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Asymptotic orbital shadowing property for diffeomorphisms

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Abstract: Let M be a closed smooth Riemannian manifold and let $f : M \to M$ be a diffeomorphism. We show that if f has the C^1 robustly asymptotic orbital shadowing property then it is an Anosov diffeomorphism. Moreover, for a C^1 generic diffeomorphism f, if f has the asymptotic orbital shadowing property then it is a transitive Anosov diffeomorphism. In particular, we apply our results to volume-preserving diffeomorphisms.

Keywords: shadowing; limit shadowing; orbital limit shadowing; asymptotic orbital shadowing; chain transitive; hyperbolic; generic

MSC: Primary 37C50; 37C20. Secondary 37D20

1 Introduction

Let M be a closed smooth Riemannian manifold with $\dim M \geq 2$, and let $\mathrm{Diff}(M)$ be the space of diffeomorphisms of M endowed with the C^1 topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM, let $f \in \mathrm{Diff}(M)$. For any $\delta > 0$, sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ in M is called a δ-*pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. For a closed f-invariant set Λ, we say that f has the *shadowing property* on Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. If $\Lambda = M$ then f has the *shadowing property*. The shadowing property is a very useful concept to investigate the hyperbolic structure (structurally stable, hyperbolic, and Anosov) of a diffeomorphism. Robinson [34] and Sakai [37] proved that f belongs to the C^1 interior of the set of all diffeomorphisms having the shadowing property if and only if it is structurally stable. Pilyugin *et al* [32] introduced another type of shadowing property which is called the orbital shadowing property. We say that f has the *orbital shadowing property* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is a point $y \in M$ such that

$$Orb(y) \subset B_\epsilon(\{x_i\}_{i\in\mathbb{Z}}) \text{ and } \{x_i\}_{i\in\mathbb{Z}} \subset B_\epsilon(Orb(y)),$$

where $Orb(y)$ is the orbit of y. It is clear that if f has the shadowing property then it has the orbital shadowing property. However, the converse is not true (see [32]). Pilyugin *et al* [32] proved that f belongs to the C^1 interior of the set of all diffeomorphisms having the orbital shadowing property if and only if it is structurally stable. Moreover, for various types of shadowing properties numerous results have been published in [9, 10, 25, 26, 37].

Let Λ be a closed f invariant set. We say that Λ is *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s_x}\| \leq C\lambda^n \text{ and } \|D_x f^{-n}|_{E^u_x}\| \leq C\lambda^n$$

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for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$, then f is called an *Anosov diffeomorphism*.

It is well known that if Λ is hyperbolic then f has the shadowing property on Λ, and so, f has the orbital shadowing property on Λ. For point $x \in M$, x is a *non-wandering point* if for any neighborhood U of x, there is $n \in \mathbb{Z}$ such that $f^n(U) \cap U \neq \emptyset$. $\Omega(f)$ denotes the set of all non-wandering points of f. It is clear $\overline{P(f)} \subset \Omega(f)$, where $P(f)$ is the set of periodic points of f, and $\overline{P(f)}$ is the closure of $P(f)$. We say that f satisfies *Axiom A* if $\Omega(f) = \overline{P(f)}$ is hyperbolic. A closed invariant set Λ is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x. If $\Lambda = M$ then f is *transitive*. We say that f is *chain transitive* if for any $\delta > 0$ and $x, y \in M$, there is a δ-pseudo orbit $\{x_i\}_{i=0}^n (n \geq 1)$ such that $x_0 = x$ and $x_n = y$. It is clear that if f is transitive then it is also chain transitive. However, the converse is not true (see [11]).

Recently, a remarkable shadowing property, called the limit shadowing property, was introduced by Eirola *et al* [12]. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a *limit pseudo orbit (or, asymptotic pseudo orbit)* of f if $d(f(x_i), x_{i+1}) \to 0$ as $i \to \pm\infty$. We say that f has the *limit shadowing property* on Λ if for any limit pseudo orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ there is a point $y \in M$ such that $d(f^i(y), x_i) \to 0$ as $i \to \pm\infty$. If $\Lambda = M$ then we say that f has the *limit shadowing property*. Carvalho and Kwietniak [10] showed that if f has the limit shadowing property then it is transitive. Note that if f is transitive then it has neither sinks nor sources. A Morse-Smale diffeomorphism has a sink and a source, and also the shadowing property, but it does not have the limit shadowing property. Carvalho [9], and Carvalho and Kwietniak [10] proved that f belongs to the C^1 interior of the set of all diffeomorphisms having the limit shadowing property if and only if it is a transitive Anosov diffeomorphism. From these results, we consider a type of shadowing property that is a generalized version of the limit shadowing property.

Now, we introduce a general concept of the limit shadowing property that was defined by Good and Meddaugh [13].

Definition 1.1. *Let $f \in \mathrm{Diff}(M)$. We say that f has the asymptotic orbital shadowing property if for each asymptotic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there exists a point $y \in M$ such that for all $\epsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that*

$$d_H(\overline{\{x_{-N_1-i}\}_{i\geq 0}}, \overline{\{f^{-N_1-i}(y)\}_{i\geq 0}}) < \epsilon \text{ and}$$
$$d_H(\overline{\{x_{N_2+i}\}_{i\geq 0}}, \overline{\{f^{N_2+i}(y)\}_{i\geq 0}}) < \epsilon.$$

Pilyugin [33] introduced the orbital limit shadowing property that is a generalized version of the limit shadowing property.

We say that f has the *orbital limit shadowing property* if for each asymptotic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is $z \in M$ such that

$$\omega(z) = \omega(\{x_i\}_{i\in\mathbb{Z}}) \text{ and } \alpha(z) = \alpha(\{x_i\}_{i\in\mathbb{Z}}),$$

where $\alpha(z)$ is the alpha limit set of z. If $g : M \to M$ being a continuous map on compact metric space M, Good and Meddaugh [13, Theorem 22] g has the asymptotic orbital shadowing property if and only if it has the orbital limit shadowing property. Base on this concept, we study the C^1 perturbation of the asymptotic orbital shadowing property. We say that f has the C^1 *robustly asymptotic orbital shadowing property* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the asymptotic orbital shadowing property. Then we have the following.

Theorem A *Let $f \in \mathrm{Diff}(M)$. f has the C^1 robustly asymptotic orbital shadowing property if and only if it is an Anosov diffeomorphism.*

A subset $\mathcal{G} \subset \mathrm{Diff}(M)$ is called *residual* if it contains a countable intersection of open and dense subsets of $\mathrm{Diff}(M)$. A dynamic property is called C^1*generic* if it holds in a residual subset of $\mathrm{Diff}(M)$. A C^1*-generic f* means that *there exists residual subset $\mathcal{G} \subset \mathrm{Diff}(M)$, and $f \in \mathcal{G}$*. Abdenur and Díaz [1] suggested the following problem: *For C^1 generic $f \in \mathrm{Diff}(M)$, if a diffeomorphism f has the shadowing property then is it hyperbolic?*

In response to the problem, Ahn *et al* [2] proved that for a C^1 generic f, if f has the shadowing property on a locally maximal homoclinic class then it is hyperbolic. Lee and Wen [20] proved that for a C^1 generic

f, if f has the shadowing property on a locally maximal chain transitive set then it is hyperbolic. Lee and Lee [19] proved that for a C^1 generic f, if f has the shadowing property on a homoclinic class then it is hyperbolic, which is a general result of the previous results ([2, 20]). For the shadowing property, the problem is still open. About the problem, several authors consider various shadowing properties (average shadowing, asymptotic average shadowing, limit shadowing, etc) which are related with the shadowing property. Very recently, the remarkable result of this problem, Lee [29] proved that for C^1 generic f of a two dimensional smooth manifold M, if f has the average shadowing property or the asymptotic average shadowing property then it is Anosov. However, the problem mentioned above, for any dimensional smooth manifold M, is still open for investigation. In this paper, we prove for another type of the shadowing property (see [26]). Lee [22] proved that C^1 generically, if f has the limit shadowing property on a homoclinic class then it is hyperbolic. Lee and Lu [21] proved that C^1 generically, if f has the limit weak shadowing property on a transitive set then the transitive set is hyperbolic. Lee and *et al* [17] proved that C^1 generically, if f has the limit weak shadowing property on the chain recurrent set then the chain recurrent set is hyperbolic. Carvalho [9] showed that for a C^1 generic f, if f has the limit shadowing property then it is a transitive Anosov diffeomorphism. Since the asymptotic orbital shadowing property is a general notion of the limit shadowing property, such as those results before, we will prove the following.

Theorem B *For C^1 generic $f \in \mathrm{Diff}(M)$, f has the asymptotic orbital shadowing property if and only if it is transitive Anosov.*

Carbalho [9, Corollary 2.3] showed that a transitive Anosov f has the limit shadowing property, and so, f also has the asymptotic orbital shadowing property. To prove Theorem B, we will show that for a C^1 generic f, if f has the asymptotic orbital shadowing property then it is a transitive Anosov diffeomorphism.

2 Proof of Theorem A

Let M be as before, and let $f \in \mathrm{Diff}(M)$. A compact invariant set Λ is *attracting* if there is a neighborhood U of Λ such that $f^n(\overline{U}) \subset U$ for all $n \geq 1$ and $\Lambda = \bigcap_{n\geq 0} f^n(\overline{U})$. An attractor of f is a transitive attracting set of f and a repeller is an attractor for f^{-1}. We say that Λ is a *proper attractor* or *repeller* if $\emptyset \neq \Lambda \neq M$. A *sink (source)* of f is an attracting (repelling) orbit of f. The following was proved by Hirsh *et al* [16, Lemma 3.2].

Lemma 2.1. *Let $f \in \mathrm{Diff}(M)$. f is chain transitive if and only if it does not contain a proper attractor.*

We also recall that the Hausdoroff distance between two compact subsets A and B of M is given by:

$$d_H(A, B) = \max\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\}$$

Lemma 2.2. *If f has the asymptotic orbital shadowing property then it is chain transitive.*

Proof. Suppose that there is a proper attractor $\Lambda \subset M$. Since Λ is a proper attractor, there is a neighborhood U of Λ such that $f^n(\overline{U}) \subset U$ for all $n \geq 1$ and $\Lambda = \bigcap_{n\geq 0} f^n(\overline{U})$. Since Λ is compact, there is $\epsilon_1 > 0$ such that $\Lambda \subset B_{\epsilon_1}(\Lambda) \subset U$. Then we have $d_H(\Lambda, \overline{U}) > \epsilon_1/2$. Now we construct an asymptotic pseudo orbit. For this, we consider two points a and b such that $a \in \Lambda$ and $b \in M \setminus U$. Now we consider sequences $x_i = f^i(a)$ for all $i \leq 0$ and $x_i = f^i(b)$ for all $i > 0$. It is clear that the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is an asymptotic pseudo orbit of f. If there are $z \in M$ and $N_1, N_2 \in \mathbb{N}$ such that $f^{-N_1}(z) \in B_{\epsilon_1/4}(a)$ then according to the proper attracting property, $f^{-N_1+i}(z) \in U$ for all $i \geq 0$. Then we have

$$Orb^+(f^{-N_1}(z)) \not\subset B_{\epsilon_1}(Orb^+(b)), \tag{1}$$

where $Orb^+(f^{-N_1}(z)) = \{f^{-N_1+i}(z) : i \geq 0\}$, and $Orb^+(f(b)) = \{f^i(b) : i > 0\}$. Thus we have $Orb^+(f^{-N_1}(z)) \cap \overline{U} = \emptyset$. Let $\epsilon = \epsilon_1/4$. Then by the asymptotic orbital shadowing property, there exist $z \in M$ and $N_1, N_2 \in \mathbb{N}$ such that f must have

$$d_H(\overline{\{x_{-N_1-i}\}_{i\geq 0}}, \overline{\{f^{-N_1-i}(z)\}_{i\geq 0}}) < \epsilon \text{ and}$$
$$d_H(\overline{\{x_{N_2+i}\}_{i\geq 0}}, \overline{\{f^{N_2+i}(z)\}_{i\geq 0}}) < \epsilon.$$

Since $Orb^+(f^{-N_1}(z)) \subset \Lambda = \bigcap_{i\geq 0} f^i(U)$ and $d_H(\Lambda, \overline{U}) > \epsilon_1/2$, this is a contradiction. Thus if f has the asymptotic orbital shadowing property then there is no proper attractor, and so, by Lemma 2.1 it is chain transitive. □

Let p be a hyperbolic periodic point of f. We define the sets

$$W^s(p) = \{x \in M : f^{\pi(p)n}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-\pi(p)n}(x) \to p \text{ as } n \to \infty\}$$

are C^1 injectively immersed submanifolds of M, where $\pi(p)$ is the period of p. We denote index(p) = dim $W^s(p)$. Note that for any hyperbolic $p, q \in P(f)$, if $W^s(p) \pitchfork W^u(q) \neq \emptyset$ and $W^u(p) \pitchfork W^s(q) \neq \emptyset$ then index(p) = index(q). For any $\epsilon > 0$, let $W^s_{\epsilon(x)}(x) = \{y \in M : d(f^n(x), f^n(y)) < \epsilon, \text{ for all } n \geq 0\}$, and $W^u_{\epsilon(x)}(x) = \{y \in M : d(f^{-n}(x), f^{-n}(y)) < \epsilon, \text{ for all } n \geq 0\}$ be the *local stable set* and the *local unstable set* of x, respectively. Note that for a hyperbolic periodic point p, we know $W^s_{\epsilon(p)}(p) \subset W^s(p)$ and $W^u_{\epsilon(p)}(p) \subset W^u(p)$.

The concept of the asymptotic orbital shadowing property can be rewritten as follows : a diffeomorphism f has the *asymptotic orbital shadowing property* if for each asymptotic pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is a point $y \in M$ such that for all $\epsilon > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

(i) $f^{-N_1-i}(z) \in B_\epsilon(\{x_{-N_1-i}\})$ and $f^{N_2+i}(z) \in B_\epsilon(\{x_{N_2+i}\})$ for $i \geq 0$,

(ii) $x_{-N_1-i} \in B_\epsilon(f^{-N_1-i}(z))$ and $x_{N_2+i} \in B_\epsilon(f^{N_2+i}(z))$ for $i \geq 0$.

Lemma 2.3. *If f has the asymptotic orbital shadowing property then for any hyperbolic periodic points p and q, we have $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$.*

Proof. Let p and q be hyperbolic periodic points of f. By Lemma 2.2, f is chain transitive. Then by [27, Lemma 2.1], f does not contain sinks nor sources. Thus p and q are saddles. For simplicity, we assume that $f(p) = p$ and $f(q) = q$. Since f is chain transitive, for any $\delta > 0$ there is a sequence $\{x_i\}_{i=0}^n (n \geq 1)$ such that $x_0 = p$, $x_n = q$, and $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, \ldots, n-1$. Then the sequence $\{x_i\}_{i=0}^n (n \geq 1)$ is a finite δ-pseudo orbit of f. Substituting $x_i = f^i(p)$ for all $i \geq 0$ and $x_i = f^i(q)$ for all $i \geq n$, the sequence $\{\ldots, p(= x_{-1}), p(= x_0), x_1, x_2, \ldots, q(= x_n), q(= x_{n+1}), \ldots\} = \{x_i\}_{i\in\mathbb{Z}}$ is an asymptotic pseudo orbit of f. Let $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$. Then base on the asymptotic orbital shadowing property, there are $z \in M$ and $N_1, N_2 \in \mathbb{N}$ such that

$$f^{-N_1-i}(z) \in B_\epsilon(p) \text{ as } i \geq 0, \text{ and}$$

$$f^{N_2+i}(z) \in B_\epsilon(q) \text{ as } i \geq 0.$$

Then we know that $d(f^{-i}(f^{-N_1}(z)), p) < \epsilon$ and $d(f^i(f^{N_2}(z)), q) < \epsilon$ for all $i \geq 0$. Therefore we know $f^{N_2}(z) \in W^s_\epsilon(q) \subset W^s(q)$ and $f^{-N_1}(z) \in W^u_\epsilon(p) \subset W^u(p)$, and so, $z \in W^u(p) \cap W^s(q)$. Thus $W^u(p) \cap W^s(q) \neq \emptyset$. The other case is similar. □

We say that f is *Kupka-Smale* if the periodic points of f are hyperbolic, and if $p, q \in P(f)$, then $W^s(p)$ is transversal to $W^u(q)$. We denote $\mathcal{KS}$ as the set of all Kupka-Smale diffeomorphsms. It is well-known that the set of all Kupka-Smale diffeomorphisms is C^1 residual in Diff(M) (see [35]).

Proposition 2.4. *If f has the C^1 robustly asymptotic orbital shadowing property then there is C^1 neighborhood $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$ and any hyperbolic $p, q \in P(g)$ we have*

$$\text{index}(p) = \text{index}(q),$$

where $P(g)$ is the set of all periodic points of g.

Proof. Let $\mathcal{U}(f)$ be a C^1 neighborhood of f. Suppose, by contradiction, that there are $g \in \mathcal{U}(f)$ such that g has two periodic points p and q with $\text{index}(p) \neq \text{index}(q)$. Then we know that $\dim W^s(p) + \dim W^u(q) < \dim M$ or $\dim W^u(p) + \dim W^s(q) < \dim M$. We may assume that $\dim W^s(p) + \dim W^u(q) < \dim M$. Note that for any hyperbolic periodic points p and q, there is a C^1 neighborhood $\mathcal{U}(f)$ such that for any $g \in \mathcal{U}(f)$, we know that $\dim W^s(p) = \dim W^s(p_g)$ and $\dim W^u(q) = \dim W^u(q_g)$, where p_g is the continuation of p and q_g is the continuation of q.

Since f has the C^1 robustly asymptotic orbital shadowing property, there is $g_1 \in \mathcal{U}(g) \cap \mathcal{KS}$ such that $\dim W^s(p_{g_1}) + \dim W^u(q_{g_1}) < \dim M$, where p_{g_1} is the continuation of p and q_{g_1} is the continuation of q. Since $g_1 \in \mathcal{KS}$, $W^s(p_{g_1}) \cap W^u(q_{g_1}) = \emptyset$. However, g_1 has the asymptotic orbital shadowing property, and so, by Lemma 2.2, g_1 is chain transitive. By Lemma 2.3, we have $W^s(p_{g_1}) \cap W^u(q_{g_1}) \neq \emptyset$. This is a contradiction. □

The following is called Franks' lemma [14] which is very useful in our proofs.

Lemma 2.5. *Let $\mathcal{U}(f)$ be any given C^1 neighborhood of f. Then there exist $\epsilon > 0$ and a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \cdots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \le \epsilon$ for all $1 \le i \le N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \le i \le N$.*

Lemma 2.6. *If a point $p \in P(f)$ is not hyperbolic then there is g C^1 close to f such that g has two hyperbolic periodic points $q, r \in P(g)$ with* $\text{index}(q) \neq \text{index}(r)$.

Proof. Suppose that $p \in P(f)$ is not hyperbolic. Then $D_p f^{\pi(p)}$ has eigenvalues whose modulus are 1, where $\pi(p)$ is the period of p. Then by Lemma 2.5, there is g C^1 close to f such that $D_{p_g} g^{\pi(p_g)}$ has only one eigenvalue λ with $|\lambda| = 1$, where $\pi(p_g)$ is the period of p_g. For simplicity, we may assume that $g^{\pi(p_g)}(p_g) = g(p_g) = p_g$. We denote $E^c_{p_g}$ as the eigenspace corresponding to λ. If $\lambda \in \mathbb{R}$ then $\dim E^c_{p_g} = 1$ and if $\lambda \in \mathbb{C}$ then $\dim E^c_{p_g} = 2$.

First, if $\lambda \in \mathbb{R}$ then $\lambda = 1$ or $\lambda = -1$. In the proof, we consider that $\lambda = 1$. The case of $\lambda = -1$ is similar. Let $\mathcal{U}(g)$ be a C^1 neighborhood of g. By Lemma 2.5, there are $\alpha > 0$ and $g_1 \in \mathcal{U}(g)$ such that (i) $g_1(p_g) = g(p_g) = p_g$, (ii) $g_1(x) = \exp_{p_g} \circ D_{p_g} g \circ \exp^{-1}_{p_g}(x)$ if $x \in B_\alpha(p_g)$, and $g(x) = g_1(x)$ if $x \notin B_{4\alpha}(p_g)$. We can consider nonzero vector v associated λ such that $\|v\| \le \alpha/2$. Then we have

$$g_1(\exp_{p_g}(v)) = \exp_{p_g} \circ D_{p_g} g \circ \exp^{-1}_{p_g}(\exp_{p_g}(v)) = \exp_{p_g}(v). \tag{2}$$

If we set $\mathcal{I}_v = \{t \cdot v : -\alpha/4 \le t \le \alpha/4\}$, then $\exp_{p_g}(\mathcal{I}_v)$ is a closed small arc, and by (2), we have $g_1(\exp_{p_g}(\mathcal{I}_v)) = \exp_{p_g}(\mathcal{I}_v)$. Thus $g_1|_{\exp_{p_g}(\mathcal{I}_v)} : \exp_{p_g}(\mathcal{I}_v) \to \exp_{p_g}(\mathcal{I}_v)$ is the identity map. Taking two endpoints $q, r \in \exp_{p_g}(\mathcal{I}_v)$, it is clear that $D_q g_1|_{E^c_q} = D_r g_1|_{E^c_r} = D_{p_g} g_1|_{E^c_{p_g}} = 1$. By Lemma 2.5, there is h C^1 close to g_1 ($h \in \mathcal{U}(g)$) such that q_h and r_h are hyperbolic with $\text{index}(q_h) \neq \text{index}(r_h)$, where q_h is the continuation of q and r_h is the continuation of r.

Finally, we consider $\lambda \in \mathbb{C}$. We assume that $g(p) = p$. As in the previous case, by Lemma 2.5, there are $\alpha > 0$ and $g_1 \in \mathcal{U}(g)$ such that (i) $g_1(p_g) = g(p_g) = p_g$, (ii) $g_1(x) = \exp_{p_g} \circ D_{p_g} g \circ \exp^{-1}_{p_g}(x)$ if $x \in B_\alpha(p_g)$, and $g(x) = g_1(x)$ if $x \notin B_{4\alpha}(p_g)$. Then there is $k > 0$ such that $D_{p_g} g_1^k(v) = v$ for any nonzero $v \in \exp^{-1}_p(E^c_{p_g}(\alpha))$. Taking $w \in \exp^{-1}_{p_g}(E^c_{p_g}(\alpha))$ such that $\|w\| = \alpha/4$, we set

$$\mathcal{L}_{p_g} = \exp_{p_g}(\{t \cdot w : 1 \le t \le 1 + \alpha/4\}).$$

Then $\mathcal{L}_{p_g}$ is a closed small arc such that (i) $g_1^i(\mathcal{L}_{p_g}) \cap g_1^j(\mathcal{L}_{p_g}) = \emptyset$ if $0 \le i \neq j \le k-1$, (ii) $g_1^k(\mathcal{L}_{p_g}) = \mathcal{L}_{p_g}$ and (iii) $g_1^k|_{\mathcal{L}_{p_g}}$ is the identity map. As in the previous case, we consder two endpoints $q, r \in \mathcal{L}_{p_g}$. Then by Lemma 2.5, there is g_2 C^1 close to g_1 such that g_2 has two hyperbolic periodic points q_{g_2} and r_{g_2} with $\text{index}(q_{g_2}) \neq \text{index}(r_{g_2})$, where q_{g_2} is the continuation of q and r_{g_2} is the continuation of r. □

We say that f satisfies the *star condition* if there is a C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, each $p \in P(g)$ is hyperbolic. $\mathcal{F}(M)$ denotes the set of all diffeomorphisms satisfying the star condition. Hayashi [15] proved that if a diffeomorphism $f \in \mathcal{F}(M)$ then f satisfies Axiom A. We will show that if a diffeomorphism f has the C^1 robustly asymptotic orbital shadowing property then f satisfies Axiom A.

Proposition 2.7. *If f has the C^1 robustly asymptotic orbital shadowing property then $f \in \mathcal{F}(M)$.*

Proof. Suppose, by contradiction, that $f \notin \mathcal{F}(M)$. Then there is g C^1 close to f such that g has a nonhyperbolic periodic point p. By Lemma 2.6, there is h C^1 close to g (h C^1 close to f) such that h has two hyperbolic periodic points q, r with $\text{index}(q) \neq \text{index}(r)$. Since f has the C^1 robustly asymptotic orbital shadowing property, by Proposition 2.4 this is a contradiction. □

A diffeomorphism f is *robustly chain transitive* if there is C^1 neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g is chain transitive. Lee [27] proved that if a diffeomorphism f is robustly chain transitive then f admits a weak hyperbolic, that is, a closed f-invariant set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E \oplus F$ and there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|Df^n|_{E_x}\| \cdot \|Df^{-n}|_{F_{f^n(x)}}\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. It is clear that if a closed f-invariant set $\Lambda \subset M$ is hyperbolic then it admits a dominated splitting for f.

Proof of Theorem A. Suppose that f has the C^1 robustly asymptotic orbital shadowing property. Since f has the asymptotic orbital shadowing property then by Lemma 2.2 f is chain transitive. Since f has the C^1 robustly asymptotic orbital shadowing property, f is robustly chain transitive. By Proposition 2.7, each periodic points of f is hyperbolic. By Proposition 2.4, the index of each periodic points of f is the same. Thus by [27, Theorem 1.1], f is Anosov. □

3 Proof of Theorem B

In this section, we show that for a C^1 generic f, if a diffeomorphism f has the asymptotic orbital shadowing property then it satisfies Axiom A. If f is chain transitive, then it contains the recurrence set $R(f)$, and so, $R(f) = M$. Since f satisfies Axiom A, it is well known that f is Anosov.

Lemma 3.1. *There is a residual set $\mathcal{G}_1 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}_1$, and if f has the asymptotic orbital shadowing property then* $\text{index}(p) = \text{index}(q)$, *for any points $p, q \in P(f)$.*

Proof. Let $f \in \mathcal{G}_1 = \mathcal{KS}$ have the asymptotic orbital shadowing property. Since f is Kupka-Smale, each periodic points of f is hyperbolic. Suppose, by contradiction, that $\text{index}(p) \neq \text{index}(q)$. Then $\dim W^s(p) + \dim W^u(q) < \dim M$ or $\dim W^u(p) + \dim W^s(q) < \dim M$. Assume that $\dim W^s(p) + \dim W^u(q) < \dim M$(Other case is similar). Since $f \in \mathcal{KS}$, we have

$$W^s(p) \cap W^u(q) = \emptyset, \text{ and } W^u(p) \cap W^s(q) = \emptyset.$$

Since f has the asymptotic orbital shadowing property, by Lemma 2.3,

$$W^s(p) \cap W^u(q) \neq \emptyset, \text{ and } W^u(p) \cap W^s(q) \neq \emptyset.$$

This is a contradiction. Thus $\text{index}(p) = \text{index}(q)$. □

Lemma 3.2. [30, Lemma 2.2] *There is a residual set $\mathcal{G}_2 \subset \text{Diff}(M)$ such that for any $f \in \mathcal{G}_2$, and if for any C^1 neighborhood $\mathcal{U}(f)$ of f there is $g \in \mathcal{U}(f)$ such that g has two periodic points p and q with* $\text{index}(p) \neq \text{index}(q)$ *then f has two periodic points p_f and q_f with* $\text{index}(p_f) \neq \text{index}(q_f)$

For any $\delta > 0$, we say that a hyperbolic $p \in P(f)$ has a δ *weak eigenvalue* if there is an eigenvalue λ of $D_p f^{\pi(p)}$ such that

$$(1 - \delta)^{\pi(p)} < |\lambda| < (1 + \delta)^{\pi(p)},$$

where $\pi(p)$ is the period of p.

Lemma 3.3. *There is a residual set $\mathcal{G}_3 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_3$, if f has the asymptotic orbital shadowing property then there is $\delta > 0$ such that for any $p \in P(f)$, p does not have a δ weak eigenvalue.*

Proof. Let $f \in \mathcal{G}_3 = \mathcal{G}_1 \cap \mathcal{G}_2$ have the asymptotic orbital shadowing property. Suppose that for any $\delta > 0$ there is $p \in P(f)$ such that p has a δ weak eigenvalue. Then there is g C^1 close to f such that $D_{p_g} g^{\pi(p_g)}$ has an eigenvalue λ with $|\lambda| = 1$. By Lemma 2.6, there is g_1 C^1 close to g(g_1 C^1 close to f) such that g_1 has two hyperbolic periodic points q, r with index$(q) \neq$ index(r). By Lemma 3.2 f has two periodic points q_f and r_f with index$(q_f) \neq$ index(r_f). This is a contradiction since $f \in \mathcal{G}_1$ and f has the asymptotic orbital shadowing property. □

Lemma 3.4. [3, Lemma 5.1] *There is a residual set $\mathcal{G}_4 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_4$, for any $\delta > 0$ and any C^1 neighborhood $\mathcal{U}(f)$ of f, if there is $g \in \mathcal{U}(f)$ and a hyperbolic $p \in P(g)$ such that p has a δ weak eigenvalue then there is a hyperbolic $p_f \in P(f)$ with a 2δ weak eigenvalue.*

Proposition 3.5. *There is a residual set $\mathcal{G}_5 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_5$, if f has the asymptotic orbital shadowing property then $f \in \mathcal{F}(M)$.*

Proof. Let $f \in \mathcal{G}_5 = \mathcal{G}_3 \cap \mathcal{G}_4$ have the asymptotic orbital shadowing property. Suppose, by contradiction, that $f \notin \mathcal{F}(M)$. Then there is g C^1 close to f such that g has a nonhyperbolic periodic point p. By Lemma 2.5, there is g_1 C^1 close to g such that g_1 has the nonhyperbolic periodic point p_{g_1} has a $\delta/2$ weak eigenvalue. By Lemma 3.4, there is a hyperbolic $p_f \in P(f)$ such that p_f has a δ weak eigenvalue. This is a contradiction by Lemma 3.3. □

Lemma 3.6. [11, Theorem 7] *There is a residual set $\mathcal{G}_6 \subset \mathrm{Diff}(M)$ such that for any $f \in \mathcal{G}_6$, f is chain transitive if and only if f is transitive.*

Proof of Theorem B. Let $f \in \mathcal{G}_5 \cap \mathcal{G}_6$ have the asymptotic orbital shadowing property. Then by Proposition 3.5, $f \in \mathcal{F}(M)$. Thus the nonwandering set $\Omega(f) = \overline{P(f)}$ is hyperbolic. Since $f \in \mathcal{G}_6$, $\Omega(f) = M$ is hyperbolic. Thus f is transitive Anosov. □

4 Volume preserving diffeomorphisms

In this section, we consider volume preserving diffeomorphisms. In fact, we apply the results of diffeomorphisms. Let M be a closed smooth Riemannian manifold with $\dim M \geq 2$ and let $\mathrm{Diff}_\mu(M)$ denote the set of volume-preserving diffeomorphisms defined on M that preserve the Lebesgue measure μ induced by the Riemannian metric (see [31]). We consider this space endowed with the C^1 Whitney topology. In the volume preserving case, if a volume preserving diffeomorphism f satisfies Axiom A, then by the Poincaré recurrence theorem, we have $\Omega(f) = M$. Thus if f satisfies Axiom A then f is an Anosov diffeomorphism. Bessa [5] proved that if a volume preserving diffeomorphism f belongs to the C^1 interior of the set of volume preserving diffeomorphisms having the shadowing property then it is Anosov. Lee and Lee [18] proved that if a volume preserving diffeomorphism f belongs to the C^1 interior of the set of volume preserving diffeomorphisms the orbital shadowing property then it is an Anosov diffeomorphism. Lee [24] proved that if a volume preserving diffeomorphism f of a compact smooth two dimension manifold has the C^1 robustly weak and limit weak shadowing property then it is Ansov. Lee [28] proved that if a volume preserving diffeomorphism f has the C^1 robustly limit shadowing property then it is an Anosov diffemorphism. From these results, we have the following.

Theorem C *If $f \in \mathrm{Diff}_\mu(M)$ has the C^1 robustly asymptotic orbital shadowing property then f is Anosov.*

In $\dim M = 2$, a C^1 generic volume preserving diffeomorphism f has a chaotic phenomenona, that is, it has a homoclinic tangency, which is related to Smale's conjecture (see [6]). Therefore, we assume that $\dim M \geq 3$. Bessa *et al* [6] proved that if a C^1 generic volume preserving diffeomorphism f has the shadowing property then it is a transitive Anosov diffeomorphism. Lee [23] proved that if a C^1 generic volume preserving diffeomorphism f has the orbital shadowing property then it is an Anosov diffeomorphism. Lee [28] proved that if a C^1 generic volume preserving diffeomorphism f has the limit shadowing property then it is a transitive Anosov diffeomorphism. From these results, we have the following.

Theorem D *For C^1 generic $f \in \mathrm{Diff}_\mu(M)$, if f has the asymptotic orbital shadowing property then f is transitive Anosov.*

4.1 Proof of Theorem C

$\mathcal{F}_\mu(M)$ denotes the set of diffeomorphisms $f \in \mathrm{Diff}_\mu(M)$ that has a C^1 neighborhood $\mathcal{U}(f) \subset \mathrm{Diff}_\mu(M)$ such that if for any $g \in \mathcal{U}(f)$, each periodic point of g is hyperbolic. Note that $\mathcal{F}_\mu(M) \subset \mathcal{F}(M)$ (see [4, Corollary 1.2]).

Theorem 4.1. [4, Theorem 1.1] *For $f \in \mathrm{Diff}_\mu(M)$, if $f \in \mathcal{F}_\mu(M)$ then it is an Anosov diffeomorphism.*

The following is a volume preserving diffeomorphism version of the Franks' lemma (see [8, Proposition 7.4]).

Lemma 4.2. *Let $f \in \mathrm{Diff}_\mu(M)$, and $\mathcal{U}(f)$ be a C^1-neighborhood of f in $\mathrm{Diff}^1_\mu(M)$. Then there exist a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f and $\epsilon > 0$ such that if $g \in \mathcal{U}_0(f)$, any finite f-invariant set $E = \{x_1, \ldots, x_m\}$, any neighborhood U of E and any volume-preserving linear maps $L_j : T_{x_j}M \to T_{g(x_j)}M$ with $\|L_j - D_{x_j}g\| \leq \epsilon$ for all $j = 1, \ldots, m$, then there is conservative diffeomorphism $g_1 \in \mathcal{U}(f)$ coinciding with f on E and out of U, and $D_{x_j}g_1 = L_j$ for all $j = 1, \ldots, m$.*

A diffeomorphism $f \in \mathrm{Diff}_\mu(M)$ is called *Kupka-Smale* if any element of $P(f)$ is hyperbolic, and its invariant manifolds intersect transversely. The Kupka-Smale volume preserving diffeomorphisms are given by Robinson's theorem (see [36]). $\mathcal{KS}_\mu$ denotes the set of all Kupka-Smale volume preserving diffeomorphisms.

Proposition 4.3. *If $f \in \mathrm{Diff}_\mu(M)$ has the C^1 robustly asymptotic orbital shadowing property then $f \in \mathcal{F}_\mu(M)$*

Proof. Suppose, by contradiction, that $f \notin \mathcal{F}_\mu(M)$. Then there is g C^1 close to f such that g has a nonhyperbolic periodic point p of g. For simplicity, we assume that $g(p) = p$.

Then there is at least one eigenvalue λ of D_pg such that $|\lambda| = 1$. By Lemma 4.2, we can have only one eigenvalue λ with $|\lambda| = 1$. Then we have $T_pM = E^s_p \oplus E^u_p \oplus E^c_p$, where E^s_p is the eigenspace corresponding to the eigenvalues smaller than 1, E^u_p is the eigenspace corresponding to the eigenvalues larger than 1, and E^c_p is the eigenspace corresponding to λ. If $\lambda \in \mathbb{R}$ then $\dim E^c_p = 1$, and if $\lambda \in \mathbb{C}$ then $\dim E^c_p = 2$.

First, we consider $\dim E^c_p = 1$. For simplicity, we may assume that $\lambda = 1$ (the other case is similar). By Lemma 4.2, we linearize g at p with respect to Moser's theorem, i.e. by choosing $\alpha > 0$ sufficiently small we construct g_1 C^1-nearby g (g_1 C^1 close to f) such that

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_pg \circ \varphi_p(x) & \text{if} \quad x \in B_\alpha(p), \\ g(x) & \text{if} \quad x \notin B_{4\alpha}(p). \end{cases}$$

Then $g_1(p) = g(p) = p$. Since the eigenvalue λ of D_pg_1 is 1, we can take $\eta = \alpha/4$ such that $D_pg_1(v) = v$ for any $v \in E^c_p(\eta)$. Take $v_0 \in E^c_p(\eta)$ such that $\|v_0\| = \eta/4$. We set

$$\mathcal{I}_{v_0} = \{t \cdot v_0 : 1 \leq t \leq 1 + \eta/4\} \subset \varphi_p(B_\eta(p)),$$

and $\varphi_p^{-1}(\mathcal{I}_{v_0}) = \mathcal{J}_p$. Let $\mathrm{diam}(\mathcal{J}_p) = \eta/4$, and let $\epsilon = \mathrm{diam}(\mathcal{J}_p)/16$.

Since $g_1(\mathcal{J}_p) = \mathcal{J}_p$ is the identity map, $\varphi_p^{-1}(\mathcal{I}_{v_0}) = \mathcal{J}_p$ is g_1-invariant and by the construction of $\mathcal{J}_p$, it is normally hyperbolic. Let $\{x_i\}_{i\in\mathbb{Z}} \subset \mathcal{J}_p$ be an asymptotic pseudo orbit of f. If the shadowing point $z \in M \setminus \mathcal{J}_p$, then by hyperbolicity we have

$$d_H(\overline{Orb(z)}, \mathcal{J}_p) > \frac{\eta}{8}.$$

This is a contradiction, since g_1 has the asymptotic orbital shadowing property. Thus shadowing point has to be taken from $\mathcal{J}_p$. Take two points $a, b \in \mathcal{J}_p$ such that $d(a, b) = 4\epsilon$. Now we construct an asymptotic pseudo orbit as follows; $g_1^i(a) = x_i$ for all $i \geq 0$ and $g_1^i(b) = x_i$ for all $i \geq 1$. Then the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is an asymptotic pseudo orbit of g_1. By the asymptotic orbital shadowing property and that $\mathcal{J}_p$ is normally hyperbolic, there are $z \in \mathcal{J}_p$ and $N_1, N_2 \in \mathbb{N}$ such that
(i) $g_1^{-N_1-i}(z) \in B_\epsilon(\{x_{-N_1-i}\})$ and $g_1^{N_2+i}(z) \in B_\epsilon(\{x_{N_2+i}\})$ for $i \geq 0$,
(ii) $x_{-N_1-i} \in B_\epsilon(Orb(g_1^{-N_1-i}(z)))$ and $x_{N_2+i} \in B_\epsilon(Orb(g_1^{N_2+i}(z)))$ for $i \geq 0$.
Since $g_1 : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map, $g_1^i(z) = z$ for all $i \in \mathbb{Z}$. Thus if $d(z, a) < \epsilon$ then $g_1^i(z) = z \notin B_\epsilon(b)$, for all $i \in \mathbb{Z}$. The other case is similar. This is a contradiction.

Finally, if $\lambda \in \mathbb{C}$, then $\dim E_p^c = 2$. For simplicity, we may assume that $g(p) = p$. As in the first case, by Lemma 2.5, there are $\alpha > 0$ and $g_1 \in \mathcal{V}(f)$ such that $g_1(p) = g(p) = p$ and

$$g_1(x) = \begin{cases} \varphi_p^{-1} \circ D_p g \circ \varphi_p(x) & \text{if} \quad x \in B_\alpha(p), \\ g(x) & \text{if} \quad x \notin B_{4\alpha}(p). \end{cases}$$

With a C^1-small modification of the map $D_p g$, we may suppose that there is $l > 0$ (the minimum number) such that $D_p g^l(v) = v$ for any $v \in \varphi_p(B_\alpha(p)) \subset T_p M$. Consider $v_0 \in \varphi_p(B_\alpha(p))$ such that $\|v_0\| = \alpha/4$, and set

$$\mathcal{L}_p = \varphi_p^{-1}(\{t \cdot v_0 : 1 \leq t \leq 1 + \alpha/4\}).$$

Then $\mathcal{L}_p$ is an arc such that
- $g_1^i(\mathcal{L}_p) \cap g_1^j(\mathcal{L}_p) = \emptyset$ for $0 \leq i \neq j \leq l - 1$,
- $g_1^l(\mathcal{L}_p) = \mathcal{L}_p$, and
- $g_1^l|_{\mathcal{L}_p}$ is the identity map.

As in the previous arguments, we can show that g_1^l does not have the asymptotic orbital shadowing property on $\mathcal{L}_p$, which contradicts the fact that $g_1 \in \mathcal{U}(f)$. Thus, if f has the C^1 robustly asymptotic orbital shadowing property, then $f \in \mathcal{F}_\mu(M)$. □

Proof of Theorem C. Suppose that $f \in \mathrm{Diff}_\mu(M)$ has the C^1 robustly asymptotic orbital shadowing property. By Proposition 4.3 $f \in \mathcal{F}_\mu(M)$ and by Theorem 4.1, f is Anosov. □

4.2 Proof of Theorem D

In this section, we will prove that if a C^1 generic volume preserving diffeomorphism f has the asymptotic orbital shadowing property then it is a transitive Anosov diffeomorphism. Note that the set of Kupka-Smale volume preserving diffeomorphisms is residualt (see [36]).

Lemma 4.4. *There is a residual set $\mathcal{R}_1 \subset \mathrm{Diff}_\mu(M)$ such that for any $f \in \mathcal{R}_1$, and if f has the asymptotic orbital shadowing property then for any $p, q \in P(f)$, we have* index(p) = index(q).

Proof. Let $f \in \mathcal{R}_1 = \mathcal{KS}_\mu$ have the asymptotic orbital shadowing property. For any $p, q \in P(f)$, by Lemma 2.3 $W^s(p) \cap W^u(q) \neq \emptyset$ and $W^u(p) \cap W^s(q) \neq \emptyset$. As in the proof of Lemma 3.1, we obtain that index(p) = index(q). □

Lemma 4.5. [6, lemma 2.5] *Let $f \in \mathrm{Diff}_\mu(M)$. If $p \in P(f)$ is not hyperbolic then there is g C^1 close to f such that g has two hyperbolic periodic points q, r with* index(q) ≠ index(r).

Lemma 4.6. [6, Proposition 2.4] *There is a residual set* $\mathcal{R}_2 \subset \mathrm{Diff}_\mu(M)$ *such that for any* $f \in \mathcal{R}_2$, *and any* C^1 *neighborhood* $\mathcal{U}(f)$ *if there is* $g \in \mathcal{U}(f)$ *such that* g *has two points* $q, r \in P(g)$ *with* $\mathrm{index}(q) \neq \mathrm{index}(r)$ *then* f *has two points* $q_f, r_f \in P(f)$ *with* $\mathrm{index}(q_f) \neq \mathrm{index}(r_f)$.

Lemma 4.7. *There is a residual set* $\mathcal{R}_3 \subset \mathrm{Diff}_\mu(M)$ *such that for any* $f \in \mathcal{R}_3$, *and if* f *has the asymptotic orbital shadowing property then there is* $\delta > 0$ *such that for any* $p \in P(f)$, p *does not have a* δ *weak eigenvalue.*

Proof. Let $f \in \mathcal{R}_3 = \mathcal{R}_1 \cap \mathcal{R}_2$ have the asymptotic orbital shadowing property. Suppose that for any $\delta > 0$, there is $p \in P(f)$ such that p has a δ weak eigenvalue. As in the proof of Lemma 3.3, we can derive a contradiction. Thus for any $p \in P(f)$, p does not have a δ weak eigenvalue. □

Lemma 4.8. [6, Lemma 2.8] *There is a residual set* $\mathcal{R}_4 \subset \mathrm{Diff}_\mu(M)$ *such that for any* $f \in \mathcal{R}_4$, *for any* $\delta > 0$ *and any* C^1 *neighborhood* $\mathcal{U}(f)$ *of* f, *if there is* $g \in \mathcal{U}(f)$ *and a hyperbolic* $p \in P(g)$ *such that* p *has a* δ *weak eigenvalue then there is a hyperbolic* $p_f \in P(f)$ *with a* 2δ *weak eigenvalue.*

Proposition 4.9. *There is a residual set* $\mathcal{R}_5 \subset \mathrm{Diff}_\mu(M)$ *such that for any* $f \in \mathcal{R}_5$, *if* f *has the asymptotic orbital shadowing property then* $f \in \mathcal{F}_\mu(M)$

Proof. Let $f \in \mathcal{R}_5 = \mathcal{R}_4 \cap \mathcal{R}_4$ have the asymptotic orbital shadowing property. Suppose that $f \notin \mathcal{F}_\mu(M)$. Then there is g C^1 close to f such that g has a nonhyperbolic periodic point p. By Lemma 4.2, there is g_1 C^1 close to g(g_1 C^1 close to f) such that g_1 has a hyperbolic $p_g \in P(g_1)$ that has a $\delta/2$ weak eigenvalue. Since $f \in \mathcal{R}_4$, p_f has a δ weak eigenvalue. This is a contradiction by Lemma 4.7. Thus if $f \in \mathcal{R}_5$ has the asymptotic orbital shadowing property, for any $p \in P(f)$, p is hyperbolic. □

Lemma 4.10. [7, Theorem 3.1] *There is a residual set* $\mathcal{R}_6 \subset \mathrm{Diff}_\mu(M)$ *such that* $f \in \mathcal{R}_6$ *is transitive.*

Proof of Theorem D. Let $f \in \mathcal{R}_7 = \mathcal{R}_5 \cap \mathcal{R}_6$ have the asymptotic orbital shadowing property. By Proposition 4.9, $f \in \mathcal{F}_\mu(M)$. By Lemma 4.10 and Theorem 4.1, f is transitive Anosov. □

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