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The fourth order strongly noncanonical operators

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Abstract: It is shown that the strongly noncanonical fourth order operator

$$\mathcal{L}\,y = \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)'$$

can be written in essentially unique canonical form as

$$\mathcal{L}\,y = q_4(t)\left(q_3(t)\left(q_2(t)\left(q_1(t)\left(q_0(t)y(t)\right)'\right)'\right)'\right)'.$$

The canonical representation essentially simplifies examination of the fourth order strongly noncanonical equations

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y(\tau(t)) = 0.$$

Keywords: canonical operator, fourth order differential equations

MSC: 34K11, 34C10

1 Introduction

In the paper, we consider the fourth order delay differential equation

$$\left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' + p(t)y(\tau(t)) = 0, \tag{E}$$

where $r_i \in C^{(4-i)}(t_0, \infty)$, $r_i(t) > 0$, $i = 1, \dots, 3$, $p > 0$, $\tau(t) \le t$, $\tau'(t) > 0$ and $\tau(t) \to \infty$ as $t \to \infty$.

Fourth-order differential equations naturally appear in models concerning physical, biological, and chemical phenomena, such as, for instance, problems of elasticity, deformation of structures, or soil settlement, for example, see [1]. In mechanical and engineering problems, questions concerning the existence of oscillatory solutions play an important role. During the past decades, there has been a constant interest in obtaining sufficient conditions for oscillatory properties of different class of fourth order differential equations with deviating argument, see [1–13].

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As far as the oscillation theory of fourth-order differential equations is concerned, the problem of investigating ways of factoring disconjugated operators

$$\mathcal{L}\, y \equiv \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)', \tag{1.1}$$

which are crucial in studying the perturbed differential equations such as (E), has been of special interest. Motivated by the famous work of George Polya, Trench [2] showed that if operator $\mathcal{L}\, y$ is *strongly noncanonical*, that is,

$$\int^{\infty} \frac{1}{r_i(s)}\, \mathrm{d}s < \infty, \qquad i = 1, 2, 3, \tag{1.2}$$

then it can be written in an essentially unique *canonical form* as

$$\mathcal{L}\, y = q_4(t)\left(q_3(t)\left(q_2(t)\left(q_1(t)\left(q_0(t)y(t)\right)'\right)'\right)'\right)',$$

so that $q_i \in C^{(4-i)}(t_0, \infty)$, $q_i(t) > 0$, $i = 0, \ldots, 4$ and

$$\int^{\infty} \frac{1}{q_i(s)}\, \mathrm{d}s = \infty, \qquad i = 1, 2, 3.$$

However, the computation using the Lemmas 1 and 2 from [2] leading to canonical representation is very complicated and does not provide closed formulas for $q_i(t)$. This brings us to the question whether is it possible to establish a closed form formulas for q_i. The aim of this paper is to positively answer to this question, showing simultaneously the advantage of the result in the investigation of oscillatory properties of strongly noncanonical equations.

2 Main results

Throughout the paper we assume that (1.2) hold and so we can employ the notation

$$\pi_i(t) = \int_t^{\infty} \frac{1}{r_i(s)}\, \mathrm{d}s, \qquad \pi_{ij}(t) = \int_t^{\infty} \frac{1}{r_i(s)}\, \pi_j(s)\, \mathrm{d}s$$

and

$$\pi_{ijk}(t) = \int_t^{\infty} \frac{1}{r_i(s)}\, \pi_{jk}(s)\, \mathrm{d}s,$$

where $i, j, k \in \{1, 2, 3\}$ are mutually different.

We start with the following auxiliary results which are elementary but very useful.

Lemma 1. *Let* (1.2) *hold. Then*

$$\pi_{ij}(t) + \pi_{ji}(t) = \pi_i(t)\pi_j(t).$$

Proof. Since

$$\left(\pi_i(t)\pi_j(t)\right)' = -\frac{\pi_j(t)}{r_i(t)} - \frac{\pi_i(t)}{r_j(t)},$$

an integration of this equality from t to ∞, yields

$$\pi_i(t)\pi_j(t) = \int_t^{\infty} \frac{1}{r_i(s)}\pi_j(s)\, \mathrm{d}s + \int_t^{\infty} \frac{1}{r_j(s)}\pi_i(s)\, \mathrm{d}s = \pi_{ij}(t) + \pi_{ji}(t).$$

□

To simplify our notation, we denote

$$\Omega(t) = \pi_{12}(t)\pi_{23}(t) - \pi_2(t)\pi_{123}(t).$$

The following result provides an alternative formula for $\Omega(t)$.

Lemma 2. *Let* (1.2) *hold. Then*

$$\pi_{12}(t)\pi_{23}(t) - \pi_2(t)\pi_{123}(t) = \pi_{21}(t)\pi_{32}(t) - \pi_2(t)\pi_{321}(t). \tag{2.1}$$

Proof. Proof of this lemma is similar to that of Lemma 1 and so it can be omitted. □

Now, we are prepared to introduce the main result.

Theorem 1. *The strongly noncanonical operator $\mathcal{L}\,y$ has the following unique canonical representation*

$$\mathcal{L}\,y = \frac{1}{\pi_{321}(t)}\left(\frac{r_3\pi_{321}^2}{\Omega}\left(\frac{r_2\Omega^2}{\pi_{321}\pi_{123}}\left(\frac{r_1\pi_{123}^2}{\Omega}\left(\frac{y}{\pi_{123}}\right)'\right)'\right)'\right)'(t). \tag{2.2}$$

Proof. Straightforward evaluation of $\mathcal{L}\,y$, with $\mathcal{L}$ as defined by (2.2) yields

$$\mathcal{L}_1\,y = \frac{r_1\pi_{123}^2}{\Omega}\left(\frac{y}{\pi_{123}}\right)' = \frac{r_1y'\pi_{123} + y\pi_{23}}{\Omega}. \tag{2.3}$$

Employing (2.1), we see that

$$\begin{aligned}\mathcal{L}_2\,y &= \frac{\Omega^2}{r_2\pi_{321}\pi_{123}}\left(\mathcal{L}_1\,y\right)' \\ &= \frac{r_2\left(r_1y'\right)'\Omega - r_1y'\left(\pi_{321} - \pi_1\pi_{32}\right) + y\pi_{32}}{\pi_{321}}.\end{aligned}$$

It follows from Lemma 2 that $\pi_{12}(t) + \pi_{21}(t) = \pi_1(t)\pi_2(t)$ and so

$$\left(\mathcal{L}_2\,y\right)' = \frac{\left(r_2\left(r_1y'\right)'\right)'\Omega - r_1y'\frac{1}{r_3}\pi_{12} - y\frac{1}{r_3}\pi_2}{\pi_{321}} + \frac{r_2\left(r_1y'\right)'\Omega\pi_{21} + r_1y'\left(\pi_1\pi_{32}\pi_{21} - \pi_{321}\pi_{21}\right) + y\pi_{32}\pi_{21}}{r_3\pi_{321}^2}$$

On the other hand, by Lemma 1 and 2

$$\begin{aligned}r_1y'\left(\pi_1\pi_{32}\pi_{21} - \pi_{321}\pi_{21} - \pi_{12}\pi_{321}\right) &= r_1y'\left(\pi_1\pi_{32}\pi_{21} - \pi_{321}\pi_1\pi_2\right) \\ &= r_1y'\pi_1\Omega.\end{aligned}$$

Consequently,

$$\left(\mathcal{L}_2\,y\right)' = \frac{\left(r_2\left(r_1y'\right)'\right)'\Omega}{\pi_{321}} + \frac{r_2\left(r_1y'\right)'\Omega\pi_{21} + r_1y'\pi_1\Omega + y\Omega}{r_3\pi_{321}^2}.$$

Then

$$\mathcal{L}_3\,y = \frac{r_3\pi_{321}^2}{\Omega}\left(\mathcal{L}_2\,y\right)'s = r_3\left(r_2\left(r_1y'\right)'\right)'\pi_{321} + r_2\left(r_1y'\right)'\pi_{21} + r_1y'\pi_1 + y.$$

Finally

$$\frac{1}{\pi_{321}(t)}\left(\mathcal{L}_3\,y\right)' = \left(r_3(t)\left(r_2(t)\left(r_1(t)y'(t)\right)'\right)'\right)' = \mathcal{L}\,y,$$

which means that the operators (1.1) and (2.2) are equivalent.

Now we shall show that operator (2.2) is canonical. Direct computation shows that

$$\int_{t_0}^{t} \frac{1}{q_1(s)}\,\mathrm{d}s = \int_{t_0}^{t} \frac{\Omega(s)}{r_1(s)\pi_{123}^2(s)}\,\mathrm{d}s = \int_{t_0}^{t} \left(\frac{\pi_{12}(s)}{\pi_{123}(s)}\right)'\mathrm{d}s = \frac{\pi_{12}(t)}{\pi_{123}(t)} - c_1.$$

Applying twice the L'Hospital rule, we get

$$\lim_{t\to\infty} \frac{\pi_{12}(t)}{\pi_{123}(t)} = \lim_{t\to\infty} \frac{1}{\pi_3(t)} = \infty$$

and so

$$\int_{t_0}^{\infty} \frac{1}{q_1(s)}\,\mathrm{d}s = \infty.$$

Similarly

$$\int_{t_0}^{t} \frac{1}{q_3(s)}\,\mathrm{d}s = \int_{t_0}^{t} \frac{\Omega(s)}{r_3(s)\pi_{321}^2(s)}\,\mathrm{d}s = \left(\frac{\pi_{32}(t)}{\pi_{321}(t)} - c_3\right) \to \infty \text{ as } t \to \infty.$$

To evaluate the last one integral it is useful to see that

$$\begin{aligned}\left(\frac{\pi_{321}(t)\pi_{23}(t)}{\pi_{32}(t)\Omega(t)} + \frac{\pi_3(t)}{\pi_{32}(t)}\right)' &= \left(\frac{\pi_3(t)\pi_{21}(t) - \pi_{321}(t)}{\Omega(t)}\right)' \\ &= \frac{\pi_{321}(t)\pi_{123}(t)}{r_2(t)\Omega^2(t)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{t_0}^{t} \frac{1}{q_2(s)}\,\mathrm{d}s &= \int_{t_0}^{t} \frac{\pi_{321}(s)\pi_{123}(s)}{r_2(s)\Omega^2(s)}\,\mathrm{d}s = \int_{t_0}^{t} \left(\frac{\pi_{321}(s)\pi_{23}(s)}{\pi_{32}(s)\Omega(s)} + \frac{\pi_3(s)}{\pi_{32}(s)}\right)'\mathrm{d}s \\ &= \frac{\pi_{321}(t)\pi_{23}(t)}{\pi_{32}(t)\Omega(t)} + \frac{\pi_3(t)}{\pi_{32}(t)} - c_2.\end{aligned}$$

Moreover,

$$\lim_{t\to\infty} \frac{\pi_3(t)}{\pi_{32}(t)} = \lim_{t\to\infty} \frac{1}{\pi_2(t)} = \infty$$

and we conclude that the operator (2.2) is canonical. By Trench's result [2] there exists the only one canonical representation of $\mathcal{L}$ (up to multiplicative constants with product 1) and so our canonical form is unique. □

We support our results with couple of illustrative examples.

Example 1. *Let us consider the following operator*

$$\mathcal{L}y = \left(t^{\gamma}\left(t^{\beta}\left(t^{\alpha}y'(t)\right)'\right)'\right)', \qquad \alpha > 1, \quad \beta > 1, \quad \gamma > 1. \tag{2.4}$$

By Theorem 1, this operator can be rewritten in canonical form as

$$\mathcal{L}y = \frac{1}{t^{3-\alpha-\beta-\gamma}}\left(t^{2-\alpha}\left(t^{2-\beta}\left(t^{2-\gamma}\left(\frac{y(t)}{t^{3-\alpha-\beta-\gamma}}\right)'\right)'\right)'\right)'.$$

Example 2. *The operator*

$$\mathcal{L}y = \left(\mathrm{e}^{\gamma t}\left(\mathrm{e}^{\beta t}\left(\mathrm{e}^{\alpha t}y'(t)\right)'\right)'\right)', \qquad \alpha > 0, \quad \beta > 0, \quad \gamma > 0.$$

can be represented in canonical form as

$$\mathcal{L}y = \mathrm{e}^{(\alpha+\beta+\gamma)t}\left(\mathrm{e}^{-\alpha t}\left(\mathrm{e}^{-\beta t}\left(\mathrm{e}^{-\gamma t}\left(\mathrm{e}^{(\alpha+\beta+\gamma)t}y(t)\right)'\right)'\right)'\right)'.$$

3 Applications to differential equations

Theorem 1 can be applied to study oscillatory properties of differential equations. We are going to outline one such application based on comparison principle.

Corollary 1. *Noncanonical differential equation* (E) *can be written in canonical form as*

$$\frac{1}{\pi_{321}(t)}\left(\frac{r_3\pi_{321}^2}{\Omega}\left(\frac{r_2\Omega^2}{\pi_{321}\pi_{123}}\left(\frac{r_1\pi_{123}^2}{\Omega}\left(\frac{y}{\pi_{123}}\right)'\right)'\right)'\right)'(t)+p(t)y(\tau(t))=0.$$

Setting $z(t) = y(t)/\pi_{123}(t)$ we get the following comparison result that reduces oscillation of strongly noncanonical equation to that of canonical equation.

Corollary 2. *Noncanonical equation* (E) *is oscillatory if and only if the canonical equation*

$$\left(\frac{r_3\pi_{321}^2}{\Omega}\left(\frac{r_2\Omega^2}{\pi_{321}\pi_{123}}\left(\frac{r_1\pi_{123}^2}{\Omega}z'\right)'\right)'\right)'(t)+\pi_{321}(t)\pi_{123}(\tau(t))p(t)z(\tau(t))=0. \tag{E_c}$$

is oscillatory.

Example 3. *Let us consider the fourth order differential equation*

$$\left(t^{\gamma}\left(t^{\beta}\left(t^{\alpha}y'(t)\right)'\right)'\right)'+p(t)y(\tau(t))=0 \tag{3.1}$$

with $\alpha>1,\beta>1,\gamma>1$*. By Corollary 2, this equation is oscillatory if and only if the canonical equation*

$$\left(t^{2-\alpha}\left(t^{2-\beta}\left(t^{2-\gamma}z'(t)\right)'\right)'\right)'+\left(t\tau(t)\right)^{3-\alpha-\beta-\gamma}p(t)z(\tau(t))=0 \tag{3.2}$$

is oscillatory. It is more convenient to study oscillation of (3.12) *instead of* (3.1).

Now, we are ready to study the properties of (E) with the help of (E_c). Without loss of generality, we can consider only with the positive solutions of (E_c). The following result is a modification of Kiguradze's lemma [3].

Let us denote

$$q_1(t)=\frac{r_1(t)\pi_{123}^2(t)}{\Omega(t)},\quad q_2(t)=\frac{r_2(t)\Omega^2(t)}{\pi_{321}(t)\pi_{123}(t)},\quad q_3(t)=\frac{r_3(t)\pi_{321}^2(t)}{\Omega(t)}$$

and

$$P(t)=\pi_{321}(t)\pi_{123}(\tau(t))p(t).$$

Then (E_c) can be rewritten as

$$\left(q_3\left(q_2\left(q_1z'\right)'\right)'\right)'(t)+P(t)z(\tau(t))=0.$$

Lemma 3. *Assume that* $z(t)$ *is an eventually positive solution of* (E_c)*, then either*

$$z(t)\in\mathcal{N}_1 \iff z'(t)>0,\quad \left(q_1z'\right)'(t)<0,\quad \left(q_2\left(q_1z'\right)'\right)'(t)>0$$

or

$$z(t)\in\mathcal{N}_3 \iff z'(t)>0,\quad \left(q_1z'\right)'(t)>0,\quad \left(q_2\left(q_1z'\right)'\right)'(t)>0$$

Consequently, the set $\mathcal{N}$ of all positive solutions of (E) has the decomposition

$$\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3.$$

To obtain oscillation of studied equation (E), we need to eliminate both cases of possible non-oscillatory solutions.

Let us denote

$$Q_1(t) = \left(\frac{1}{q_2(t)} \int_t^\infty \frac{1}{q_3(u)} \int_u^\infty P(s) \mathrm{d}s \, \mathrm{d}u \right) \int_{t_1}^{\tau(t)} \frac{1}{q_1(u)} \mathrm{d}u$$

and

$$Q_2(t) = P(t) \int_{t_1}^{t} \frac{1}{q_1(s_1)} \int_{t_1}^{s_1} \frac{1}{q_2(u)} \int_{t_1}^{s} \frac{1}{q_3(s)} \mathrm{d}s \, \mathrm{d}u \, \mathrm{d}s_1.$$

Theorem 2. *Let* (1.2) *hold. Assume that both first-order delay differential equations*

$$x'(t) + Q_1(t)x(\tau(t)) = 0 \tag{3.3}$$

and

$$x'(t) + Q_2(t)x(\tau(t)) = 0. \tag{3.4}$$

are oscillatory. Then (E) *is oscillatory.*

Proof. Assume that $y(t)$ is an eventually positive solution of (E), say for $t \ge t_1$. Then by Corollary 2, $z(t) = y(t)/\pi_{123}(t)$ is a solution of (E_c).

It follows from Lemma 3 that either $z(t) \in \mathcal{N}_1$ or $z(t) \in \mathcal{N}_3$. At first, we admit that $z(t) \in \mathcal{N}_1$. Noting that $q_1(t)z'(t)$ is decreasing, we see that

$$z(t) \ge \int_{t_1}^{t} \frac{1}{q_1(u)} q_1(u)z'(u) \, du \ge q_1(t)z'(t) \int_{t_1}^{t} \frac{1}{q_1(u)} \, du. \tag{3.5}$$

Integrating (E_c) from t to ∞, we have

$$q_3 \left(q_2 \left(q_1 z' \right)' \right)'(t) \ge \int_t^\infty P(s)z(\tau(s)) \, \mathrm{d}s. \tag{3.6}$$

Taking into account that $z(\tau(t))$ is increasing, the last inequality yields

$$\left(q_2 \left(q_1 z' \right)' \right)'(t) \ge z(\tau(t)) \frac{1}{q_3(t)} \int_t^\infty P(s) \, \mathrm{d}s. \tag{3.7}$$

Integrating once more, we are led to

$$-\left(q_1 z' \right)'(t) \ge z(\tau(t)) \frac{1}{q_2(t)} \int_t^\infty \frac{1}{q_3(u)} \int_u^\infty P(s) \mathrm{d}s \, \mathrm{d}u. \tag{3.8}$$

Combining the last inequality with (3.5), one gets

$$\begin{aligned} -\left(q_1 z' \right)'(t) &\ge \left(z'(\tau(t)) \frac{q_1(\tau(t))}{q_2(t)} \int_t^\infty \frac{1}{q_3(u)} \int_u^\infty P(s) \mathrm{d}s \, \mathrm{d}u \right) \int_{t_1}^{\tau(t)} \frac{1}{q_1(u)} \mathrm{d}u \\ &= q_1(\tau(t))z'(\tau(t)Q_1(t). \end{aligned} \tag{3.9}$$

Thus, the function $x(t) = q_1(t)z'(t)$ is a positive solution of the differential inequality

$$x'(t) + Q_1(t)x(\tau(t)) \le 0.$$

Hence, by Philos theorem [4], we conclude that the corresponding differential equation (3.3) also has a positive solution, which contradicts the assumptions of the theorem.

Now, we shall assume that $z(t) \in \mathcal{N}_3$. Since $q_3\left(q_2\left(q_1 z'\right)'\right)'$ is decreasing, we are led to

$$q_2\left(q_1 z'\right)'(t) \ge \int_{t_1}^{t} \frac{1}{q_3(s)} q_3(s)\left(q_2\left(q_1 z'\right)'\right)'(s)\mathrm{d}s$$
$$\ge q_3\left(q_2\left(q_1 z'\right)'\right)'(t) \int_{t_1}^{t} \frac{1}{q_3(s)}\mathrm{d}s.$$

Integrating the above inequality, one can verify that

$$z'(t) \ge q_3\left(q_2\left(q_1 z'\right)'\right)'(t) \frac{1}{q_1(t)} \int_{t_1}^{t} \frac{1}{q_2(u)} \int_{t_1}^{s} \frac{1}{q_3(s)}\mathrm{d}s\,\mathrm{d}u$$

Integrating once more, we see that $x(t) = q_3\left(q_2\left(q_1 z'\right)'\right)'(t)$ satisfies

$$z(t) \ge x(t) \int_{t_1}^{t} \frac{1}{q_1(s_1)} \int_{t_1}^{s_1} \frac{1}{q_2(u)} \int_{t_1}^{s} \frac{1}{q_3(s)}\mathrm{d}s\,\mathrm{d}u\,\mathrm{d}s_1.$$

Setting the last estimate into (E_c), we see that $x(t)$ is a positive solution of the differential inequality

$$x'(t) + Q_2(t)x(\tau(t)) \le 0,$$

which in view of Philos theorem in [4] guarantees that the corresponding differential equation (3.4) has also a positive solution. This is a contradiction and the proof is complete now. □

Applying suitable criteria for oscillation of (3.3) and (3.4), we immediately obtain the criteria for oscillation of (E). We use the one which is due to Ladde et al. [5].

Corollary 3. *Let* (1.2) *hold. Assume that for* $i = 1, 2$

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} Q_i(s)\mathrm{d}s > \frac{1}{\mathrm{e}} \tag{3.10}$$

hold. Then (*E*) *is oscillatory.*

We support our results by another example.

Example 4. *Let us consider the general Euler delay differential equation*

$$\left(t^{\gamma}\left(t^{\beta}\left(t^{\alpha}y'(t)\right)'\right)'\right)' + a\,t^{\alpha+\beta+\gamma-4}y(\lambda t) = 0 \tag{3.11}$$

with $\alpha > 1$, $\beta > 1$, $\gamma > 1$, $a > 0$, *and* $0 < \lambda < 1$. *By Corollary 2 and Example 3, this equation is oscillatory if and only if the canonical equation*

$$\left(t^{2-\alpha}\left(t^{2-\beta}\left(t^{2-\gamma}z'(t)\right)'\right)'\right)' + a\lambda^{\alpha+\beta+\gamma-3}t^{\alpha+\beta+\gamma-4}z(\lambda t) = 0 \tag{3.12}$$

is oscillatory. The straightforward computation yields that

$$Q_1(t) \sim \frac{a\lambda^{2-\alpha-\beta}}{(\alpha+\beta+\gamma-3)(\beta+\gamma-2)(\gamma-1)}\, t^{-1}$$

and

$$Q_2(t) \sim \frac{a\lambda^{3-\alpha-\beta-\gamma}}{(\alpha+\beta+\gamma-3)(\alpha+\beta-2)(\alpha-1)}\, t^{-1}.$$

By Corollary 4 considered equation is oscillatory, provided that both conditions

$$\frac{a\lambda^{2-\alpha-\beta}}{(\alpha+\beta+\gamma-3)(\beta+\gamma-2)(\gamma-1)} \ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}}$$

and

$$\frac{a\lambda^{3-\alpha-\beta-\gamma}}{(\alpha+\beta+\gamma-3)(\alpha+\beta-2)(\alpha-1)} \ln\left(\frac{1}{\lambda}\right) > \frac{1}{\mathrm{e}}$$

are satisfied.

4 Summary

In this paper we provided canonical representation for strongly noncanonical operator. This canonical transformation is easy and immediate. Moreover, we point out its application in the oscillation theory.

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