x^{\square} \in E(S^{\square})$ and so $x^{\square} = xx^{\square}$. Similarly, $x \in I_o$ implied that $x = xx^o$ for some $x^o \in V_{S^o}(x)$ with $x^o \in E(S^o)$ and so $x^o = x^ox$. Let $x^{\square o} \in V_{S^o}(x^{\square})$. From $x^o \mathcal{L} x \mathcal{R} x^{\square} \mathcal{R} x^{\square}x^{\square o}$, by Lemma 2.1, we deduce that $x^o \mathcal{R} x^ox^{\square}x^{\square o} \mathcal{L} x^{\square}x^{\square o} \mathcal{L} x^{\square o}$ with $x^ox^{\square}x^{\square o} \in E^oI_o \subseteq E^o$ since S^o is a quasi-ideal and S satisfies the regularity condition. Thus $x^o \mathcal{L} x^{\square o}x^o \mathcal{R} x^{\square o}$. Certainly, $x^o \mathcal{R} x^ox^{\square} \mathcal{L} x^{\square}$ and so by Lemma 2.1, $x^{\square o} \mathcal{R} x^{\square o}x^ox^{\square} \mathcal{L} x^ox^{\square}$ and $x^{\square o}x^ox^{\square} \mathcal{H}^* x^{\square o}x^{\square} \in I_o \bigcap \Lambda_{\square}$. Consequently, $x^{\square}x^{\square o}x^o \mathcal{H} x$ and so $x^{\square}x^{\square o}x^o = x$ since $x \in E$ and $x^{\square}x^{\square o} \cdot x^o \in I_oE^o \subseteq E$. Also $(x^{\square o}x^ox^{\square})^2 = x^{\square o}x^o(x^{\square}x^{\square o}x^o)x^{\square} = x^{\square o}x^oxx^{\square} = x^{\square o}x^ox^{\square}$ and $x^{\square o}x^ox^{\square} \in E$. Therefore

$$x^{\square} \cdot x^{\square o}x^o \cdot x^{\square} = xx^{\square} = x^{\square} \quad \text{and} \quad x^{\square o}x^o \cdot x^{\square} \cdot x^{\square o}x^o = x^{\square o}x^ox = x^{\square o}x^o$$

and so $x^{\square o}x^o \in V_{S^o}(x^{\square})$. Hence $x = x^{\square} \cdot x^{\square o}x^o \in I(S^{\square}, S^o) = \Sigma$. □

Theorem 4.3. *Suppose that $S^{\square}$ and S^o are quasi-ideal RGQA transversals of the abundant semigroup S satisfying the regularity condition. Then $S^{\square}S^o$ is a quasi-ideal RGQA transversal of S.*

Proof. It is easy to see that $S^{\square}S^o$ is a quasi-ideal and a subsemigroup of S.

For any $x \in S^{\square}S^o$, there exist $s^{\square} \in S^{\square}$, $t^o \in S^o$ such that $x = s^{\square}t^o$. We deduce from S^o is a quasi-ideal of S that $e_{s^{\square}}(\overline{s^{\square}}f_{s^{\square}}e_{t^o})^+ \in I_{s^{\square}t^o} = I_x$ and we denote it by e_x. It is obvious that $i_{s^{\square}} \in E_{S^{\square}}$ since $(s^{\square}) \in S^{\square}$ and so from $e_{s^{\square}}\mathcal{R}^*i_{s^{\square}} \in E_{S^{\square}}$ we deduce that $e_{s^{\square}} \in I_o \bigcap \Lambda_{\square}$. It follows from Theorem 4.2 that there exists $a \in Reg(S^{\square})$ such that $e_{s^{\square}} = aa^o$ and so

$$e_x = e_{s^{\square}}(\overline{s^{\square}}f_{s^{\square}}e_{t^o})^+ = aa^o(\overline{s^{\square}}f_{s^{\square}}e_{t^o})^+ \in S^{\square}S^o.$$

Dually, $\lambda_x \in S^{\square}S^o$. Thus $e_x, \lambda_x \in E_{S^{\square}S^o}$, and it follows from $e_x\mathcal{R}^*x\mathcal{L}^*\lambda_x$ that $S^{\square}S^o$ is abundant. Through simple calculation we way show that $e_x\mathcal{R}^*(S)x\mathcal{L}^*(S)\lambda_x$, hence $S^{\square}S^o$ is a abundant *-subsemigroup of S.

Suppose that e is an idempotent of $S^{\square}S^o$. Then e can decompose into as for some $a \in S^{\square}$, $s \in S^o$. Since $(sas)(asa)(sas) = sas$, $(asa)(sas)(asa) = asa$ and $sas \in S^o$, we have $sas \in V_{S^o}(asa)$, and so $e = asasas = asa(asa)^o$. By $asa \in S^{\square}$, every idempotent of $S^{\square}S^o$ is of the form bb^o for some b in $Reg(S^{\square})$. Suppose that e and f are idempotents of $S^{\square}S^o$. Then $e = bb^o$ and $f = cc^o$ for some $b, c \in Reg(S^{\square})$ with $b^o \in V_{S^o}(b)$ and $c^o \in V_{S^o}(c)$. For any $l \in E_{S^o}$, by the regularity condition, lcc^o is regular and so $lcc^o \in E_S$

since S^o is a RGQA transversal of S. Thus $lcc^o \in E_S \bigcap S^o = E_{S^o}$ since S^o is also a quasi-ideal of S. Therefore $ef = bb^o cc^o = bb^o(b^{o\star} cc^o) \in I_o E_{S^o} \subseteq E_S$ and $S^\square S^o$ is a quasi-adequate semigroup.

For every $x \in S$, there exist $a, b \in Reg(S)$ with $e_x = aa^o, \lambda_x = b^\square b$, where $a^o \in V_{S^o}(a), b^\square \in V_{S^\square}(b)$. Then

$$x = e_x x \lambda_x = aa^o x b^\square b = aa^o(a^{o\square} a^o x b^\square b^{\square o}) b^\square b,$$

where $a^{o\square} \in V_{S^\square}(a^o),\ b^{\square o} \in V_{S^o}(b^\square)$, and so

$$e_x = aa^o \mathcal{L} a^{o\square} a^o \in E_{S^\square S^o},\ \lambda_x = b^\square b \mathcal{R} b^\square b^{\square o} \in E_{S^\square S^o}.$$

Since $a^{o\square} a^o x b^\square b^{\square o} \lambda_x = a^{o\square} a^o x \lambda_x = a^{o\square} a^o x$, hence $a^{o\square} a^o x b^\square b^{\square o} \mathcal{R}^\star a^{o\square} a^o x$. Since $x \mathcal{R}^\star e_x$ with $\mathcal{R}^\star$ a left congruence, it follows that

$$a^{o\square} a^o x \mathcal{R}^\star a^{o\square} a^o e_x = a^{o\square} a^o \in E_{S^\square S^o}.$$

Similarly,

$$a^{o\square} a^o x b^\square b^{\square o} \mathcal{L}^\star x b^\square b^{\square o} \mathcal{L}^\star b^\square b^{\square o} \in E_{S^\square S^o}.$$

Therefore, $x = e_x(a^{o\square} a^o x b^\square b^{\square o})\lambda_x$, where $e_x, \lambda_x \in E_S, e_x \mathcal{L}(a^{o\square} a^o x b^\square b^{\square o})^+ = a^{o\square} a^o \in E_{S^\square S^o}$ and $\lambda_x \mathcal{R}(a^{o\square} a^o x b^\square b^{\square o})^\star = b^\square b^{\square o} \in E_{S^\square S^o}$. Consequently, $S^\square S^o$ is a generalised quasi-adequate transversal of S.

If $a \in S, b \in S^\square S^o$ are regular, take $a' \in V_{S^\square S^o}(a), b' \in V_{S^\square S^o}(b)$, it follows from Lemma 4.2 that there exist $a^\square \in V_{S^\square}(a), a^o \in V_{S^o}(a), b^\square \in V_{S^\square}(b), b^o \in V_{S^o}(b)$, such that $a' = a^\square aa^o, b' = b^\square bb^o$. Since $b \in S^\square S^o, b^o \in V_{S^o}(b)$, we have $b \in V_{S^\square S^o}(b^o)$. By Lemma 4.2, there exist $(b^o)^\square \in V_{S^\square}(b^o), (b^o)^o \in V_{S^o}(b^o)$, such that $b = (b^o)^\square b^o (b^o)^o$. Thus

$$a'abb' = a^\square aa^o abb^\square bb^o = a^\square abb^o = a^\square a(b^o)^\square b^o(b^o)^o b^o = a^\square a(b^o)^\square b^o \in \Lambda_\square \Lambda_\square \subseteq \Lambda_\square,$$

and $a'abb'$ is idempotent. Meanwhile

$$bb'a'a = bb^\square bb^o a^\square aa^o a = bb^o a^\square a = (b^o)^\square b^o (b^o)^o b^o a^\square a = (b^o)^\square b^o a^\square a \in \Lambda_\square \Lambda_\square \subseteq \Lambda_\square,$$

and $bb'a'a \in E(S)$. Therefore

$$ab(b'a')ab = a(a'abb')(a'abb')b = (aa'a)(bb'b) = ab,$$

$$b'a'(ab)b'a' = b'(bb'a'a)(bb'a'a)a' = (b'bb')(aa'a) = b'a',$$

that is,

$$V_{S^\square S^o}(b) V_{S^\square S^o}(a) \subseteq V_{S^\square S^o}(ab).$$

Similarly,

$$V_{S^\square S^o}(a) V_{S^\square S^o}(b) \subseteq V_{S^\square S^o}(ba).$$

Therefore by Theorem 3.1 $S^\square S^o$ is refined. Combining with $S^\square S^o$ a quasi-ideal implied that $S^\square S^o$ is a quasi-ideal RGQA transversal of S. □

Theorem 4.4. *Suppose that S is an abundant semigroup satisfying the regularity condition. If S has quasi-ideal RGQA transversals, then all quasi-ideal RGQA transversals of S compose a rectangular band.*

Proof. Suppose that S^o is a quasi-ideal RGQA transversal of S, then $S^o S^o = S^o$. In fact, for any $s^o \in S^o,\ s^o = s^o(s^o)^\star \in S^o S^o$, hence $S^o \subseteq S^o S^o$ and it is clear that the reverse inclusion valids. By Theorem 4.3, all quasi-ideal RGQA transversals of S compose a semigroup and so compose a band.

Let $S^o, S^\square, S^\diamond$ be quasi-ideal RGQA transversals of S. For any $s^o \in S^o, x \in S, t^\square \in S^\square$, we have

$$s^o x t^\square = s^o x e_{t^\square} (\overline{t^\square})^+ t^\square \in S^o SSS^o S^\square \subseteq S^o S^\square,\ s^o t^\square = s^o(s^o)^\star t^\square \in S^o S^o S^\square \subseteq S^o SS^\square,$$

where $t^\square \mathcal{R}^\star e_{t^\square} \in E_S$ and $e_{t^\square} \mathcal{L}(\overline{t^\square})^+ \in E_{S^o}$. Thus $S^o SS^\square = S^o S^\square$ and so

$$S^o S^\diamond S^\square \subseteq S^o SS^\square = S^o S^\square.$$

For any $s^o \in S^o$, $t^\Box \in S^\Box$, we have

$$s^o t^\Box = s^o f_{s^o}(f_{s^o})^\diamond f_{s^o} t^\Box \in S^o S S^\diamond S S^\Box = S^o S^\diamond S^\Box,$$

where $(f_{s^o})^\diamond$ denotes an inverse of f_{s^o} in $S^\diamond$. Hence $S^o S^\diamond S^\Box = S^o S^\Box$ and so all quasi-ideal RGQA transversals of S compose a rectangular band. □

5 An Example

In [18], Kong and Wang gave an example to show that S satisfy the regularity condition cannot be removed in Theorem 3.3 of [18]. In fact this example also demonstrates that S satisfying the regularity condition cannot be removed in Theorem 4.3. We only need to notice that Example 1 in [18]
$S^o S^\Box = \{i, j, o, w\}$ is not a quasi-adequate subsemigroup of S as $ji = w \notin E_{S^o S^\Box}$ and therefore not a RGQA transversal of S.

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