Open Mathematics

Research Article

Feng Liu* and Lei Xu

Regularity of one-sided multilinear fractional maximal functions

https://doi.org/10.1515/math-2018-0129
Received April 26, 2018; accepted November 26, 2018

Abstract: In this paper we introduce and investigate the regularity properties of one-sided multilinear fractional maximal operators, both in continuous case and in discrete case. In the continuous setting, we prove that the one-sided multilinear fractional maximal operators $\mathfrak{M}_\beta^+$ and $\mathfrak{M}_\beta^-$ map $W^{1,p_1}(\mathbb{R})\times\cdots\times W^{1,p_m}(\mathbb{R})$ into $W^{1,q}(\mathbb{R})$ with $1 < p_1,\dots,p_m < \infty$, $1 \le q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \beta$, boundedly and continuously. In the discrete setting, we show that the discrete one-sided multilinear fractional maximal operators are bounded and continuous from $\ell^1(\mathbb{Z})\times\cdots\times\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. Here $\mathrm{BV}(\mathbb{Z})$ denotes the set of functions of bounded variation defined on $\mathbb{Z}$. Our main results represent significant and natural extensions of what was known previously.

Keywords: One-sided multilinear fractional maximal operators, Sobolev spaces, discrete maximal operators, bounded variation, continuity

MSC: 42B25, 46E35

1 Introduction and the main results

Over the last several years a considerable amount of attention has been given to investigate the behavior of differentiability of maximal function. A good start was due to Kinnunen [1] who showed that the usual centered Hardy-Littlewood maximal function $\mathcal{M}$ is bounded on $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \le \infty$, where $W^{1,p}(\mathbb{R}^d)$ is the first order Sobolev space, which consists of functions $f \in L^p(\mathbb{R}^d)$, whose first weak partial derivatives $D_i f$, $i = 1, 2, \dots, d$, belong to $L^p(\mathbb{R}^d)$. We endow $W^{1,p}(\mathbb{R}^d)$ with the norm

$$\|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)},$$

where $\nabla f = (D_1 f, D_2 f, \dots, D_d f)$ is the weak gradient of f. Later on, Kinnunen's result was extended to a local version in [2], to a fractional version in [3], to a multilinear version in [4, 5] and to a one-sided version in [6]. Meanwhile, the continuity of $\mathcal{M} : W^{1,p} \to W^{1,p}$ for $1 < p < \infty$ was proved by Luiro in [7] and in [8] for its local version. Since Kinnunen's result does not hold for $p = 1$, an important question was posed by Hajłasz and Onninen in [9]: Is the operator $f \mapsto |\nabla \mathcal{M} f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$? Progress on the above problem has been restricted to dimension $d = 1$. In 2002, Tanaka [10] showed that if $f \in W^{1,1}(\mathbb{R})$, then the uncentered Hardy-Littlewood maximal function $\widetilde{\mathcal{M}}f$ is weakly differentiable and

$$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \le 2\|f'\|_{L^1(\mathbb{R})}. \tag{1.1}$$

***Corresponding Author: Feng Liu:** College of Mathematics and Systems Sciences, Shandong University of Science and Technology, Qingdao 266590, China, E-mail: Fliu@sdust.edu.cn
Lei Xu: College of Mathematics and Systems Sciences, Shandong University of Science and Technology, Qingdao 266590, China, E-mail: skmathxl@163.com

This result was later sharpened by Aldaz and Pérez Lázaro [11] who proved that if f is of bounded variation on $\mathbb{R}$, then $\widetilde{\mathcal{M}}f$ is absolutely continuous and its total variation satisfies

$$\mathrm{Var}(\widetilde{\mathcal{M}}f) \le \mathrm{Var}(f). \tag{1.2}$$

The above result implies directly (1.1) with constant $C = 1$ (also see [12] for a simple proof). In remarkable work [13], Kurka obtained that (1.1) and (1.2) hold for $\mathcal{M}$ (with constant $C = 240,004$). Recently, Carneiro and Madrid [14] extended (1.1) and (1.2) to a fractional setting. Very recently, Liu and Wu [15] extended the partial result of [14] to a multilinear setting. For other interesting works related to this theory, we refer the reader to [16–25], among others.

In this paper we focus on the regularity properties of the one-sided multilinear fractional maximal operators. More precisely, let m be a positive integer. For $0 \le \beta < m$, we define the one-sided multilinear fractional maximal operators $\mathfrak{M}_\beta^+$ and $\mathfrak{M}_\beta^-$ by

$$\mathfrak{M}_\beta^+(\vec{f})(x) = \sup_{s>0} \frac{1}{s^{m-\beta}} \prod_{i=1}^m \int_x^{x+s} |f_i(y)|dy \ \text{ and } \mathfrak{M}_\beta^-(\vec{f})(x) = \sup_{r>0} \frac{1}{r^{m-\beta}} \prod_{i=1}^m \int_{x-r}^{x} |f_i(y)|dy,$$

where $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in L^1_{\mathrm{loc}}(\mathbb{R})$. When $\beta = 0$, the operator $\mathfrak{M}_\beta^+$ (resp., $\mathfrak{M}_\beta^-$) reduces to the one-sided multilinear Hardy-Littlewood maximal operator $\mathfrak{M}^+$ (resp., $\mathfrak{M}^-$). When $m = 1$, the operator $\mathfrak{M}_\beta^+$ (resp., $\mathfrak{M}_\beta^-$) reduces to the one-sided fractional maximal operator $\mathcal{M}_\beta^+$ (resp., $\mathcal{M}_\beta^-$). Especially, the one-sided Hardy-Littlewood maximal operator $\mathcal{M}^+$ (resp., $\mathcal{M}^-$) corresponds to the operator $\mathcal{M}_\beta^+$ (resp., $\mathcal{M}_\beta^-$) in this case $\beta = 0$.

As we all known, the reasons to study one-sided operators involve not only the generalization of the theory of the two-sided operators but also the close connection between the one-sided operators and two-sided operators. The one-sided Hardy-Littlewood maximal operator $\mathcal{M}^+$ can be seen as the special case of the ergodic maximal operator. Furthermore, there is a close connection between the one-sided fractional maximal functions and the well-known Riemann-Liourille fractional integral that can be viewed as the one-sided version of Riesz potential and the Weyl fractional integral (see [26]). It was known that both $\mathcal{M}_\beta^+$ and $\mathcal{M}_\beta^-$ are of type (p, q) for $1 < p < \infty$, $0 \le \beta < 1/p$ and $q = p/(1-p\beta)$. Moreover, both $\mathcal{M}_\beta^+$ and $\mathcal{M}_\beta^-$ are of weak type $(1, q)$ for $0 \le \beta < 1$ and $q = 1/(1-\beta)$. Observing that the following inequalities are valid:

$$\mathfrak{M}_\beta^+(\vec{f})(x) \le \prod_{i=1}^m \mathcal{M}_{\beta_i}^+ f_i(x), \quad \forall x \in \mathbb{R}, \tag{1.3}$$

where $\vec{f} = (f_1, \dots, f_m)$ and $\beta = \sum_{i=1}^m \beta_i$ with $\beta_i \ge 0\,(i = 1, 2, \dots, m)$. By (1.3), the L^p bounds for $\mathcal{M}_\beta^+$ and Hölder's inequality, one has

$$\|\mathfrak{M}_\beta^+(\vec{f})\|_{L^q(\mathbb{R})} \le C(\beta, p_1, \dots, p_m) \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R})} \tag{1.4}$$

for $1/q = \sum_{i=1}^m 1/p_i - \beta$, provided that (i) $\beta = 0$, $1 \le q \le \infty$ and $1 < p_1, \dots, p_m \le \infty$; (ii) $0 < \beta < m$, $1 \le q < \infty$ and $1 < p_1, \dots, p_m < \infty$. The same result holds for $\mathfrak{M}_\beta^-$.

The investigation on the regularity of one-sided maximal operator began with Tanaka [10] in 2002 when he observed that if $f \in W^{1,1}(\mathbb{R})$, then the distributional derivatives of $\mathcal{M}^+ f$ and $\mathcal{M}^- f$ are integrable functions, and

$$\|(\mathcal{M}^+ f)'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})} \ \text{ and } \|(\mathcal{M}^- f)'\|_{L^1(\mathbb{R})} \le \|f'\|_{L^1(\mathbb{R})}.$$

By a combination of arguments in [10, 12], both $\mathcal{M}^+ f$ and $\mathcal{M}^- f$ are absolutely continuous on $\mathbb{R}$. Recently, Liu and Mao [6] proved that both $\mathcal{M}^+$ and $\mathcal{M}^-$ are bounded and continuous on $W^{1,p}(\mathbb{R})$ for $1 < p < \infty$. Very recently, Liu [27] extended the main results of [6] to the fractional case. More precisely, Liu proved the following result.

Theorem A ([27]). *Let $1 < p < \infty$, $0 \le \beta < 1/p$ and $q = p/(1-p\beta)$. Then both $\mathcal{M}_\beta^+$ and $\mathcal{M}_\beta^-$ map $W^{1,p}(\mathbb{R})$ into $W^{1,q}(\mathbb{R})$ boundedly and continuously. Moreover, if $f \in W^{1,p}(\mathbb{R})$, then*

$$|(\mathcal{M}_\beta^+ f)'(x)| \le \mathcal{M}_\beta^+ f'(x) \ \text{ and } |(\mathcal{M}_\beta^- f)'(x)| \le \mathcal{M}_\beta^- f'(x)$$

for almost every $x \in \mathbb{R}$.

In this paper we shall extended Theorem A to the multilinear case. We now formulate our main results as follows.

Theorem 1.1. *Let* $1 < p_1, \dots, p_m < \infty$, $0 \le \beta < \sum_{i=1}^m 1/p_i$, $1/q = \sum_{i=1}^m 1/p_i - \beta$ *and* $1 \le q < \infty$. *Then* $\mathfrak{M}_\beta^+$ *maps* $W^{1,p_1}(\mathbb{R}) \times \dots \times W^{1,p_m}(\mathbb{R})$ *into* $W^{1,q}(\mathbb{R})$ *boundedly and continuously. Especially, if* $\vec{f} = (f_1, \dots, f_m)$ *with each* $f_i \in W^{1,p_i}(\mathbb{R})$, *then the weak derivative* $(\mathfrak{M}_\beta^+(\vec{f}))'$ *exists almost everywhere. More precisely,*

$$|(\mathfrak{M}_\beta^+(\vec{f}))'(x)| \le \sum_{j=1}^m \mathfrak{M}_\beta^+(\vec{f}^j)(x)$$

for almost every $x \in \mathbb{R}$, *where* $\vec{f}^j = (f_1, \dots, f_{j-1}, f_j', f_{j+1}, \dots, f_m)$. *Moreover,*

$$\|\mathfrak{M}_\beta^+(\vec{f})\|_{1,q} \le C(\beta, p_1, \dots, p_m) \prod_{i=1}^m \|f_i\|_{1,p_i}.$$

The same results hold for $\mathfrak{M}_\beta^-$.

Theorem 1.2. *Let* $\vec{f} = (f_1, \dots, f_m)$ *with each* $f_i \in L^{p_i}(\mathbb{R})$ *for* $1 < p_1, \dots, p_m < \infty$ *and* $1 \le \beta < \sum_{i=1}^m 1/p_i$.

(i) *Then the weak derivative* $(\mathfrak{M}_\beta^+(\vec{f}))'$ *exists almost everywhere. Precisely,*

$$|(\mathfrak{M}_\beta^+(\vec{f}))'(x)| \le C(m, \beta)\mathfrak{M}_{\beta-1}^+(\vec{f})(x)$$

for almost every $x \in \mathbb{R}$.

(ii) *Let* $1/q = \sum_{i=1}^m 1/p_i - \beta + 1$. *Then*

$$\|(\mathfrak{M}_\beta^+(\vec{f}))'\|_{L^q(\mathbb{R})} \le C(m, \beta, p_1, \dots, p_m) \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

The same results hold for $\mathfrak{M}_\beta^-$.

Remark 1.1. Theorem 1.1 extends Theorems 1.1-1.2 in [6], which correspond to the case $m = 1$ and $\beta = 0$. Theorem 1.1 also extends Theorem A, which corresponds to the case $m = 1$.

On the other hand, the investigation of the regularity properties of discrete maximal operators has also attracted the attention of many authors (see [6, 14, 16, 27–33] for example). Let us recall some notation and relevant results. For $1 \le p < \infty$ and a discrete function $f : \mathbb{Z} \to \mathbb{R}$, we define the ℓ^p-norm and the ℓ^∞-norm of f by $\|f\|_{\ell^p(\mathbb{Z})} = (\sum_{n\in\mathbb{Z}} |f(n)|^p)^{1/p}$ and $\|f\|_{\ell^\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |f(n)|$. We also define the first derivative of f by $f'(n) = f(n+1) - f(n)$ for any $n \in \mathbb{Z}$. For $f : \mathbb{Z} \to \mathbb{R}$, we define the total variation of f by

$$\mathrm{Var}(f) = \|f'\|_{\ell^1(\mathbb{Z})}.$$

We denote by $\mathrm{BV}(\mathbb{Z})$ the set of all functions $f : \mathbb{Z} \to \mathbb{R}$ satisfying $\mathrm{Var}(f) < \infty$.

In 2011, Bober et al. [28] first studied the regularity properties of discrete Hardy-Littlewood maximal operators and proved that

$$\mathrm{Var}(\widetilde{M}f) \le \mathrm{Var}(f) \tag{1.5}$$

and

$$\mathrm{Var}(Mf) \le \Big(2 + \frac{146}{315}\Big)\|f\|_{\ell^1(\mathbb{Z})}. \tag{1.6}$$

Here M (resp., $\widetilde{M}$) denotes the discrete centered (resp., uncentered) Hardy-Littlewood maximal operator, which are defined by

$$Mf(n) = \sup_{r\in\mathbb{N}} \frac{1}{2r+1} \sum_{k=-r}^{r} |f(n+k)| \ \text{ and } \ \widetilde{M}f(n) = \sup_{r,\,s\in\mathbb{N}} \frac{1}{r+s+1} \sum_{k=-r}^{s} |f(n+k)|,$$

where $\mathbb{N} = \{0, 1, 2, 3, \dots, \}$. We note that inequality (1.5) is sharp. It was known that inequality $\mathrm{Var}(Mf) \le 294{,}912{,}004\mathrm{Var}(f)$ was established by Temur in [32]. Inequality (1.6) is not optimal, and it was asked in [28] whether the sharp constant for (1.6) is in fact $C = 2$, which was addressed by Madrid in [31]. Recently, Carneiro and Madrid [14] extended (1.5) to the fractional setting. They also pointed out that the discrete fractional maximal operators M_β and $\widetilde{M}_\beta$ are bounded and continuous from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$ (also see [29, 34]). Here M_β and $\widetilde{M}_\beta$ are the discrete centered and uncentered fractional maximal operators, which are defined by

$$M_\beta f(n) = \sup_{r\in\mathbb{N}} \frac{1}{(2r+1)^{1-\beta}} \sum_{k=-r}^{r} |f(n+k)| \quad\text{and}\quad \widetilde{M}_\beta f(n) = \sup_{r,\, s\in\mathbb{N}} \frac{1}{(r+s+1)^{1-\beta}} \sum_{k=-r}^{s} |f(n+k)|.$$

Our second aim of this paper is to consider the discrete one-sided multilinear fractional maximal operators

$$\mathrm{M}_\beta^+(\vec{f})(n) = \sup_{s\in\mathbb{N}} \frac{1}{(s+1)^{m-\beta}} \prod_{i=1}^{m} \sum_{k=0}^{s} |f_i(n+k)|,$$

$$\mathrm{M}_\beta^-(\vec{f})(n) = \sup_{r\in\mathbb{N}} \frac{1}{(r+1)^{m-\beta}} \prod_{i=1}^{m} \sum_{k=-r}^{0} |f_i(n+k)|,$$

where $0 \le \beta < m$ and $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in L^1_{\mathrm{loc}}(\mathbb{Z})$. When $\beta = 0$, the operators M_β^+ and M_β^- reduce to the discrete one-sided multilinear Hardy-Littlewood maximal operators M^+ and M^-, respectively. When $m = 1$, the operators M_β^+ and M_β^- reduce to the discrete one-sided fractional maximal operators M_β^+ and M_β^-, respectively. Particularly, the discrete one-sided Hardy-Littlewood maximal operators M^+ and M^- correspond to the special case of M_β^+ and M_β^- when $\beta = 0$, respectively. Recently, Liu and Mao [6] proved that both M^+ and M^- are bounded and continuous from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. Moreover, if $f \in \mathrm{BV}(\mathbb{Z})$, then

$$\max\{\mathrm{Var}(M^+f), \mathrm{Var}(M^-f)\} \le \mathrm{Var}(f). \tag{1.7}$$

We notice that the constant $C = 1$ in inequality (1.7) is sharp. Very recently, Liu [27] pointed out that M_β^+ and M_β^- are not bounded from $\mathrm{BV}(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$ when $0 < \beta < 1$. However, Liu established the following result.

Theorem B ([27]). *Let $0 \le \beta < 1$. Then M_β^+ is bounded and continuous from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. Moreover, if $f \in \ell^1(\mathbb{Z})$, then*

$$\mathrm{Var}(M_\beta^+ f) \le 2\|f\|_{\ell^1(\mathbb{Z})},$$

and the constant $C = 2$ is the best possible. The same results hold for M_β^-.

In this paper we shall extended Theorem B to the following.

Theorem 1.3. *Let $0 \le \beta < m$. Then M_β^+ is bounded and continuous from $\ell^1(\mathbb{Z}) \times \cdots \times \ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. Moreover, if $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in \ell^1(\mathbb{Z})$, then*

$$\mathrm{Var}(\mathrm{M}_\beta^+(\vec{f})) \le 2m \prod_{i=1}^{m} \|f_i\|_{\ell^1(\mathbb{Z})}.$$

The same results hold for M_β^-.

Remark 1.2. When $m = 1$, Theorem 1.3 implies Theorem B.

The rest of this paper is organized as follows. Section 2 contains some notation and preliminary lemmas, which can be used to prove the continuity part in Theorem 1.1. Motivated by the ideas in [5, 7], we give the proofs of Theorems 1.1-1.2 in Section 3. Finally, we prove Theorem 1.3 in Section 4. It should be pointed out that the proof of the boundedness part in Theorem 1.3 is based on the method of [31]. The proof of the continuity part in Theorem 1.3 relies on the previous boundedness result and a useful application of the Brezis-Lieb lemma in [35]. Throughout this paper, the letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but independent of the essential variables.

2 Preliminary notation and lemmas

In this section we shall introduce some notation and lemmas, which play key roles in the proof of the continuity part in Theorem 1.1. Let $A \subset \mathbb{R}$ and $r \in \mathbb{R}$. We define

$$d(r,A) := \inf_{a\in A} |r-a| \text{ and } A_{(\lambda)} := \{x \in \mathbb{R} : d(x,A) \leq \lambda\} \text{ for } \lambda \geq 0.$$

Denote $\|f\|_{p,A}$ by the L^p-norm of $f\chi_A$ for all measurable sets $A \subset \mathbb{R}$. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in L^{p_i}(\mathbb{R})$ for $1 < p_i < \infty$ and $1 \leq q < \infty$ with $1/q = \sum_{i=1}^m 1/p_i - \beta$. In what follows, we only consider the operator $\mathfrak{M}_\beta^+$ and the other case is analogous. Fix $x \in \mathbb{R}$, we define the set $\mathfrak{R}_\beta^+(\vec{f})(x)$ by

$$\mathfrak{R}_\beta^+(\vec{f})(x) := \Big\{ s \geq 0 : \mathfrak{M}_\beta^+(\vec{f})(x) = \limsup_{k\to\infty} \frac{1}{s_k^{m-\beta}} \prod_{i=1}^m \int_x^{x+s_k} |f_i(y)|dy \text{ for some } s_k > 0,\ s_k \to s \Big\}.$$

We also define the function $u^+_{x,\vec{f},\beta} : [0,\infty) \mapsto \mathbb{R}$ by

$$u^+_{x,\vec{f},\beta}(0) = \begin{cases} \prod_{i=1}^m |f_i(x)|, & \text{if } \beta = 0; \\ 0, & \text{if } 0 < \beta < m, \end{cases}$$

$$u^+_{x,\vec{f},\beta}(s) = \frac{1}{s^{m-\beta}} \prod_{i=1}^m \int_x^{x+s} |f_i(y)|dy \text{ for } s \in (0,\infty).$$

We notice that the followings are valid.
(i) $u^+_{x,\vec{f},\beta}$ is continuous on $(0,\infty)$ for all $x \in \mathbb{R}$ and at $r = 0$ for almost everywhere $x \in \mathbb{R}$;
(ii) $\lim_{s\to\infty} u^+_{x,\vec{f},\beta}(s) = 0$ since $u^+_{x,\vec{f},\beta}(s) \leq \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R})} s^{-1/q}$;
(iii) The set $\mathfrak{R}_\beta^+(\vec{f})(x)$ is nonempty and closed for any $x \in \mathbb{R}$;
(iv) Almost every point is a Lebesgue point.

From the above observations we have

$$\mathfrak{M}_\beta^+(\vec{f})(x) = u^+_{x,\vec{f},\beta}(s) \text{ if } 0 < s \in \mathfrak{R}_\beta^+(\vec{f})(x), \quad \forall x \in \mathbb{R},$$

$$\mathfrak{M}_\beta^+(\vec{f})(x) = u^+_{x,\vec{f},\beta}(0) \text{ for almost every } x \in \mathbb{R} \text{ such that } 0 \in \mathfrak{R}_\beta^+(\vec{f})(x).$$

Lemma 2.1. *Let $1 < p_1, \dots, p_m < \infty$ and $1 \leq q < \infty$ with $1/q = \sum_{i=1}^m 1/p_i - \beta$. Let $\vec{f_j} = (f_{1,j}, \dots, f_{m,j})$ and $\vec{f} = (f_1, \dots, f_m)$ such that $f_{i,j} \to f_i$ in $L^{p_i}(\mathbb{R})$ when $j \to \infty$. Then, for all $R > 0$ and $\lambda > 0$, it holds that*

$$\lim_{j\to\infty} |\{x \in (-R,R) : \mathfrak{R}_\beta^+(\vec{f_j})(x) \not\subseteq \mathfrak{R}_\beta^+(\vec{f})(x)_{(\lambda)}\}| = 0. \tag{2.1}$$

Proof. Without loss of generality, we may assume that all $f_{i,j} \geq 0$ and $f_i \geq 0$. By the similar argument as in the proof of Lemma 2.2 in [7], we can conclude that the set $\{x \in \mathbb{R} : \mathfrak{R}_\beta^+(\vec{f_j})(x) \not\subseteq \mathfrak{R}_\beta^+(\vec{f})(x)_{(\lambda)}\}$ is measurable for any $j \in \mathbb{Z}$. Let $\lambda > 0$ and $R > 0$. We first claim that for almost every $x \in (-R,R)$, there exists $\gamma(x) \in \mathbb{N} \setminus \{0\}$ such that

$$u^+_{x,\vec{f},\beta}(s) < \mathfrak{M}_\beta^+(\vec{f})(x) - \frac{1}{\gamma(x)} \text{ when } d(s, \mathfrak{R}_\beta^+(\vec{f})(x)) > \lambda. \tag{2.2}$$

Otherwise, for almost every $x \in (-R,R)$, there exists a bounded sequence of radii $\{s_k\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} u^+_{x,\vec{f},\beta}(s_k) = \mathfrak{M}_\beta^+(\vec{f})(x) \text{ and } d(s_k, \mathfrak{R}_\beta^+(\vec{f})(x)) > \lambda.$$

We can choose a subsequence $\{r_k\}_{k=1}^{\infty}$ of $\{s_k\}_{k=1}^{\infty}$ such that $r_k \to s$ as $k \to \infty$. Then we have $s \in \mathfrak{R}_{\beta}^{+}(\vec{f})(x)$ and $d(s, \mathfrak{R}_{\beta}^{+}(\vec{f})(x)) \geq \lambda$, which is a contradiction. Thus (2.2) holds. Given $\epsilon \in (0, 1)$, (2.2) yields that there exists $\gamma = \gamma(R, \lambda, \epsilon) \in \mathbb{N} \setminus \{0\}$ and a measurable set E with $|E| < \epsilon$ such that

$$(-R, R) \subset \{x \in \mathbb{R} : u_{x,\vec{f},\beta}^{+}(s) < \mathfrak{M}_{\beta}^{+}(\vec{f})(x) - \gamma^{-1} \ \text{ if } d(s, \mathfrak{R}_{\beta}^{+}(\vec{f})(x)) > \lambda\} \cup E.$$

Notice that

$$\mathfrak{M}_{\beta}^{+}(\vec{f})(x) - u_{x,\vec{f},\beta}^{+}(s) \leq |\mathfrak{M}_{\beta}^{+}(\vec{f}_j)(x) - \mathfrak{M}_{\beta}^{+}(\vec{f})(x)| + |u_{x,\vec{f}_j,\beta}^{+}(s) - u_{x,\vec{f},\beta}^{+}(s)| + \mathfrak{M}_{\beta}^{+}(\vec{f}_j)(x) - u_{x,\vec{f}_j,\beta}^{+}(s).$$

It yields that

$$\{x \in \mathbb{R} : u_{x,\vec{f},\beta}^{+}(s) < \mathfrak{M}_{\beta}^{+}(\vec{f})(x) - \gamma^{-1} \ \text{ if } d(s, \mathfrak{R}_{\beta}^{+}(\vec{f})(x)) > \lambda\} \subset A_{1,j} \cup A_{2,j} \cup A_{3,j},$$

where

$$A_{1,j} := \{x \in \mathbb{R} : |\mathfrak{M}_{\beta}^{+}(\vec{f}_j)(x) - \mathfrak{M}_{\beta}^{+}(\vec{f})(x)| \geq (4\gamma)^{-1}\},$$

$$A_{2,j} := \{x \in \mathbb{R} : |u_{x,\vec{f}_j,\beta}^{+}(s) - u_{x,\vec{f},\beta}^{+}(s)| \geq (2\gamma)^{-1} \ \text{ for some } s \text{ such that } d(s, \mathfrak{R}_{\beta}^{+}(\vec{f})(x)) > \lambda\},$$

$$A_{3,j} := \{x \in \mathbb{R} : u_{x,\vec{f}_j,\beta}^{+}(s) < \mathfrak{M}_{\beta}^{+}(\vec{f}_j)(x) - (4\gamma)^{-1} \ \text{ if } d(s, \mathfrak{R}_{\beta}^{+}(\vec{f})(x)) > \lambda\}.$$

Hence,

$$(-R, R) \subset A_{1,j} \cup A_{2,j} \cup A_{3,j} \cup E. \tag{2.3}$$

Let $\bar{A}$ be the set of all points x such that x is a Lebesgue point of all f_j. Note that $|\mathbb{R} \setminus \bar{A}| = 0$ and $A_{3,j} \cap \bar{A} \subset \{x \in \mathbb{R} : \mathfrak{R}_{\beta}^{+}(\vec{f}_j)(x) \subset \mathfrak{R}_{\beta}^{+}(\vec{f})(x)_{(\lambda)}\}$. This together with (2.3) implies

$$\{x \in (-R, R) : \mathfrak{R}_{\beta}^{+}(\vec{f}_j)(x) \not\subset \mathfrak{R}_{\beta}^{+}(\vec{f})(x)_{(\lambda)}\} \subset A_{1,j} \cup A_{2,j} \cup E \cup (\mathbb{R} \setminus \bar{A}).$$

It follows that

$$|\{x \in (-R, R) : \mathfrak{R}_{\beta}^{+}(\vec{f}_j)(x) \not\subset \mathfrak{R}_{\beta}^{+}(\vec{f})(x)_{(\lambda)}\}| \leq |A_{1,j}| + |A_{2,j}| + \epsilon. \tag{2.4}$$

We can write

$$\begin{aligned}
&|\mathfrak{M}_{\beta}^{+}(\vec{f}_j)(x) - \mathfrak{M}_{\beta}^{+}(\vec{f})(x)| \\
&\leq \sup_{s>0} \frac{1}{s^{m-\beta}} \Big| \prod_{i=1}^{m} \int_{x}^{x+s} f_{i,j}(y)dy - \prod_{i=1}^{m} \int_{x}^{x+s} f_i(y)dy \Big| \\
&\leq \sum_{l=1}^{m} \sup_{s>0} \frac{1}{s^{m-\beta}} \prod_{\mu=1}^{l-1} \int_{x}^{x+s} f_{\mu}(y)dy \prod_{\nu=l+1}^{m} \int_{x}^{x+s} f_{\nu,j}(y)dy \int_{x}^{x+s} |f_{l,j}(y) - f_l(y)|dy \\
&\leq \sum_{l=1}^{m} \mathfrak{M}_{\beta}^{+}(\vec{f}_j^{\,l})(x)
\end{aligned} \tag{2.5}$$

for any $x \in \mathbb{R}$, where $\vec{f}_j^{\,l} = (f_1, \ldots, f_{l-1}, f_{l,j} - f_l, f_{l+1,j}, \ldots, f_{m,j})$. From (2.5) we have

$$\begin{aligned}
|A_{1,j}| &\leq \Big| \Big\{ x \in \mathbb{R} : \sum_{l=1}^{m} \mathfrak{M}_{\beta}^{+}(\vec{f}_j^{\,l})(x) \geq (4\gamma)^{-1} \Big\} \Big| \\
&\leq \sum_{l=1}^{m} |\{x \in \mathbb{R} : \mathfrak{M}_{\beta}^{+}(\vec{f}_j^{\,l})(x) \geq (4m\gamma)^{-1}\}| \\
&\leq (4m\gamma)^{q} \sum_{l=1}^{m} \|\mathfrak{M}_{\beta}^{+}(\vec{f}_j^{\,l})\|_{L^q(\mathbb{R})}^{q}.
\end{aligned} \tag{2.6}$$

Since $f_{i,j} \to f_i$ in $L^{p_i}(\mathbb{R})$ as $j \to \infty$, then there exists $N_0 = N_0(\epsilon, \gamma) \in \mathbb{N} \setminus \{0\}$ such that

$$\|f_{i,j} - f_i\|_{L^{p_i}(\mathbb{R})} < \frac{\epsilon}{\gamma} \ \text{ and } \ \|f_{i,j}\|_{L^{p_i}(\mathbb{R})} \leq \|f_i\|_{L^{p_i}(\mathbb{R})} + 1, \quad \forall j \geq N_0. \tag{2.7}$$

(2.7) together with (2.6) and (1.4) yields that

$$|A_{1,j}| \le C(m, q, \beta, p_1, \dots, p_m, \vec{f})\epsilon, \quad \forall j \ge N_0. \tag{2.8}$$

On the other hand, one can easily check that

$$|u^+_{x,\vec{f}_j,\beta}(s) - u^+_{x,\vec{f},\beta}(s)| \le \sum_{l=1}^{m} \mathfrak{M}^+_\beta(\vec{f}^{\,l}_j)(x), \quad \forall s > 0.$$

This together with the argument similar to those used in deriving (2.8) implies

$$|A_{2,j}| \le C(m, q, \beta, p_1, \dots, p_m, \vec{f})\epsilon, \quad \forall j \ge N_0. \tag{2.9}$$

It follows from (2.4), (2.8) and (2.9) that

$$|\{x \in (-R, R) : \mathfrak{R}^+_\beta(\vec{f}_j)(x) \not\subseteq \mathfrak{R}^+_\beta(\vec{f})(x)_{(\lambda)}\}| \le C(m, q, \beta, p_1, \dots, p_m, \vec{f})\epsilon, \quad \forall j \ge N_0,$$

which gives (2.1) and completes the proof of Lemma 2.1. □

We now define the Hausdorff distance between two sets A and B by

$$\pi(A, B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}.$$

The following result can be obtained by Lemma 2.1 and a similar argument as in the proof of Corollary 2.3 in [7], we omit the details.

Lemma 2.2. *Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in L^{p_i}(\mathbb{R})$ for $1 < p_1, \dots, p_m < \infty$. Let $1 \le q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \beta$. Then, for all $R > 0$ and $\lambda > 0$, we have*

$$\lim_{h\to 0} |\{x \in (-R, R) : \pi(\mathfrak{R}^+_\beta(\vec{f})(x), \mathfrak{R}^+_\beta(\vec{f})(x + h)) > \lambda\}| = 0.$$

The following result presents some formulas for the derivatives of the one-sided multilinear fractional maximal functions, which play the key roles in the proof of the continuity part in Theorem 1.1.

Lemma 2.3. *Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,p_i}(\mathbb{R})$ for $1 < p_i < \infty$. Let $1 \le q < \infty$ and $1/q = \sum_{i=1}^m 1/p_i - \beta$. Then, for almost every $x \in \mathbb{R}$, we have*

$$(\mathfrak{M}^+_\beta(\vec{f}))'(x) = \sum_{l=1}^{m} \frac{1}{s^{m-\beta}} \prod_{\substack{1\le j\le m\\ j\ne l}} \int_x^{x+s} |f_j(y)|dy \int_x^{x+s} |f_l|'(y)dy \text{ for all } 0 < s \in \mathfrak{R}^+_\beta(\vec{f})(x); \tag{2.10}$$

$$(\mathfrak{M}^+_\beta(\vec{f}))'(x) = \begin{cases} \sum\limits_{l=1}^{m} |f_l|'(x) \prod\limits_{\substack{1\le j\le m\\ j\ne l}} |f_j(x)|, & \text{if } \beta = 0 \text{ and } 0 \in \mathfrak{R}^+_\beta(\vec{f})(x), \\ 0, & \text{if } 0 < \beta < m \text{ and } 0 \in \mathfrak{R}^+_\beta(\vec{f})(x). \end{cases} \tag{2.11}$$

Proof. We may assume that all $f_i \ge 0$ since $|f| \in W^{1,p}(\mathbb{R})$ if $f \in W^{1,p}(\mathbb{R})$ with $1 < p < \infty$. By the boundedness part in Theorem 1.1 we see that $\mathfrak{M}^+_\beta(\vec{f}) \in W^{1,q}(\mathbb{R})$. Invoking Lemma 2.2, we can choose a sequence $\{s_k\}_{k=1}^\infty$, $s_k > 0$ such that $\lim_{k\to\infty} s_k = 0$ and $\lim_{k\to\infty} \pi(\mathfrak{R}^+_\beta(\vec{f})(x), \mathfrak{R}^+_\beta(\vec{f})(x + s_k)) = 0$ for almost every $x \in (-R, R)$. For $1 \le i \le m$ and $h \in \mathbb{R}$, we set

$$f^i_h(x) = \frac{f^i_{\tau(h)}(x) - f_i(x)}{h} \text{ and } f^i_{\tau(h)}(x) = f_i(x + h).$$

It was known that

$$\|f^i_{\tau(s_k)} - f_i\|_{L^{p_i}(\mathbb{R})} \to 0 \text{ as } k \to \infty,$$

$$\|f^i_{s_k} - (f_i)'\|_{L^{p_i}(\mathbb{R})} \to 0 \text{ as } k \to \infty,$$

$$\|\mathcal{M}^+(f^i_{\tau(s_k)} - f_i)\|_{L^{p_i}(\mathbb{R})} \to 0 \text{ as } k \to \infty,$$
$$\|\mathcal{M}^+(f^i_{s_k} - f_i^{\prime})\|_{L^{p_i}(\mathbb{R})} \to 0 \text{ as } k \to \infty,$$
$$\|(\mathfrak{M}^+_\beta(\vec{f}))_{s_k} - (\mathfrak{M}^+_\beta(\vec{f}))^{\prime}\|_{L^q(\mathbb{R})} \to 0 \text{ as } k \to \infty.$$

Here $(\mathfrak{M}^+_\beta(\vec{f}))_{s_k}(x) = \frac{1}{s_k}(\mathfrak{M}^+_\beta(\vec{f})(x+s_k) - \mathfrak{M}^+_\beta(\vec{f})(x))$. Furthermore, there exists a subsequence $\{h_k\}_{k=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ and a measurable set $A_1 \subset (-R, R)$ with $|(-R, R)\backslash A_1| = 0$ such that

(i) $f^i_{\tau(h_k)}(x) \to f_i(x), f^i_{h_k}(x) \to f_i^{\prime}(x), \mathcal{M}^+(f^i_{\tau(h_k)} - f_i)(x) \to 0, \mathcal{M}^+(f^i_{h_k} - f_i^{\prime})(x) \to 0$ and $(\mathfrak{M}^+_\beta(\vec{f}))_{h_k}(x) \to (\mathfrak{M}^+_\beta(\vec{f}))^{\prime}(x)$ when $k \to \infty$ for any $x \in A_1$ and $1 \le i \le m$;

(ii) $\lim_{k\to\infty} \pi(\mathfrak{R}^+_\beta(\vec{f})(x), \mathfrak{R}^+_\beta(\vec{f})(x + h_k)) = 0$ for any $x \in A_1$.

Let

$$A_2 := \bigcap_{k=1}^{\infty}\{x \in \mathbb{R} : \mathfrak{M}^+_\beta(\vec{f})(x + h_k) \ge u^+_{x+h_k,\vec{f},\beta}(0)\},$$

$$A_3 := \bigcap_{k=1}^{\infty}\{x \in \mathbb{R} : \mathfrak{M}^+_\beta(\vec{f})(x + h_k) = u^+_{x+h_k,\vec{f},\beta}(0) \text{ if } 0 \in \mathfrak{R}^+_\beta(\vec{f})(x + h_k)\},$$

$$A_4 := \{x \in \mathbb{R} : \mathfrak{M}^+_\beta(\vec{f})(x) = u^+_{x,\vec{f},\beta}(0) \text{ if } 0 \in \mathfrak{R}^+_\beta(\vec{f})(x)\}.$$

It is obvious that $|(-R, R)\backslash A_j| = 0$ for $j = 2, 3, 4$. Let $x \in A_1 \cap A_2 \cap A_3 \cap A_4$ be a Lebesgue point of all f_i and $f_i^{\prime}$. Fix $s \in \mathfrak{R}^+_\beta(\vec{f})(x)$, there exists radii $r_k \in \mathfrak{R}^+_\beta(\vec{f})(x + h_k)$ such that $\lim_{k\to\infty} r_k = s$. We consider the following two cases:

Case A ($s > 0$). Without loss of generality we may assume that all $r_k > 0$. Then

$$\begin{aligned}(\mathfrak{M}^+_\beta(\vec{f}))^{\prime}(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathfrak{M}^+_\beta(\vec{f})(x + h_k) - \mathfrak{M}^+_\beta(\vec{f})(x))\\ &\le \lim_{k\to\infty} \frac{1}{h_k}\frac{1}{r_k^{m-\beta}}\Big(\prod_{i=1}^m \int_{x+h_k}^{x+h_k+r_k} f_i(y)dy - \prod_{i=1}^m \int_x^{x+r_k} f_i(y)dy\Big)\\ &= \sum_{l=1}^m \lim_{k\to\infty} \frac{1}{r_k^{m-\beta}} \prod_{\mu=1}^{l-1} \int_x^{x+r_k} f_\mu(y)dy \prod_{\nu=l+1}^m \int_x^{x+r_k} f^\nu_{\tau(h_k)}(y)dy \int_x^{x+r_k} f^l_{h_k}(y)dy.\end{aligned} \tag{2.12}$$

Since $f^\nu_{\tau(h_k)}\chi_{(x,x+r_k)} \to f_\nu\chi_{(x,x+s)}$ and $f^l_{h_k}\chi_{(x,x+r_k)} \to f_l^{\prime}\chi_{(x,x+s)}$ in $L^1(\mathbb{R})$ as $k \to \infty$. Then (2.12) yields that

$$(\mathfrak{M}^+_\beta(\vec{f}))^{\prime}(x) \le \sum_{l=1}^m \frac{1}{s^{m-\beta}} \prod_{\substack{1\le j\le m\\ j\ne l}} \int_x^{x+s} f_j(y)dy \int_x^{x+s} f_l^{\prime}(y)dy. \tag{2.13}$$

On the other hand,

$$\begin{aligned}(\mathfrak{M}^+_\beta(\vec{f}))^{\prime}(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathfrak{M}^+_\beta(\vec{f})(x + h_k) - \mathfrak{M}^+_\beta(\vec{f})(x))\\ &\ge \lim_{k\to\infty} \frac{1}{h_k}\frac{1}{s^{m-\beta}}\Big(\prod_{i=1}^m \int_{x+h_k}^{x+h_k+s} f_i(y)dy - \prod_{i=1}^m \int_x^{x+s} f_i(y)dy\Big)\\ &= \sum_{l=1}^m \lim_{k\to\infty} \frac{1}{s^{m-\beta}} \prod_{\mu=1}^{l-1} \int_x^{x+s} f_\mu(y)dy \prod_{\nu=l+1}^m \int_x^{x+s} f^\nu_{\tau(h_k)}(y)dy \int_x^{x+s} f^l_{h_k}(y)dy\\ &= \sum_{l=1}^m \frac{1}{s^{m-\beta}} \prod_{\substack{1\le j\le m\\ j\ne l}} \int_x^{x+s} f_j(y)dy \int_x^{x+s} f_l^{\prime}(y)dy.\end{aligned} \tag{2.14}$$

Combining (2.14) with (2.13) yields that (2.10) holds for almost every $x \in (-R, R)$.

Case B ($s = 0$). We shall discuss the following two cases:

(i) When $0 < \beta < m$. Since $\mathfrak{M}^+_\beta(\vec{f})(x) = 0$, then all $f_i(y) \equiv 0$ for almost every $y \in (x, \infty)$. Thus $\mathfrak{M}^+_\beta(\vec{f})(y) \equiv 0$ for $y \geq x$. It follows that

$$(\mathfrak{M}^+_\beta(\vec{f}))'(x) = \lim_{k\to\infty} \frac{1}{h_k}(\mathfrak{M}^+_\beta(\vec{f})(x+h_k) - \mathfrak{M}^+_\beta(\vec{f})(x)) = 0.$$

This yields that (2.11) holds for almost every $x \in (-R, R)$ in this case $0 < \beta < m$.

(ii) When $\beta = 0$. We notice that

$$\begin{aligned}\lim_{k\to\infty} \frac{1}{h_k}\Big(\prod_{i=1}^m f_i(x+h_k) - \prod_{i=1}^m f_i(x)\Big) &= \sum_{l=1}^m \lim_{k\to\infty} f^l_{h_k}(x) \prod_{\mu=1}^{l-1} f_\mu(x) \prod_{\nu=l+1}^m f_\nu(x+h_k) \\ &= \sum_{l=1}^m f_l'(x) \prod_{\substack{1\leq j\leq m\\ j\neq l}} f_j(x).\end{aligned} \tag{2.15}$$

It follows that

$$\begin{aligned}(\mathfrak{M}^+_\beta(\vec{f}))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathfrak{M}^+_\beta(\vec{f})(x+h_k) - \mathfrak{M}^+_\beta(\vec{f})(x)) \\ &\geq \lim_{k\to\infty} \frac{1}{h_k}\Big(\prod_{i=1}^m f_i(x+h_k) - \prod_{i=1}^m f_i(x)\Big) \\ &= \sum_{l=1}^m f_l'(x) \prod_{\substack{1\leq j\leq m\\ j\neq l}} f_j(x).\end{aligned} \tag{2.16}$$

Below we estimate the upper bound of $(\mathfrak{M}^+_\beta(\vec{f}))'(x)$. If there exists $k_0 \in \mathbb{N}\setminus\{0\}$ such that $s_k > 0$ for any $k \geq k_0$, then, by the argument similar to those used in deriving (2.12),

$$\begin{aligned}(\mathfrak{M}^+_\beta(\vec{f}))'(x) \leq \sum_{l=1}^m \lim_{k\to\infty} &\Big(\prod_{\mu=1}^{l-1} \frac{1}{r_k} \int_x^{x+r_k} f_\mu(y)dy\Big)\Big(\prod_{\nu=l+1}^m \frac{1}{r_k}\int_x^{x+r_k} f^\nu_{\tau(h_k)}(y)dy\Big) \\ &\times\Big(\frac{1}{r_k}\int_x^{x+r_k} f^l_{h_k}(y)dy\Big).\end{aligned} \tag{2.17}$$

Since

$$\begin{aligned}&\Big|\lim_{k\to\infty} \frac{1}{r_k}\int_x^{x+r_k} f^\nu_{\tau(h_k)}(y)dy - f_\nu(x)\Big| \\ &\leq \lim_{k\to\infty} \frac{1}{r_k}\int_x^{x+r_k} |f^\nu_{\tau(h_k)}(y) - f_\nu(y)|dy \leq \lim_{k\to\infty} \mathcal{M}^+(f^\nu_{\tau(h_k)} - f_\nu)(x) = 0.\end{aligned}$$

It follows that

$$\lim_{k\to\infty} \frac{1}{r_k}\int_x^{x+r_k} f^\nu_{\tau(h_k)}(y)dy = f_\nu(x). \tag{2.18}$$

Similarly,

$$\lim_{k\to\infty} \frac{1}{r_k}\int_x^{x+r_k} f^l_{h_k}(y)dy = f_l'(x). \tag{2.19}$$

It follows from (2.17)-(2.19) that

$$(\mathfrak{M}^+_\beta(\vec{f}))'(x) \leq \sum_{l=1}^m f_l'(x) \prod_{\substack{1\leq j\leq m\\ j\neq l}} f_j(x). \tag{2.20}$$

If $s_k = 0$ for infinitely many k, then, by (2.15) we have

$$\begin{aligned}(\mathfrak{M}^+_\beta(\vec{f}))'(x) &= \lim_{k\to\infty} \frac{1}{h_k}(\mathfrak{M}^+_\beta(\vec{f})(x+h_k) - \mathfrak{M}^+_\beta(\vec{f})(x)) \\ &= \lim_{k\to\infty} \frac{1}{h_k}\Big(\prod_{i=1}^m f_i(x+h_k) - \prod_{i=1}^m f_i(x)\Big) = \sum_{l=1}^m f_l'(x) \prod_{\substack{1\leq j\leq m\\ j\neq l}} f_j(x).\end{aligned}$$

This together with (2.16) and (2.20) yields that (2.11) holds for almost every $x \in (-R, R)$ in the case $\beta = 0$. Since R was arbitrary, this proves Lemma 2.3. □

3 Proofs of Theorems 1.1-1.2

In this section we shall prove Theorems 1.1-1.2. Let us begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. We only prove Theorem 1.1 for $\mathfrak{M}_\beta^+$ and the other case is analogous. Let $\{s_k\}_{k\ge1}$ be an enumeration of positive rational numbers. Then we can write

$$\mathfrak{M}_\beta^+(\vec{f})(x) = \sup_{k\ge1} \frac{1}{s_k^{m-\beta}} \prod_{i=1}^m \int_x^{x+s_k} |f_i(y)|dy.$$

Define the family of operators $\{T_k\}_{k\ge1}$ by

$$T_k(\vec{f})(x) = \max_{1\le i\le k} \frac{1}{s_i^{m-\beta}} \prod_{j=1}^m \int_x^{x+s_i} |f_j(y)|dy.$$

Fix $x,\ h \in \mathbb{R}$, one has

$$\begin{aligned}
&|T_k(\vec{f})(x+h) - T_k(\vec{f})(x)|\\
&\le \max_{1\le i\le k} \frac{1}{s_i^{m-\beta}} \Big| \prod_{j=1}^m \int_{x+h}^{x+h+s_i} |f_j(y)|dy - \prod_{j=1}^m \int_x^{x+s_i} |f_j(y)|dy \Big|\\
&\le \sum_{l=1}^m \max_{1\le i\le k} \frac{1}{s_i^{m-\beta}} \prod_{\mu=1}^{l-1} \int_x^{x+s_i} |f_\mu(y)|dy \prod_{\nu=l+1}^m \int_x^{x+s_i} |f_{\tau(h)}^\nu(y)|dy \int_x^{x+s_i} |f_{\tau(h)}^l(y) - f_l(y)|dy.
\end{aligned}$$

It follows that

$$(T_k(\vec{f}))'(x) \le \sum_{l=1}^m \mathfrak{M}_\beta^+(\vec{f}^l)(x) \tag{3.1}$$

for almost every $x \in \mathbb{R}$, where $\vec{f}^l = (f_1, \dots, f_{l-1}, f_l', f_{l+1}, \dots, f_m)$. Here we used the fact that $||f|'(x)| = |f'(x)|$ for almost every $x \in \mathbb{R}$. By (3.1), (1.4) and Minkowski's inequality, we obtain

$$\begin{aligned}
\|T_k(\vec{f})\|_{1,q} &\le \|T_k(\vec{f})\|_{L^q(\mathbb{R})} + \|(T_k(\vec{f}))'\|_{L^q(\mathbb{R})}\\
&\le \|\mathfrak{M}_\beta^+(\vec{f})\|_{L^q(\mathbb{R})} + \Big\| \sum_{l=1}^m \mathfrak{M}_\beta^+(\vec{f}^l) \Big\|_{L^q(\mathbb{R})}\\
&\le C(\beta, p_1, \dots, p_m) \Big(\prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R})} + \sum_{l=1}^m \|f_l'\|_{L^{p_l}(\mathbb{R})} \prod_{\substack{1\le j\le m\\ j\ne l}} \|f_j\|_{L^{p_j}(\mathbb{R})} \Big)\\
&\le C(m, \beta, p_1, \dots, p_m) \prod_{i=1}^m \|f_i\|_{1,p_i}.
\end{aligned}$$

Therefore, $\{T_k(\vec{f})\}$ is a bounded sequence in $W^{1,q}(\mathbb{R})$ which converges to $\mathfrak{M}_\beta^+(\vec{f})$ pointwise. The weak compactness of Sobolev spaces implies that $\mathfrak{M}_\beta^+(\vec{f}) \in W^{1,q}(\mathbb{R})$, $T_k(\vec{f})$ converges to $\mathfrak{M}_\beta^+(\vec{f})$ weakly in $L^q(\mathbb{R})$ and $(T_k(\vec{f}))'$ converges to $(\mathfrak{M}_\beta^+(\vec{f}))'$ weakly in $L^q(\mathbb{R})$. This together with (3.1) yields that

$$|(\mathfrak{M}_\beta^+(\vec{f}))'(x)| \le \sum_{l=1}^m \mathfrak{M}_\beta^+(\vec{f}^l)(x) \tag{3.2}$$

for almost every $x \in \mathbb{R}$. It follows from (3.2) and (1.4) that

$$\|\mathfrak{M}_\beta^+(\vec{f})\|_{1,q} = \|\mathfrak{M}_\beta^+(\vec{f})\|_{L^q(\mathbb{R})} + \|(\mathfrak{M}_\beta^+(\vec{f}))'\|_{L^q(\mathbb{R})} \le C(m, \beta, p_1, \dots, p_m) \prod_{i=1}^m \|f_i\|_{1,p_i}.$$

This completes the boundedness part of Theorem 1.1.

We now prove the continuity for $\mathfrak{M}_\beta^+$ by employing the idea in [20]. Let $\beta, m, p_1, \dots, p_m, q$ be given as in Theorem 1.1. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in W^{1,p_i}(\mathbb{R})$ and $\vec{f}_j = (f_{1,j}, \dots, f_{m,j})$ such that $f_{i,j} \to f_i$ in $W^{1,p_i}(\mathbb{R})$ when $j \to \infty$. We get from (2.5) that

$$|\mathfrak{M}_\beta^+(\vec{f}_j)(x) - \mathfrak{M}_\beta^+(\vec{f})(x)| \le \sum_{l=1}^{m} \mathfrak{M}_\beta^+(\vec{f}_j^{\,l})(x) \tag{3.3}$$

for any $x \in \mathbb{R}$, where $\vec{f}_j^{\,l}$ is given as in (2.5). (3.3) together with (1.4) implies that

$$\begin{aligned}&\|\mathfrak{M}_\beta^+(\vec{f}_j) - \mathfrak{M}_\beta^+(\vec{f})\|_{L^q(\mathbb{R})}\\ &\le \sum_{l=1}^{m} \|\mathfrak{M}_\beta^+(\vec{f}_j^{\,l})\|_{L^q(\mathbb{R})} \le C(m, \beta, p_1, \dots, p_m) \sum_{l=1}^{m} \|f_{l,j} - f_l\|_{L^{p_l}(\mathbb{R})} \prod_{\mu=1}^{l-1} \|f_\mu\|_{L^{p_\mu}(\mathbb{R})} \prod_{\nu=l+1}^{m} \|f_{\nu,j}\|_{L^{p_\nu}(\mathbb{R})}.\end{aligned}$$

It follows that

$$\|\mathfrak{M}_\beta^+(\vec{f}_j) - \mathfrak{M}_\beta^+(\vec{f})\|_{L^q(\mathbb{R})} \to 0 \ \text{ when } \ j \to \infty.$$

Hence, to prove the continuity for $\mathfrak{M}_\beta^+$, it suffices to show that

$$\|(\mathfrak{M}_\beta^+(\vec{f}_j))' - (\mathfrak{M}_\beta^+(\vec{f}))'\|_{L^q(\mathbb{R})} \to 0 \ \text{ when } \ j \to \infty. \tag{3.4}$$

Below we prove (3.4). We may assume that all $f_{i,j} \ge 0$ and $f_i \ge 0$. For $1 \le l \le m$, we set $\vec{f}^{\,l} = (f_1, \dots, f_{l-1}, f_l', f_{l+1}, \dots, f_m)$. Fix $\epsilon \in (0,1)$. We can choose $R > 0$ such that $\sum_{l=1}^m \|\mathfrak{M}_\beta^+(\vec{f}^{\,l})\|_{q,B_1} < \epsilon$ with $B_1 = (-\infty, -R) \cup (R, \infty)$. The absolute continuity implies that there exists $\eta > 0$ such that $\sum_{l=1}^m \|\mathfrak{M}_\beta^+(\vec{f}^{\,l})\|_{q,B} < \epsilon$ for any measurable subset B of $(-R, R)$ with $|B| < \eta$. As already observed, for almost every $x \in \mathbb{R}$, the function $u^+_{x,\vec{f}^{\,l},\beta}$ is uniformly continuous on $[0, \infty)$. Therefore, for almost every $x \in \mathbb{R}$, the function $\sum_{l=1}^m u^+_{x,\vec{f}^{\,l},\beta}$ is uniformly continuous on $[0, \infty)$. We can find $\delta(x) > 0$ such that

$$\Big|\sum_{l=1}^{m} u^+_{x,\vec{f}^{\,l},\beta}(s_1) - \sum_{l=1}^{m} u^+_{x,\vec{f}^{\,l},\beta}(s_2)\Big| < R^{-1/q}\epsilon \ \text{ if } |s_1 - s_2| \le \delta(x).$$

We can write $(-R, R)$ as

$$(-R, R) = \Big(\bigcup_{k=1}^{\infty} \Big\{x \in (-R, R) : \delta(x) > \frac{1}{k}\Big\}\Big) \cup \mathcal{N},$$

where $|\mathcal{N}| = 0$. We can choose $\delta > 0$ such that

$$\begin{aligned}&\Big|\Big\{x \in (-R, R) : \Big|\sum_{l=1}^{m} u^+_{x,\vec{f}^{\,l},\beta}(s_1) - \sum_{l=1}^{m} u^+_{x,\vec{f}^{\,l},\beta}(s_2)\Big| \ge R^{-1/q}\epsilon \text{ for some } s_1, s_2 \text{ with } |s_1 - s_2| \le \delta\Big\}\Big|\\ &=: |B_2| < \tfrac{\eta}{2}.\end{aligned}$$

By Lemma 2.1, there exists $N_1 \in \mathbb{N} \setminus \{0\}$ such that

$$|\{x \in (-R, R) : \mathfrak{R}_\beta^+(\vec{f}_j)(x) \not\subseteq \mathfrak{R}_\beta^+(\vec{f})(x)_{(\delta)}\}| =: |B^j| < \frac{\eta}{2} \quad \forall j \ge N_1.$$

Fix $j \ge N_1$. Let $\vec{f}_{l,j} = (f_{1,j}, \dots, f_{l-1,j}, f_{l,j}', f_{l+1,j}, \dots, f_{m,j})$ and $s \in \mathfrak{R}_\beta^+(\vec{f}_j)(x)$. We consider the following two cases:

(i) $s > 0$. We can write

$$\begin{aligned}&|u^+_{x,\vec{f}_{l,j},\beta}(s) - u^+_{x,\vec{f}^{\,l},\beta}(s)|\\ &= \frac{1}{s^{m-\beta}}\Big|\prod_{\substack{1\le\mu\le m\\ \mu\ne l}} \int_x^{x+s} f_{\mu,j}(y)dy \int_x^{x+s} f_{l,j}'(y)dy - \prod_{\substack{1\le\mu\le m\\ \mu\ne l}} \int_x^{x+s} f_\mu(y)dy \int_x^{x+s} f_l'(y)dy\Big|\\ &\le \sum_{\mu=1}^{l-1} \mathfrak{M}_\beta^+(\vec{F}_{\mu,j})(x) + \sum_{\nu=l+1}^{m} \mathfrak{M}_\beta^+(\vec{G}_{\nu,j})(x) + \mathfrak{M}_\beta^+(\vec{H}_{l,j})(x) =: \mathfrak{G}_{l,j}(x),\end{aligned} \tag{3.5}$$

where

$$\vec{F}_{\mu,j} = (f_1, \ldots, f_{\mu-1}, f_{\mu,j} - f_\mu, f_{\mu+1,j}, \ldots, f_{l-1,j}, f'_{l,j}, f_{l+1,j}, \ldots, f_{m,j}),$$
$$\vec{G}_{\nu,j} = (f_1, \ldots, f_{l-1}, f'_l, f_{l+1}, \ldots, f_{\nu-1}, f_{\nu,j} - f_\nu, f_{\nu+1,j}, \ldots, f_{m,j}),$$
$$\vec{H}_{l,j} = (f_1, \ldots, f_{l-1}, f'_{l,j} - f'_l, f_{l+1,j}, \ldots, f_{m,j}).$$

(ii) $s = 0$. If $0 < \beta < m$, then $|u_{x,\vec{f}_{l,j},\beta}(s) - u_{x,\vec{f}^l,\beta}(s)| = 0$. If $\beta = 0$, then we have

$$\begin{aligned}
&|u^+_{x,\vec{f}_{l,j},\beta}(s) - u^+_{x,\vec{f}^l,\beta}(s)| \\
&\leq \sum_{\mu=1}^{l-1} \Big(\prod_{l_1=1}^{\mu-1} f_{l_1}(x)\Big)(f_{\mu,j}(x) - f_\mu(x))\Big(\prod_{l_2=\mu+1}^{l-1} f_{l_2,j}(x)\Big)|f'_{l,j}(x)|\Big(\prod_{l_3=l+1}^{m} f_{l_3,j}(x)\Big) \\
&+ \sum_{\nu=l+1}^{m} \Big(\prod_{l_1=1}^{l-1} f_{l_1}(x)\Big)|f'_l(x)|\Big(\prod_{l_2=l+1}^{\nu-1} f_{l_2}(x)\Big)|f_{\nu,j}(x) - f_\nu(x)|\Big(\prod_{l_3=\nu+1}^{m} f_{l_3,j}(x)\Big) \\
&+ \Big(\prod_{l_1=1}^{l-1} f_{l_1}(x)\Big)|f'_{l,j}(x) - f'_l(x)|\Big(\prod_{l_2=l+1}^{m} f_{l_2,j}(x)\Big).
\end{aligned}$$

This together with (3.5) and the Lebesgue differentiation theorem leads to

$$|u^+_{x,\vec{f}_{l,j},\beta}(s) - u^+_{x,\vec{f}^l,\beta}(s)| \leq \mathfrak{G}_{l,j}(x) \tag{3.6}$$

for almost every $x \in \mathbb{R}$ and $s \in \mathfrak{R}^+_\beta(\vec{f}_j)(x)$. By (3.6) and Lemma 2.3, we obtain

$$\begin{aligned}
&|(\mathfrak{M}^+_\beta(\vec{f}_j))'(x) - (\mathfrak{M}^+_\beta(\vec{f}))'(x)| \\
&= \Big|\sum_{l=1}^m u_{x,\vec{f}_{l,j},\beta}(s_1) - \sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_2)\Big| \\
&\leq \Big|\sum_{l=1}^m u_{x,\vec{f}_{l,j},\beta}(s_1) - \sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_1)\Big| + \Big|\sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_1) - \sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_2)\Big| \\
&\leq \sum_{l=1}^m \mathfrak{G}_{l,j}(x) + \Big|\sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_1) - \sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_2)\Big|
\end{aligned} \tag{3.7}$$

for almost every $x \in \mathbb{R}$ and any $s_1 \in \mathfrak{R}^+_\beta(\vec{f}_j)(x)$ and $s_2 \in \mathfrak{R}^+_\beta(\vec{f})(x)$. On can easily check that

$$\lim_{j\to\infty} \|\mathfrak{G}_{l,j}\|_{L^q(\mathbb{R})} = 0, \quad \forall 1 \leq l \leq m.$$

It follows that there exists $N_2 \in \mathbb{N} \setminus \{0\}$ such that $\sum_{l=1}^m \|\mathfrak{G}_{l,j}\|_{L^q(\mathbb{R})} < \epsilon$ for any $j \geq N_2$.

If $x \notin B_1 \cup B_2 \cup B^j$, we can choose $s_1 \in \mathfrak{R}^+_\beta(\vec{f}_j)(x)$ and $s_2 \in \mathfrak{R}^+_\beta(\vec{f})(x)$ such that $|s_1 - s_2| \leq \delta$ and

$$\Big|\sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_1) - \sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_2)\Big| < |R|^{-1/q}\epsilon.$$

On the other hand, we have that for any $l = 1, 2, \ldots, m$, $s_1 \in \mathfrak{R}^+_\beta(\vec{f}_j)(x)$ and $s_2 \in \mathfrak{R}^+_\beta(\vec{f})(x)$,

$$\Big|\sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_1) - \sum_{l=1}^m u_{x,\vec{f}^l,\beta}(s_2)\Big| \leq 2\sum_{l=1}^m \mathfrak{M}^+_\beta(\vec{f}^l)(x).$$

Note that $|B_2 \cup B^j| < \eta$ for any $j \geq N_1$. Thus we get from (3.7) that

$$\begin{aligned}
&\|(\mathfrak{M}^+_\beta(\vec{f}_j))' - (\mathfrak{M}^+_\beta(\vec{f}))'\|_{L^q(\mathbb{R})} \\
&\leq \Big\|\sum_{l=1}^m \mathfrak{G}_{l,j}\Big\|_{L^q(\mathbb{R})} + \||R|^{-1/q}\epsilon\|_{q,(-R,R)} + 2\Big\|\sum_{l=1}^m \mathfrak{M}^+_\beta(\vec{f}^l)\Big\|_{q,B_1\cup B_2\cup B^j} \leq C\epsilon,
\end{aligned}$$

for any $j \geq \max\{N_1, N_2\}$, which leads to

$$\lim_{j\to\infty} \|(\mathfrak{M}^+_\beta(\vec{f}_j))' - (\mathfrak{M}^+_\beta(\vec{f}))'\|_{L^q(\mathbb{R})} = 0.$$

This yields (3.4) and completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. The proof is similar to the proof of Theorem 2.3 in [5]. We omit the details. □

4 Proof of Theorem 1.3

We only prove Theorem 1.3 for M_β^+ and the other case is analogous.

Step 1: proof of the boundedness for M_β^+. We shall adopt the method in [31] to prove the boundedness for M_β^+. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_i \in \ell^1(\mathbb{Z})$. Without loss of generality, we may assume $f_i \geq 0$. For convenience, let $\Gamma(x) = (x+1)^{\beta-m} - (x+2)^{\beta-m}$ for any $x \geq 0$. One can easily check that $\Gamma(x)$ is decreasing on $[0, \infty)$ and $\sum_{n\in\mathbb{N}} \Gamma(n) = 1$. Since all $f_i \in \ell^1(\mathbb{Z})$, then, for any $n \in \mathbb{Z}$, there exists $s_n \in \mathbb{N}$ such that $\mathrm{M}_\beta^+(\vec{f})(n) = \mathrm{A}_{s_n}(\vec{f})(n)$, where

$$\mathrm{A}_s(\vec{f})(n) = (s+1)^{\beta-m} \prod_{i=1}^m \sum_{k=0}^s f_i(n+k)$$

for any $s \in \mathbb{N}$ and $n \in \mathbb{Z}$. Let

$$X^+ = \{n \in \mathbb{Z} : \mathrm{M}_\beta^+(\vec{f})(n+1) > \mathrm{M}_\beta^+(\vec{f})(n)\} \quad \text{and} \quad X^- = \{n \in \mathbb{Z} : \mathrm{M}_\beta^+(\vec{f})(n) \geq \mathrm{M}_\beta^+(\vec{f})(n+1)\}.$$

Then we can write

$$\begin{aligned} &\operatorname{Var}(\mathrm{M}_\beta^+(\vec{f})) \\ &= \sum_{n\in X^+} (\mathrm{M}_\beta^+(\vec{f})(n+1) - \mathrm{M}_\beta^+(\vec{f})(n)) + \sum_{n\in X^-} (\mathrm{M}_\beta^+(\vec{f})(n) - \mathrm{M}_\beta^+(\vec{f})(n+1)) \\ &\leq \sum_{n\in X^+} (\mathrm{A}_{s_{n+1}}(\vec{f})(n+1) - \mathrm{A}_{s_{n+1}+1}(\vec{f})(n)) + \sum_{n\in X^-} (\mathrm{A}_{s_n}(\vec{f})(n) - \mathrm{A}_{s_n+1}(\vec{f})(n+1)). \end{aligned} \tag{4.1}$$

Fix $n \in \mathbb{Z}$, by direct computations we obtain

$$\begin{aligned} &\mathrm{A}_{s_{n+1}}(\vec{f})(n+1) - \mathrm{A}_{s_{n+1}+1}(\vec{f})(n) \\ &= (s_{n+1}+1)^{\beta-m} \prod_{i=1}^m \sum_{k=0}^{s_{n+1}} f_i(n+1+k) - (s_{n+1}+2)^{\beta-m} \prod_{i=1}^m \sum_{k=0}^{s_{n+1}+1} f_i(n+k) \\ &\leq \sum_{l=1}^m \Big((s_{n+1}+1)^{\beta-m} \sum_{k=0}^{s_{n+1}} f_l(n+1+k) - (s_{n+1}+2)^{\beta-m} \sum_{k=0}^{s_{n+1}+1} f_l(n+k) \Big) \\ &\quad \times \prod_{\mu=1}^{l-1} \sum_{k=0}^{s_{n+1}+1} f_\mu(n+k) \prod_{\nu=l}^m \sum_{k=0}^{s_{n+1}} f_\nu(n+1+k). \end{aligned} \tag{4.2}$$

Since

$$\begin{aligned} &(s_{n+1}+1)^{\beta-m} \sum_{k=0}^{s_{n+1}} f_l(n+1+k) - (s_{n+1}+2)^{\beta-m} \sum_{k=0}^{s_{n+1}+1} f_l(n+k) \\ &\leq (s_{n+1}+1)^{\beta-m} \sum_{k\in\mathbb{Z}} f_l(k)\chi_{[n+1, n+s_{n+1}+1]}(k) - (s_{n+1}+2)^{\beta-m} \sum_{k\in\mathbb{Z}} f_l(k)\chi_{[n, n+s_{n+1}+1]}(k) \\ &\leq \sum_{k\in\mathbb{Z}} f_l(k)\Gamma(s_{n+1})\chi_{[n+1, n+s_{n+1}+1]}(k) \\ &\leq \sum_{k\in\mathbb{Z}} f_l(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k). \end{aligned} \tag{4.3}$$

Combining (4.3) with (4.2) yields that

$$\mathrm{A}_{s_{n+1}}(\vec{f})(n+1) - \mathrm{A}_{s_{n+1}+1}(\vec{f})(n) \leq \sum_{l=1}^m \prod_{\substack{1\leq j\leq m \\ j\neq l}} \|f_j\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_l(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k) \Big). \tag{4.4}$$

On the other hand, one finds

$$\begin{aligned} \mathrm{A}_{s_n}(\vec{f})(n) - \mathrm{A}_{s_n+1}(\vec{f})(n+1) &= (s_n+1)^{\beta-m} \prod_{i=1}^m \sum_{k=0}^{s_n} f_i(n+k) - (s_n+2)^{\beta-m} \prod_{i=1}^m \sum_{k=0}^{s_n+1} f_i(n+1+k) \\ &= \sum_{l=1}^m \Big((s_n+1)^{\beta-m} \sum_{k=0}^{s_n} f_l(n+k) - (s_n+2)^{\beta-m} \sum_{k=0}^{s_n+1} f_l(n+1+k) \Big) \\ &\quad \times \prod_{\mu=1}^{l-1} \sum_{k=0}^{s_n+1} f_\mu(n+1+k) \prod_{\nu=l+1}^{m} \sum_{k=0}^{s_n} f_\nu(n+k). \end{aligned}$$

It follows that

$$
\begin{aligned}
&\mathrm{A}_{s_n}(\vec{f})(n) - \mathrm{A}_{s_n+1}(\vec{f})(n+1) \\
&\le \sum_{l=1}^{m} \Big((s_n+1)^{\beta-m} \sum_{k\in\mathbb{Z}} f_l(k)\chi_{[n,n+s_n]}(k) - (s_n+2)^{\beta-m} \sum_{k\in\mathbb{Z}} f_l(k)\chi_{[n+1,n+s_n+2]}(k) \Big) \\
&\quad \times \prod_{\substack{1\le j\le m\\ j=l}} \|f_j\|_{\ell^1(\mathbb{Z})} \\
&\le \sum_{l=1}^{m} \prod_{\substack{1\le j\le m\\ j=l}} \|f_j\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_l(k)\Gamma(s_n)\chi_{[n+1,n+s_n+1]}(k) + f_l(n) \Big) \\
&\le \sum_{l=1}^{m} \prod_{\substack{1\le j\le m\\ j=l}} \|f_j\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_l(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k) + f_l(n) \Big).
\end{aligned}
\tag{4.5}
$$

(4.1) and (4.4)-(4.5) imply that

$$
\begin{aligned}
\mathrm{Var}(\mathrm{M}_\beta^+(\vec{f})) &\le \sum_{l=1}^{m} \prod_{\substack{1\le j\le m\\ j=l}} \|f_j\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_l(k) \Big(\sum_{\substack{n\in X^+\\ n<k}} \Gamma(k-n-1) \\
&\quad + \sum_{\substack{n\in X^-\\ n<k}} \Gamma(k-n-1) \Big) + \sum_{n\in X^-} f_l(n) \Big) \\
&\le \sum_{l=1}^{m} \prod_{\substack{1\le j\le m\\ j=l}} \|f_j\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_l(k) \sum_{n<k} \Gamma(k-n-1) + \|f_l\|_{\ell^1(\mathbb{Z})} \Big) \\
&\le 2m \prod_{1\le j\le m} \|f_j\|_{\ell^1(\mathbb{Z})}.
\end{aligned}
$$

Step 2: proof of the continuity for M_β^+. Let $\vec{f} = (f_1, \dots, f_m)$ with each $f_j \in \ell^1(\mathbb{Z})$ and $\vec{f_j} = (f_{1,j}, \dots, f_{m,j})$ such that $f_{i,j} \to f_i$ in $\ell^1(\mathbb{Z})$ as $j \to \infty$. By the boundedness part in Theorem 1.3, we know that $(\mathrm{M}_\beta^+(\vec{f}))' \in \ell^1(\mathbb{Z})$. Without loss of generality we may assume that all $f_{i,j} \ge 0$ and $f_i \ge 0$ since $\big||f_j| - |f|\big| \le |f_j - f|$. We want to show that

$$\lim_{j\to\infty} \|(\mathrm{M}_\beta^+(\vec{f_j}))' - (\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})} = 0. \tag{4.6}$$

Given $\epsilon \in (0,1)$, there exists $N_1 = N_1(\epsilon, \vec{f}) > 0$ such that

$$\|f_{i,j} - f_i\|_{\ell^1(\mathbb{Z})} < \epsilon \tag{4.7}$$

and

$$\|f_{i,j}\|_{\ell^1(\mathbb{Z})} \le \|f_{i,j} - f_i\|_{\ell^1(\mathbb{Z})} + \|f_i\|_{\ell^1(\mathbb{Z})} < \|f_i\|_{\ell^1(\mathbb{Z})} + 1 \tag{4.8}$$

for any $j \ge N_1$ and all $1 \le i \le m$. We get from (4.7)-(4.8) that

$$
\begin{aligned}
&|\mathrm{M}_\beta^+(\vec{f_j})(n) - \mathrm{M}_\beta^+(\vec{f})(n)| \\
&\le \sup_{s\in\mathbb{N}} (s+1)^{\beta-m} \Big| \prod_{i=1}^{m} \sum_{k=0}^{s} f_{i,j}(n+k) - \prod_{i=1}^{m} \sum_{k=0}^{s} f_i(n+k) \Big| \\
&\le \sup_{s\in\mathbb{N}} (s+1)^{\beta-m} \sum_{l=1}^{m} \sum_{k=0}^{s} |f_{i,j}(n+k) - f_i(n+k)| \prod_{\mu=1}^{l-1} \sum_{k=0}^{s} f_\mu(n+k) \prod_{\nu=l+1}^{m} \sum_{k=0}^{s} f_{\nu,j}(n+k) \\
&\le \sum_{l=1}^{m} \|f_{l,j} - f_l\|_{\ell^1(\mathbb{Z})} \prod_{\mu=1}^{l-1} \|f_\mu\|_{\ell^1(\mathbb{Z})} \prod_{\nu=l+1}^{m} \|f_{\nu,j}\|_{\ell^1(\mathbb{Z})} \\
&\le c(\vec{f})\epsilon
\end{aligned}
$$

for any $n \in \mathbb{Z}$ and $j \ge N_1$, which implies that $\mathrm{M}_\beta^+(\vec{f_j}) \to \mathrm{M}_\beta^+(\vec{f})$ pointwise as $j \to \infty$ and

$$\lim_{j\to\infty} (\mathrm{M}_\beta^+(\vec{f_j}))'(n) = (\mathrm{M}_\beta^+(\vec{f}))'(n) \tag{4.9}$$

for all $n \in \mathbb{Z}$. By the fact that $(\mathrm{M}_\beta^+(\vec{f}))' \in \ell^1(\mathbb{Z})$ and the classical Brezis-Lieb lemma in [35], to prove (4.6), it suffices to show that

$$\lim_{j\to\infty} \|(\mathrm{M}_\beta^+(\vec{f}_j))'\|_{\ell^1(\mathbb{Z})} = \|(\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})}. \tag{4.10}$$

By (4.9) and Fatou's lemma, one finds

$$\|(\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})} \le \liminf_{j\to\infty} \|(\mathrm{M}_\beta^+(\vec{f}_j))'\|_{\ell^1(\mathbb{Z})}.$$

Thus, to prove (4.10), it suffices to show that

$$\limsup_{j\to\infty} \|(\mathrm{M}_\beta^+(\vec{f}_j))'\|_{\ell^1(\mathbb{Z})} \le \|(\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})}. \tag{4.11}$$

We now prove (4.11). Since each $f_i \in \ell^1(\mathbb{Z})$, then there exists a sufficiently large positive integer $R_1 = R_1(\epsilon, \vec{f})$ such that

$$\sup_{1\le i\le m} \sum_{|n|\ge R_1} f_i(n) < \epsilon. \tag{4.12}$$

Note that

$$\lim_{|n|\to\infty} \mathrm{M}_\beta^+(\vec{f})(n) = 0.$$

It follows that there exists an integer $R_2 = R_2(\epsilon) > 0$ such that $\mathrm{M}_\beta^+(\vec{f})(n) < \epsilon$ for all $|n| \ge R_2$. Moreover, there exists an integer $R_3 > 0$ such that $s^{\beta-m} < \epsilon$ if $s \ge R_3$ since $\beta < m$. Let $R = \max\{R_1, R_2, R_3\}$. (4.9) yields that there exists an integer $N_2 = N(\epsilon, R) > 0$ such that

$$|(\mathrm{M}_\beta^+(\vec{f}_j))'(n) - (\mathrm{M}_\beta^+(\vec{f}))'(n)| \le \frac{\epsilon}{4R+2} \tag{4.13}$$

for any $j \ge N_2$ and $|n| \le 2R$. From (4.13) we have

$$\begin{aligned} &\|(\mathrm{M}_\beta^+(\vec{f}_j))'\|_{\ell^1(\mathbb{Z})} \\ &\le \sum_{|n|\le 2R} |(\mathrm{M}_\beta^+(\vec{f}_j))'(n) - (\mathrm{M}_\beta^+(\vec{f}))'(n)| + \|(\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})} + \sum_{|n|\ge 2R} |(\mathrm{M}_\beta^+(\vec{f}_j))'(n)| \\ &\le \|(\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})} + \epsilon + \sum_{|n|\ge 2R} |(\mathrm{M}_\beta^+(\vec{f}_j))'(n)| \end{aligned} \tag{4.14}$$

for any $j \ge N_2$. Fix $j \ge N_2$ and set

$$X_j^+ = \{|n| \ge 2R : \mathrm{M}_\beta^+(\vec{f}_j)(n+1) > \mathrm{M}_\beta^+(\vec{f}_j)(n)\},$$

$$X_j^- = \{|n| \ge 2R : \mathrm{M}_\beta^+(\vec{f}_j)(n) \ge \mathrm{M}_\beta^+(\vec{f}_j)(n+1)\}.$$

Since all $f_{i,j} \in \ell^1(\mathbb{Z})$, then, for any $n \in \mathbb{Z}$, there exists $r_n \in \mathbb{N}$ such that $\mathrm{M}_\beta^+(\vec{f}_j)(n) = \mathrm{A}_{r_n}(\vec{f}_j)(n)$. Then we have

$$\begin{aligned} &\sum_{|n|\ge 2R} |(\mathrm{M}_\beta^+(\vec{f}_j)'(n)| \\ &= \sum_{n\in X_j^+} (\mathrm{M}_\beta^+(\vec{f}_j)(n+1) - \mathrm{M}_\beta^+(\vec{f}_j)(n)) + \sum_{n\in X_j^-} (\mathrm{M}_\beta^+(\vec{f}_j)(n) - \mathrm{M}_\beta^+(\vec{f}_j)(n+1)) \\ &\le \sum_{n\in X_j^+} (\mathrm{A}_{r_{n+1}}(\vec{f}_j)(n+1) - \mathrm{A}_{r_{n+1}+1}(\vec{f}_j)(n)) + \sum_{n\in X_j^-} (\mathrm{A}_{r_n}(\vec{f}_j)(n) - \mathrm{A}_{r_n+1}(\vec{f}_j)(n+1)). \end{aligned} \tag{4.15}$$

By the arguments similar to those used in deriving (4.4) and (4.5), one has

$$\mathrm{A}_{r_{n+1}}(\vec{f}_j)(n+1) - \mathrm{A}_{r_{n+1}+1}(\vec{f}_j)(n) \le \sum_{l=1}^{m} \prod_{\substack{1\le\mu\le m\\ \mu\ne l}} \|f_{\mu,j}\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_{l,j}(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k) \Big), \tag{4.16}$$

$$\mathrm{A}_{r_n}(\vec{f}_j)(n) - \mathrm{A}_{r_n+1}(\vec{f}_j)(n+1) \le \sum_{l=1}^{m} \prod_{\substack{1\le\mu\le m\\ \mu\ne l}} \|f_{\mu,j}\|_{\ell^1(\mathbb{Z})} \Big(\sum_{k\in\mathbb{Z}} f_{l,j}(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k) + f_{l,j}(n) \Big). \tag{4.17}$$

It follows from (4.13)-(4.15) that

$$\begin{aligned}
&\sum_{|n|\geq 2R} |(\mathrm{M}_\beta^+(\vec{f}_j)'(n)| \\
&\leq \sum_{l=1}^{m} \prod_{\substack{1\leq\mu\leq m\\ \mu\neq l}} \|f_{\mu,j}\|_{\ell^1(\mathbb{Z})}\Big(\sum_{n\in X_j^+}\sum_{k\in\mathbb{Z}} f_{l,j}(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k)\Big) \\
&\quad+\sum_{l=1}^{m} \prod_{\substack{1\leq\mu\leq m\\ \mu\neq l}} \|f_{\mu,j}\|_{\ell^1(\mathbb{Z})}\Big(\sum_{n\in X_j^-}\sum_{k\in\mathbb{Z}} f_{l,j}(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k)+\sum_{n\in X_j^-} f_{l,j}(n)\Big) \\
&\leq \sum_{l=1}^{m} \prod_{\substack{1\leq\mu\leq m\\ \mu\neq l}} \|f_{\mu,j}\|_{\ell^1(\mathbb{Z})}\Big(\sum_{|n|\geq 2R}\sum_{k\in\mathbb{Z}} f_{l,j}(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k)+\sum_{|n|\geq 2R} f_{l,j}(n)\Big).
\end{aligned} \tag{4.18}$$

By (4.7)-(4.8) and (4.12), we obtain

$$\begin{aligned}
&\sum_{|n|\geq 2R}\sum_{k\in\mathbb{Z}} f_{l,j}(k)\Gamma(k-n-1)\chi_{(n,\infty)}(k) \\
&\leq \sum_{k\in\mathbb{Z}} f_{l,j}(k)\sum_{\substack{|n|\geq 2R\\ n<k}}\Gamma(k-n-1) \\
&\leq \sum_{|k|\geq R} f_{l,j}(k)\sum_{\substack{|n|\geq 2R\\ n<k}}\Gamma(k-n-1)+\sum_{|k|<R} f_{l,j}(k)\sum_{n\leq -2R}\Gamma(k-n-1) \\
&\leq \sum_{|k|\geq R} f_{l,j}(k)+\sum_{|k|<R} f_{l,j}(k)\sum_{n=2R}^{\infty}\Gamma(n-R) \\
&\leq \|f_{l,j}-f_l\|_{\ell^1(\mathbb{Z})}+\|f_l\chi_{|n|\geq 2R}\|_{\ell^1(\mathbb{Z})}+R^{\beta-m}\|f_{l,j}\|_{\ell^1(\mathbb{Z})}\leq C(f_l)\epsilon
\end{aligned} \tag{4.19}$$

for any $j\geq N_1$. It follows from (4.8), (4.12) and (4.18)-(4.19) that

$$\sum_{|n|\geq 2R} |(\mathrm{M}_\beta^+(\vec{f}_j))'(n)| \leq C(\vec{f})\epsilon \tag{4.20}$$

for any $j\geq N_1$. Combining (4.20) with (4.14) yields that

$$\|(\mathrm{M}_\beta^+(\vec{f}_j))'\|_{\ell^1(\mathbb{Z})} \leq \|(\mathrm{M}_\beta^+(\vec{f}))'\|_{\ell^1(\mathbb{Z})}+C\epsilon$$

for any $j\geq \max\{N_1,N_2\}$. This proves (4.11) and finishes the proof of Theorem 1.3. □

Acknowledgement: This work was supported partly by the National Natural Science Foundation of China (Grant No. 11701333) and the Support Program for Outstanding Young Scientific and Technological Top-notch Talents of College of Mathematics and Systems Science (Grant No. Sxy2016K01).

References

[1] Kinnunen J., The Hardy-Littlewood maximal function of a Sobolev function, Israel J. Math., 1997, 100, 117–124

[2] Kinnunen J., Lindqvist P., The derivative of the maximal function, J. Reine. Angew. Math., 1998, 503, 161–167

[3] Kinnunen J., Saksman E., Regularity of the fractional maximal function, Bull. London Math. Soc., 2003, 35(4), 529–535

[4] Carneiro E., Moreira D., On the regularity of maximal operators, Proc. Amer. Math. Soc., 2008, 136(12), 4395–4404

[5] Liu F., Wu H., On the regularity of the multisublinear maximal functions, Canad. Math. Bull., 2015, 58(4), 808–817

[6] Liu F., Mao S., On the regularity of the one-sided Hardy-Littlewood maximal functions, Czech. Math. J., 2017, 67(142), 219–234

[7] Luiro H., Continuity of the maixmal operator in Sobolev spaces, Proc. Amer. Math. Soc., 2007, 135(1), 243–251

[8] Luiro H., On the regularity of the Hardy-Littlewood maximal operator on subdomains of $\mathbb{R}^n$, Proc. Edinburgh Math. Soc., 2010, 53(1), 211–237

[9] Hajłasz P., Onninen J., On boundedness of maximal functions in Sobolev spaces, Ann. Acad. Sci. Fenn. Math., 2004, 29, 167–176

[10] Tanaka H., A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function, Bull. Austral. Math. Soc., 2002, 65(2), 253–258
[11] Aldaz J.M., Pérez Lázaro J., Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities, Trans. Amer. Math. Soc., 2007, 359(5), 2443–2461
[12] Liu F., Chen T., Wu H., A note on the end-point regularity of the Hardy-Littlewood maximal functions, Bull. Austral. Math. Soc., 2016, 94(1), 121–130
[13] Kurka O., On the variation of the Hardy-Littlewood maximal function, Ann. Acad. Sci. Fenn. Math., 2015, 40, 109–133
[14] Carneiro E., Madrid J., Derivative bounds for fractional maximal functions, Trans. Amer. Math. Soc., 2017, 369(6), 4063–4092
[15] Liu F., Wu H., Endpoint regularity of multisublinear fractional maximal functions, Canad. Math. Bull., 2017, 60(3), 586–603
[16] Carneiro E., Mardid J., Pierce L.B., Endpoint Sobolev and BV continuity for maximal operators, J. Funct. Anal., 2017, 273(10), 3262–3294
[17] Liu F., A note on Marcinkiewicz integrals associated to surfaces of revolution, J. Austral. Math. Soc., 2018, 104, 380–402
[18] Carneiro E., Svaiter B.F., On the variation of maximal operators of convolution type, J. Funct. Anal., 2013, 265(10), 837–865
[19] Li X., Zhao Q., A new integrable symplectic map by the binary nonlinearization to the super AKNS system, J. Geome. Phy., 2017, 121, 123–137
[20] Liu F., Continuity and approximate differentiability of multisublinear fractional maximal functions, Math. Inequal. Appl., 2018, 21(1), 25–40
[21] Zhao Q., Li X., A Bargmann System and The Involutive Solutions Associated With A New 4-Order Lattice Hierarchy, Anal. Math. Physics, 2016, 6(3), 237–254
[22] Liu F., Xue Q., Yabuta K., Boundedness and continuity of maximal singular integrals and maximal functions on Triebel-Lizorkin spaces, Sci. China Math. (to appear), doi:10.1007/s11425-017-9416-5
[23] Hajłasz P., Maly J., On approximate differentiability of the maximal function, Proc. Amer. Math. Soc., 2010, 138(1), 165–174
[24] Liu F., Xue Q., Yabuta K., Rough maximal singular integral and maximal operators supported by subvarieties on Triebel-Lizorkin spaces, Nonlinear Anal., 2018, 171, 41–72
[25] Liu F., Wu H., On the regularity of maixmal operators supported by submanifolds, J. Math. Anal. Appl., 2017, 453, 144–158
[26] Andersen K.F., Sawyer E.T., Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators, Trans. Amer. Math. Soc., 1988, 308(2), 547–558
[27] Liu F., On the regularity of one-sided fractional maximal functions, Math. Slovaca, 2018, 68(5), 1097–1112
[28] Bober J., Carneiro E., Hughes K., Pierce L.B., On a discrete version of Tanaka's theorem for maximal functions, Proc. Amer. Math. Soc., 2012, 140(5), 1669–1680
[29] Liu F., A remark on the regularity of the discrete maximal operator, Bull. Austral. Math. Soc., 2017, 95(1), 108–120
[30] Carneiro E., Hughes K., On the endpoint regularity of discrete maximal operators, Math. Res. Lett., 2012, 19(6), 1245–1262
[31] Madrid J., Sharp inequalities for the variation of the discrete maximal function, Bull. Austral. Math. Soc., 2017, 95(1), 94–107
[32] Temur F., On regularity of the discrete Hardy-Littlewood maximal function, preprint at http://arxiv.org/abs/1303.3993
[33] Liu F., Wu H., Regularity of discrete multisublinear fractional maximal functions, Sci. China Math., 2017, 60(8), 1461–1476
[34] Liu F., Wu H., A note on the endpoint regularity of the discrete maximal operator, Proc. Amer. Math. Soc., 2019, 147(2), 583–596
[35] Brezis H., Lieb E., A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc., 1983, 88(3), 486–490