Open Mathematics

Research Article

Josef Šlapal*

Galois connections between sets of paths and closure operators in simple graphs

https://doi.org/10.1515/math-2018-0128
Received July 31, 2018; accepted December 6, 2018

Abstract: For every positive integer n, we introduce and discuss an isotone Galois connection between the sets of paths of lengths n in a simple graph and the closure operators on the (vertex set of the) graph. We consider certain sets of paths in a particular graph on the digital line $\mathbb{Z}$ and study the closure operators associated, in the Galois connection discussed, with these sets of paths. We also focus on the closure operators on the digital plane $\mathbb{Z}^2$ associated with a special product of the sets of paths considered and show that these closure operators may be used as background structures on the plane for the study of digital images.

Keywords: Simple graph; Closure operator; Galois connection; digital space; Khalimsky topology; Jordan curve theorem

MSC 2010: 05C10; 54A05; 54D05

Introduction

It is useful to find new relationships between different mathematical structures or theories because they demonstrate the interconnectedness of mathematics and enable us to use tools of one theory to solve problems of another. In the present paper, we will discuss certain relationships between graph theory and topology. We will introduce and study Galois connections between the sets of paths of the same length in a graph and closure operators on the vertex set of the graph. We will focus on the connectedness provided by the closure operators associated, in the Galois connections introduced, to certain sets of paths in the 2-adjacency graph on the digital line $\mathbb{Z}$. The closure operators will be shown to generalize the connected ordered topologies [1], hence the Khalimsky topology, on $\mathbb{Z}$. Further, we will discuss the closure operators on the digital plane $\mathbb{Z}^2$ associated with special products of the sets of paths with the same length in the 2-adjacency graph on $\mathbb{Z}$. These closure operators include the Khalimsky topology on $\mathbb{Z}^2$ and we will show that they allow for a digital analogue of the Jordan curve theorem (recall that the classical Jordan curve theorem states that a simple closed curve in the real (Euclidean) plane separates the plane into precisely two components). It follows that the closure operators may be used as background structures on the digital plane $\mathbb{Z}^2$ for the study of digital images.

The idea of studying connectedness in a graph with respect to sets of paths is taken from [2] where certain special sets of paths, called path partitions, were used to obtain connectedness with a convenient geometric behavior. In the present note, we will employ arbitrary sets of paths (of given lengths) and investigate connectedness with respect to the closure operators associated with the sets in a Galois connection.

For the graph-theoretic terminology, we refer to [3]. By a *graph* $G = (V, E)$, we understand an (undirected simple) graph (without loops) with $V \neq \emptyset$ as the *vertex* set and $E \subseteq \{\{x, y\};\ x, y \in V,\ x \neq y\}$ as the set

***Corresponding Author: Josef Šlapal:** IT4Innovations Centre of Excellence, Brno University of Technology, 612 66 Brno, Czech Republic, E-mail: slapal@fme.vutbr.cz

of *edges*. We will say that G is a graph *on* V. Two vertices $x, y \in V$ are said to be *adjacent* (to each other) if $\{x, y\} \in E$. Recall that a *path* in G is a (finite) sequence $(x_i|\ i \leq n)$, i.e., $(x_0, x_1, \ldots, x_n)$, of pairwise different vertices of V such that x_i is adjacent to x_{i+1} whenever $i < n$. The non-negative integer n is called the *length* of the path $(x_i|\ i \leq n)$. A sequence $(x_i|\ i \leq n)$ of vertices of G is called a *circle* if $n > 2$, $x_0 = x_n$, and $(x_i|\ i < n)$ is a path. Given graphs $G_1 = (V_1, E_1,)$ and $G_2 = (V_2, E_2)$, we say that G_1 is a *subgraph* of G_2 if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. If, moreover, $V_1 = V_2$, then G_1 is called a *factor* of G_2.

Recall [4] that the *strong product* of a pair of graphs $G_1 = (V_1, E_1,)$ and $G_2 = (V_2, E_2,)$ is the the graph $G_1 \otimes G_2 = (V_1 \times V_2, E)$ with the set of edges $E = \{\{(x_1, x_2), (y_1, y_2)\} \subseteq V_1 \times V_2;$ there exists a nonempty subset $J \subseteq \{1, 2\}$ such that $\{x_j, y_j\} \in E_j$ for every $j \in J$ and $x_j = y_j$ for every $j \in \{1, 2\} - J\}$. Note that the strong product differs from the cartesian product of G_1 and G_2, i.e., from the graph $(V_1 \times V_2, F)$ where $F = \{\{(x_1, x_2), (y_1, y_2)\};\ \{x_j, y_j\} \in E_j$ for every $j \in \{1, 2\}\}$ (the cartesian product is a factor of the strong product).

By a *closure operator* u on a set X, we mean a map $u\colon \exp X \to \exp X$ (where $\exp X$ denotes the power set of X) which is
(i) grounded (i.e., $u\emptyset = \emptyset$),
(ii) extensive (i.e., $A \subseteq X \Rightarrow A \subseteq uA$), and
(iii) monotone (i.e., $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$).
The pair (X, u) is then called a *closure space*. Closure spaces were studied by E. Čech [5] (who called them topological spaces).

A closure operator u on X that is
(iv) additive (i.e., $u(A \cup B) = uA \cup uB$ whenever $A, B \subseteq X$) and
(v) idempotent (i.e., $uuA = uA$ whenever $A \subseteq X$)
is called a *Kuratowski closure operator* or a *topology* and the pair (X, u) is called a *topological space*.

Given a cardinal $m > 1$, a closure operator u on a set X and the closure space (X, u) are called an *S_m-closure operator* and an *S_m-closure space* (briefly, an *S_m-space*), respectively, if the following condition is satisfied:

$A \subseteq X \Rightarrow uA = \bigcup\{uB; B \subseteq A, \operatorname{card} B < m\}.$

In [6], S_2-closure operators and S_2-closure spaces are called *quasi-discrete*. S_2-topologies (S_2-topological spaces) are usually called *Alexandroff topologies* (*Alexandroff spaces*) - see [7]. Every S_2-closure operator is of course additive and every S_α-closure operator is an S_β-closure operator whenever $\alpha < \beta$. If $\alpha \leq \aleph_0$, then every additive S_α-closure operator is an S_2-closure operator.

Many concepts defined for topological spaces (see, e.g., [8]) can naturally be extended to closure spaces. Let us mention some of them. Given a closure space (X, u), a subset $A \subseteq X$ is called *closed* if $uA = A$, and it is called *open* if $X - A$ is closed. A closure space (X, u) is said to be a *subspace* of a closure space (Y, v) if $uA = vA \cap X$ for each subset $A \subseteq X$. We will briefly speak about a subspace X of (Y, v). A closure space (X, u) is said to be *connected* if $\emptyset$ and X are the only clopen (i.e., both closed and open) subsets of X. A subset $X \subseteq Y$ is connected in a closure space (Y, v) if the subspace X of (Y, v) is connected. A maximal connected subset of a closure space is called a *component* of this space. The *connectedness graph* of a closure operator u on a set X is the graph with the vertex set X whose edges are the connected two-element subsets of X. All the basic properties of connected sets and components in topological spaces are also preserved in closure spaces. In particular, we will employ the fact that the union of a sequence (finite or infinite) of connected subsets is connected if every pair of consecutive members of the sequence has a nonempty intersection.

If u, v are closure operators on a set X, then we put $u \leq v$ if $uA \subseteq vA$ for every subset $A \subseteq X$ (clearly, $\leq$ is a partial order on the set of all closure operators on X).

In this note, the concept of a Galois connection is understood in the isotone sense. Thus, a Galois connection between partially ordered sets $G = (G, \leq)$ and $H = (H, \leq)$ is a pair (f, g) where $f : G \to H$ and $g : H \to G$ are isotone (i.e., order preserving) maps such that $f(g(y)) \leq y$ for every $y \in H$ and $x \leq g(f(x))$ for every $x \in G$. For more details concerning Galois connections see [9].

1 Closure operators associated with sets of paths

In the sequel, n will denote a positive integer.

Given a graph G, we denote by $\mathcal{P}_n(G)$ the set of all paths of length n in G. For every subset $\mathcal{B} \subseteq \mathcal{P}_n(G)$, we put $\hat{\mathcal{B}} = \{(x_i|\ i \le m);\ 0 < m \le n$ and there exists $(y_i|\ i \le n) \in \mathcal{B}$ such that $x_i = y_i$ for every $i \le m\}$ (so that the elements of $\hat{\mathcal{B}}$ are the paths of lengths less than or equal to n that are initial parts of the paths belonging to $\mathcal{B}$; we clearly have $\mathcal{B} \subseteq \hat{\mathcal{B}}$).

Let $G_j = (V_j, E_j)$ be a graph and $\mathcal{B}_j \subseteq \mathcal{P}_n(G_j)$ for every $j = 1, 2$. Then, we put $\mathcal{B}_1 \otimes \mathcal{B}_2 = \{((x_i^1, x_i^2)|\ i \le n);\ (x_i^1, x_i^2) \in V_1 \times V_2$ for all $i \le n$, there is a nonempty subset $J \subseteq \{1, 2\}$ such that $(x_i^j|\ i \le n) \in \mathcal{B}_j$ for every $j \in J$, and $(x_i^j|\ i \le n)$ is a constant sequence for every $j \in \{1, 2\} - J\}$. We will need the obvious fact that $\mathcal{B}_1 \otimes \mathcal{B}_2 \subseteq \mathcal{P}_n(G_1 \otimes G_2)$.

Let $G = (V, E)$ be a graph. Given a subset $\mathcal{B} \subseteq \mathcal{P}_n(G)$ ($n > 0$ an ordinal), we put $f_n(\mathcal{B})X = X \cup \{x \in V$; there exists $(x_i|\ i \le m) \in \hat{\mathcal{B}}$ with $\{x_i;\ i < m\} \subseteq X$ and $x_m = x\}$ for every $X \subseteq V$. It may easily be seen that $f_n(\mathcal{B})$ is an S_{n+1}-closure operator on G. Thus, denoting by $\mathcal{U}(G)$ the set of all closure operators on G (i.e., on the vertex set V of G), we get a map $f_n : \exp \mathcal{P}_n(G) \to \mathcal{U}(G)$. The closure operator $f_n(\mathcal{B})$ is said to be *associated* to $\mathcal{B}$. It is evident that every path belonging to $\hat{\mathcal{B}}$ is a connected subset of the closure space $(V, f_n(\mathcal{B}))$.

Further, given a closure operator u on G, we put $g_n(u) = \{(x_i|\ i \le n) \in \mathcal{P}_n(G);\ x_j \in u\{x_i;\ i < j\}$ for every j with $0 < j \le n\}$. Thus, we get a map $g_n : \mathcal{U}(G) \to \exp \mathcal{P}_n(G)$.

Theorem 1.1. *Let $G = (V, E)$ be a graph. Then, the pair (f_n, g_n) constitutes a Galois connection between the partially ordered sets $(\exp \mathcal{P}_n(G), \subseteq)$ and $(\mathcal{U}(G), \le)$.*

Proof. It is evident that f_n and g_n are isotone. Let $\mathcal{B} \subseteq \mathcal{P}_n(G)$ and let $(y_i|\ i \le n) \in \mathcal{B}$ be a path. Put $f_n(\mathcal{B}) = u$. Then, u is the closure operator on G given by $uX = X \cup \{x \in V;$ there exists $(x_i|\ i \le m) \in \hat{\mathcal{B}}$ with $\{x_i;\ i < m\} \subseteq X$ and $x_m = x\}$ for every $X \subseteq V$. We have $(y_i|\ i \le j) \in \hat{\mathcal{B}}$ for every j, $0 < j \le n$, hence $y_j \in u\{y_i;\ i < j\}$. Consequently, $(y_i|\ i \le n) \in g_n(u)$. Therefore, $\mathcal{B} \subseteq g_n(u) = g_n(f_n(\mathcal{B}))$.
Let $u \in \mathcal{U}(G)$ and let $X \subseteq V$ be a subset. Put $g_n(u) = \mathcal{B}$ and let $y \in f_n(g_n(u))X$ be an element. If $y \in X$, then $y \in uX$. Let $y \notin X$. Then, there exists $(x_i|\ i \le m) \in \hat{\mathcal{B}}$ such that $\{x_i;\ i < m\} \subseteq X$ and $x_m = y$. Since $(x_i|\ i \le m) \in \hat{\mathcal{B}}$, there is $(y_i|\ i < n) \in \mathcal{B} = g_n(u)$ with $x_i = y_i$ for every $i \le m$. Thus, $y_j \in u\{x_i;\ i < j\}$ for every j, $0 < j < n$. In particular, $y = y_m \in u\{x_i;\ i < m\} \subseteq uX$. Therefore, $f_n(g_n(u)) \le u$. □

In what follows, we will investigate, for a given graph G, the (isomorphic) partially ordered sets $f_n(\exp \mathcal{P}_n(G))$ and $g_n(\mathcal{U}(G))$. If the graph G is complete, then of course $f_1(\exp \mathcal{P}_1(G))$ is the set of all S_2-closure operators on G.

Proposition 1.2. *Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$. Then, $\mathcal{B} \in g_n(\mathcal{U}(G))$ if and only if the following condition is satisfied:*

(*) If $(x_i|\ i \le n) \in \mathcal{P}_n(G)$ has the property that, for every i_0, $0 < i_0 \le n$, there exist $(y_j|\ j \le n) \in \mathcal{B}$ and j_0, $0 < j_0 \le n$, such that $x_{i_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq \{x_i;\ i < i_0\}$, then $(x_i|\ i \le n) \in \mathcal{B}$.

Proof. Let $\mathcal{B} \in g_n(\mathcal{U}(G))$, let $(x_i|\ i \le n) \in \mathcal{P}_(G)$, and let, for any $i_0, 0 < i_0 \le n$, there be $(y_j|j \le n) \in \mathcal{B}$ and $j_0, 0 < j_0 \le n$, such that $x_{i_0} = y_{j_0}$ and $\{y_j; j < j_0\} \subseteq \{x_i; i < i_0\}$. Then, $x_{i_0} \in f_n(\mathcal{B})\{y_j; j < j_0\} \subseteq f_n(\mathcal{B})\{x_i; i < i_0\}$ for every i_0, $0 < i_0 \le n$. Therefore, $(x_i|\ i \le n) \in g_n(f_n(\mathcal{B})) = \mathcal{B}$. Thus, the condition (*) is satisfied.
Conversely, let the condition (*) be satisfied and let $(x_i|i \le n) \in g_n(f_n(\mathcal{B}))$. Then, $x_{i_0} \in f_n(\mathcal{B})\{x_i;\ i < i_0\}$ for each i_0, $0 < i_0/= n$. Hence, for every i_0, $0 < i_0 \le n$, there exist $(y_j|\ j \le n) \in \mathcal{B}$ and j_0, $0 < j_0 \le n$, such that $x_{i_0} = y_{j_0}$ and $\{y_j;\ j < j_0\} \subseteq \{x_i;\ i < i_0\}$. Therefore, $(x_i|\ i \le n) \in \mathcal{B}$ and we have shown that $g_n(f_n(\mathcal{B})) \subseteq \mathcal{B}$. Consequently, $\mathcal{B} = g_n(f_n(\mathcal{B}))$, so that $\mathcal{B} \in g_n(\mathcal{U}(G))$. The proof is complete. □

Example 1.3. Note that every subset $\mathcal{B} \subseteq \mathcal{P}_1(G)$ satisfies the condition (*). A subset $\mathcal{B} \subseteq \mathcal{P}_2(G)$ satisfies (*) if and only if each of the following six conditions implies $(x, y, z) \in \mathcal{B}$:

(1) $(x, y, t) \in \mathcal{B}$, $(x, z, u) \in \mathcal{B}$,
(2) $(x, y, t) \in \mathcal{B}$, $(y, z, u) \in \mathcal{B}$,

(3) $(x, y, t) \in \mathcal{B}$, $(y, x, z) \in \mathcal{B}$.

The following assertion is obvious:

Proposition 1.4. *Let $G = (V, E)$ be a graph and $u \in \mathcal{U}(G)$. Then $u \in f_n(\exp \mathcal{P}_n(G))$ if and only if the following condition is satisfied:*

If $X \subseteq V$ and $x \in uX - X$, then there exist $(x_i | i \le n) \in \mathcal{P}_n(G)$ and a positive integer $m \le n$, such that $\{x_i;\ i < m\} \subseteq X$, $x_j \in u\{x_i;\ i < j\}$ for each j, $0 < j \le n$, and $x_m = x$.

Though $f_n(\mathcal{B})$ is neither additive nor idempotent in general, we have:

Proposition 1.5. *Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$ a subset. The union of a system of closed subsets of $(V, f_n(\mathcal{B}))$ is a closed subset of $(V, f_n(\mathcal{B}))$.*

Proof. Put $f_n(\mathcal{B}) = u$. Let $\{X_j;\ j \in J\}$ be a system of closed subsets of (V, u) and let $x \in u \bigcup_{j \in J} X_j$. Then, there are $(x_i | \ i \le n) \in \mathcal{B}$ and i_0, $0 < i_0 \le n$, such that $x_{i_0} = x$ and $x_i \in \bigcup_{j \in J} X_j$ for all $i \le i_0$. In particular, we have $x_0 \in \bigcup_{j \in J} X_j$ so that there is $j_0 \in J$ such that $x_0 \in X_{j_0}$. If there exists k, $0 < k \le n$, such that $x_i \in X_{j_0}$ for all $i < k$, then $x_k \in u\{x_i;\ i < k\} \subseteq uX_{j_0} = X_{j_0}$. Consequently, we have $\{x_i;\ i \le n\} \subseteq X_{j_0}$. Thus, $x = x_{i_0} \in X_{j_0} \subseteq \bigcup_{j \in J} X_j$. We have shown that $u \bigcup_{j \in J} X_j \subseteq \bigcup_{j \in J} X_j$, which completes the proof. □

Proposition 1.6. *Let $G = (V, E)$ be a graph and $\mathcal{B} \subseteq \mathcal{P}_n(G)$ a subset. Then, the closure operator $f_n(\mathcal{B})$ is idempotent if and only if $(V, f_n(\mathcal{B}))$ is an Alexandroff space.*

Proof. Put $f_n(\mathcal{B}) = u$. Let u be idempotent and let $X \subseteq V$ be a subset and $x \in u$ a point. If $x \in X$, then $x \in u\{x\} \subseteq \bigcup_{y \in X} u\{y\}$. Suppose that $x \notin X$. Then there are $(x_i | \ i \le n) \in \mathcal{B}$ and i_0, $0 < i_0 \le n$, such that $x_{i_0} = x$ and $x_i \in X$ for all $i \le i_0$. Clearly, we have $x \in u\{x_i;\ i < i_0\}$. If $i_0 = 1$, then $x \in u\{x_0\}$. Let $i_0 > 1$. We have $u\{x_i;\ i < k\} \subseteq uu\{x_i;\ i < k-1\} = u\{x_i;\ i < k-1\}$ for every k, $1 < k \le n$. Consequently, $u\{x_i;\ i < i_0\} \subseteq u\{x_i;\ i < i_0 - 1\} \subseteq \ldots \subseteq u\{x_0\}$, so that $x \in u\{x_0\}$. Therefore, $x \in \bigcup_{y \in X} u\{y\}$ and the proof is complete. □

2 Sets of paths in and the associated closure operators on the 2-adjacency graph

Recall that the 2-adjacency graph (on $\mathbb{Z}$) is the graph $\mathbb{Z}_2 = (\mathbb{Z}, A_2)$ where $A_2 = \{\{p, q\};\ p, q \in \mathbb{Z},\ |p - q| = 1\}$.

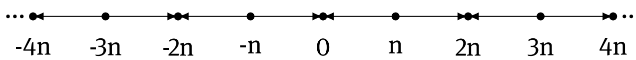
For every $l \in \mathbb{Z}$, we put

$$I_l = \begin{cases} (ln + i | \ i \le n) \text{ if } l \text{ is odd,} \\ ((l+1)n - i | \ i \le n) \text{ if } l \text{ is even.} \end{cases}$$

Throughout this section, $\mathcal{B} \subseteq \mathcal{P}_n(\mathbb{Z}_2)$ will denote the set $\mathcal{B} = \{I_l;\ l \in \mathbb{Z}\}$. Thus, all paths I_l belonging to $\mathcal{B}$ are just the arithmetic sequences $(x_i | i \le n)$ of integers with the difference equal to 1 or -1 and with $x_0 = ln$ if l is odd and $x_0 = (l+1)n$ if l is even. Note that each element $z \in \mathbb{Z}$ belongs to at least one and at most two paths in $\mathcal{B}$. It belongs to two (different) paths from $\mathcal{B}$ if and only if there is $l \in \mathbb{Z}$ with $z = ln$ (in which case z is the first member of each of the paths I_l and I_{l-1} if l is odd, and z is the last member of each of the two paths if l is even). The graph $\mathbb{Z}_2$ with the set $\mathcal{B}$ of paths is demonstrated in Figure 1 where only the vertices kn, $k \in \mathbb{Z}$, are marked (by bold dots) so that, between any two neighboring vertices marked, there are $n - 1$ more vertices that are not marked out. The paths in $\mathcal{B}$ are represented by the line segments oriented from the first to the last members of the paths (and every directed line segment represents n edges of the graph).

Clearly, the closure operator $f_n(\mathcal{B})$ is additive if and only if $n = 1$. The closure operator $f_1(\mathcal{B})$ coincides with the Khalimsky topology on $\mathbb{Z}$ generated by the subbase $\{\{2k - 1, 2k, 2k + 1\};\ k \in \mathbb{Z}\}$ - cf. [1].

Figure 1: A section of the graph $\mathbb{Z}_2$ with the set $\mathcal{B}$ of paths demonstrated.



Proposition 2.1. *In the closure space $(\mathbb{Z}, f_n(\mathcal{B}))$, the points ln, $l \in \mathbb{Z}$ odd, are open while all the other points are closed.*

Proof. Let $l_0 \in \mathbb{Z}$ be an arbitrary odd number and put $z = l_0 n$. Let $x \in f_n(\mathcal{B})(\mathbb{Z} - \{z\})$ be a point and suppose that $x \notin \mathbb{Z} - \{z\}$, i.e., that $x = z$. Then, there are $(x_i|\ i \le n) \in \mathcal{B}$ and i_0, $0 < i_0 \le n$, such that $x = x_{i_0}$ and $\{x_i;\ i < i_0\} \in \mathbb{Z} - \{z\}$. Let $m \in \mathbb{Z}$ be the odd number with $x_i = mn + i$ for all $i \le n$ or $x_i = mn - i$ for all $i \le n$. Then, $x_{i_0} = mn + i_0$ or $x_{i_0} = mn - i_0$. Since $0 < i_0 \le n$, we have $0 < i_0 < 2n$ and, consequently, $mn < x_{i_0} < (m+2)n$ or $(m-2)n < x_{i_0} < mn$. Hence, $x_{i_0} \ne ln$ for any odd number $l \in \mathbb{Z}$. Thus, $x_{i_0} \ne z$, so that $x \in \mathbb{Z} - \{z\}$. Therefore, the set $\mathbb{Z} - \{z\}$ is closed, i.e., $\{z\}$ is open in $(\mathbb{Z}, f_n(\mathcal{B}))$.
Let $z \in \mathbb{Z}$ be an arbitrary point with $z \ne ln$ for any odd number $l \in \mathbb{Z}$. Let $x \in f_n(\mathcal{B})\{z\}$ and suppose that $x \ne y$. Then, there are $(x_i|\ i \le n) \in \mathcal{B}$ and i_0, $0 < i_0 \le n$, such that $x = x_{i_0}$ and $\{x_i;\ i < i_0\} = \{z\}$. Thus, $x_0 = z$, which contradicts the existence of an odd number $l_0 \in \mathbb{Z}$ with $x_0 = l_0 n$. Hence, $x = y$ and, therefore, $\{z\}$ is closed in $(\mathbb{Z}, f_n(\mathcal{B}))$. □

Theorem 2.2. *$(\mathbb{Z}, f_n(\mathcal{B}))$ is a connected closure space.*

Proof. Since every path belonging to $\mathcal{B}$ is connected in $(\mathbb{Z}, f_n(\mathcal{B}))$, the set $A = I_0 \cup I_1 \cup I_2 \cup \dots$ of non-negative integers and the set $B = I_{-1} \cup I_{-2} \cup I_{-3} \cup \dots$ of non-positive integers are connected (note that $I_k \cap I_{k+1} = \{k_n\}$). Thus, $\mathbb{Z} = A \cup B$ is connected because $A \cap B = \{0\}$. □
Given a totally ordered set $(X, \le)$ and a point $x \in X$, we put $L(x) = \{y \in \mathbb{Z};\ y < x\}$ and $U(x) = \{y \in \mathbb{Z};\ y > x\}$. The set of integers $\mathbb{Z}$ is considered to be totally ordered by the natural order. Clearly, for every $z \in \mathbb{Z}$, both $L(z)$ and $U(z)$ are closed in the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, f_n(\mathcal{B}))$.

Theorem 2.3. *Let $z \in \mathbb{Z}$ be a point. Then, there are points $z_1, z_2 \in \mathbb{Z}$ such that $L(z_1)$ and $U(z_2)$ are components of the subspace $\mathbb{Z} - \{z\}$ of $(\mathbb{Z}, f_n(\mathcal{B}))$ and all the other components of $\mathbb{Z} - \{z\}$ are singletons. If $z = ln + i$ where $l, i \in \mathbb{Z}$, l even and $|i| \le 1$, then $z_1 = z_2 = z$ (so that $\mathbb{Z} - \{z\}$ has no singleton components).*

Proof. There are $l_0, i_0 \in \mathbb{Z}$, l odd and $|i_0| \le n$, such that $z = l_0 n + i_0$. Let $|i_0| = n$. Then, $z = (l_0 + 1)n$ or $z = (l_0 - 1)n$ so that there is an even number $m \in \mathbb{Z}$ with $z = mn$. We clearly have $L(z) = \bigcup\{I_l;\ l \le (m-2)n\} \cup \{i;\ (m-1)n \le i < z\}$. $L(z)$ is connected in $(\mathbb{Z}, f_n(\mathcal{B}))$ because $(I_l|\ l \le (m-2)n)$ is a sequence of paths belonging to $\mathcal{B}$ with every pair of consecutive paths having a point in common, $\{i;\ (m-1)n \le i < z\} \in \hat{\mathcal{B}}$ and $(m-1)n \in I_{(m-2)n} \cap \{i;\ (m-1)n \le i < z\}$. Similarly, $U(z) = \bigcup\{I_l;\ l \ge (m+1)n\} \cup \{i;\ z < i \le (m+1)n\}$ is connected in $(\mathbb{Z}, f_n(\mathcal{B}))$. Since $L(z)$ and $U(z)$ are closed, disjoint, and satisfying $L(z) \cup U(z) = \mathbb{Z} - \{z\}$, they are the components of $\mathbb{Z} - \{z\}$.
Let $|i_0| < n$ and suppose that $i_0 \ge 0$. Then, $L(l_0 n)$ is connected in $(\mathbb{Z}, f_n(\mathcal{B}))$ because it is the union of a sequence of paths belonging to $\mathcal{B}$, namely the sequence $(I_l|\ l \le l_0)$ in which every pair of consecutive paths has a point in common. Further, since $(l_0 n + i|\ i < l_0 n + i_0) \in \hat{\mathcal{B}}$ and $L(l_0 n) \cap \{l_0 n + i;\ i < l_0 n + i_0\} \ne \emptyset$, the set $L(z) = L(l_0 n) \cup \{l_0 n + i;\ i < l_0 n + i_0\}$ is connected in $(\mathbb{Z}, f_n(\mathcal{B}))$. It is also evident that $L(z)$ is closed in the subspace $\mathbb{Z} - \{z\}$. Further, $U((l_0+1)n - 1)$ is connected because it is the union of a sequence of paths belonging to $\mathcal{B}$, namely, the sequence $(I_l|\ l \ge l_0 + 1)$ in which every pair of consecutive paths has a point in common. It is also evident that $U(z)$ is closed in the subspace $\mathbb{Z} - \{z\}$. Clearly, we have $\mathbb{Z} - \{z\} = L(z) \cup \{i;\ z < i < 2l_0 n\} \cup U((l_0+1)n - 1)$ where the sets $L(z)$, $\{i;\ z < i < 2l_0 n\}$, and $U((l_0+1)n - 1)$ are pairwise disjoint. The singleton subsets of $\{i;\ z < i < (l_0+1)n\}$ are closed in $\mathbb{Z} - \{z\}$. We have shown that $L(z)$, $U((l_0+1)n - 1)$,

and the singletons $\{i\}$, $z < i < (l_0+1)n$, are the components of $\mathbb{Z} - \{z\}$. We may show in an analogous way that $U(z)$, $L((l_0+1)n-1)$, and the singletons $\{i\}$, $(l_0-1)n < i < z$, are the components of $\mathbb{Z} - \{z\}$ if $i_0 \le 0$.

To prove the second part of the Theorem, suppose that $z = ln + i$ where $l, i \in \mathbb{Z}$, l even and $|i| \le 1$. It was shown in the first part of the proof that $L(z)$ and $U(z)$ are the (only) components of $\mathbb{Z} - \{z\}$ if $i = 0$ (because then $z = l_0 n + i_0$ where $l_0 = l_1$ is odd and $i_0 = n$). Suppose that $i = 1$. Then, $z = l_0 n + i_0$ where $l_0 = l + 1$ and $i_0 = 1 - n$. We have $(l_0 - 1)n + 1 = ln + 1 = z$, so that $L((l_0-1)n+1) = L(z)$. Since $\{i;\ (l_0-1)n < i < z\} \neq \emptyset$, $L(z)$ and $U(z)$ are the (only) components of $\mathbb{Z} - \{z\}$ according to the previous part of the proof. Using similar arguments, we may show that $L(z)$ and $U(z)$ are the (only) components of $\mathbb{Z} - \{z\}$ if $i = -1$. The proof is complete. □

Remark 2.4. Recall [1] that a connected topological space (X, u) is called a connected ordered topological space (COTS for short) if, for any three-point subset $Y \subseteq X$, there is a point $x \in Y$ such that Y meets two components of the subspace $X - \{x\}$. It was shown in [1] that a connected topological space (X, u) is a COTS if and only if there is a total order on X such that, for every $x \in X$, the sets $L(x)$ and $U(x)$ are components of the subspace $X - \{x\}$. Thus, Theorem 3 results in the well-known fact that the Khalimsky space $(\mathbb{Z}, f_1(\mathcal{B}))$ is a COTS ([1]). Hence, the closure operators $f_n(\mathcal{B})$ may be regarded as generalizations of the connected ordered topologies on $\mathbb{Z}$.

3 Closure operators associated with sets of paths in the digital plane

In the sequel, we will discuss the graph $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the set $\mathcal{B} \otimes \mathcal{B}$ of paths of length n. The graph is demonstrated in Figure 2 where, as in Figure 1, only the vertices (kn, ln), $k, l \in \mathbb{Z}$, are marked (by bold dots) and only the paths between these vertices are marked (by line segments oriented from the first to the last members of the paths). Thus, between any pair of neighboring parallel horizontal or vertical line segments (having the same orientation), there are $n - 1$ more parallel line segments with the same orientation that are not displayed in order to make the Figure transparent. And, of course, every oriented line segment represents n edges of the graph.

Observe that the closure space $(\mathbb{Z}^2, f_n(\mathcal{B} \otimes \mathcal{B}))$ is connected. Indeed, the set $\mathbb{Z} \times \{k\}$ is connected for every $k \in \mathbb{Z}$ by applying arguments similar to those used in the proof of Theorem 2 Therefore, the set $\mathbb{Z}^2 = A \cup B$ where $A = (\mathbb{Z} \times \{0\}) \cup (\mathbb{Z} \times \{1\}) \cup (\mathbb{Z} \times \{2\}) \cup \dots$ and $B = (\mathbb{Z} \times \{0\}) \cup (\mathbb{Z} \times \{-1\}) \cup (\mathbb{Z} \times \{-2\}) \cup \dots$ is connected, again, by applying arguments similar to those used in the proof of Theorem 2.

Note that $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ is nothing but the well-known 8-adjacency graph on $\mathbb{Z}^2$. Let H_n denote the factor of the graph $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ with exactly those edges $\{(x_1, y_1), (x_2, y_2)\}$ of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ that satisfy one of the following four conditions for some $k \in \mathbb{Z}$:

Figure 2: A section of the graph $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the set $\mathcal{B} \otimes \mathcal{B}$ of paths demonstrated.

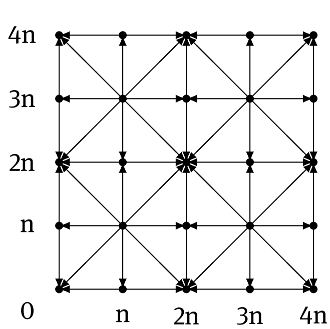


Figure 3: A section of the graph H_n.

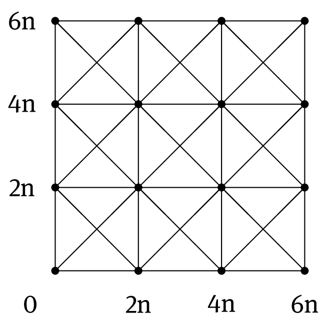
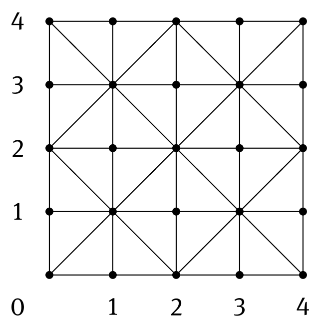


Figure 4: A section of the connectedness graph of the Khalimsky topology on $\mathbb{Z}^2$.



$x_1 - y_1 = x_2 - y_2 = 2kn,$
$x_1 + y_1 = x_2 + y_2 = 2kn,$
$x_1 = x_2 = 2kn,$
$y_1 = y_2 = 2kn.$

A section of the graph H_n is shown in Figure 3 where only the vertices $(2kn, 2ln)$, $k, l \in \mathbb{Z}$, are marked (by bold dots) and thus, on every edge drawn between two such vertices, there are $2n - 1$ more (non-displayed) vertices so that the edges represent $2n$ edges in the graph H_n. Note that every circle C in H_n is a connected subset of $(\mathbb{Z}^2, f_n(\mathcal{B} \otimes \mathcal{B}))$. Indeed, C consists (is the union) of a finite sequence of paths in $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ belonging to $\mathcal{B} \otimes \mathcal{B}$ such that every pair of consecutive paths in the sequence has a point in common.

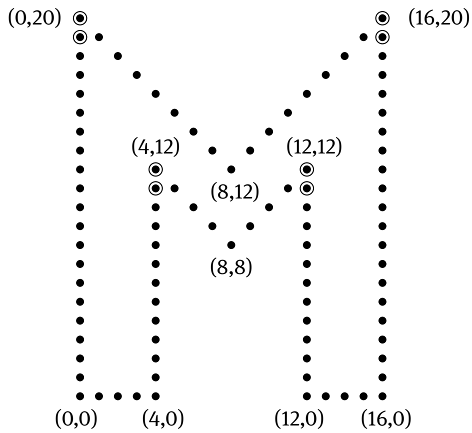
The closure operator $f_1(\mathcal{B} \otimes \mathcal{B})$ (which is a topology) is called the *Khalimsky topology* on $\mathbb{Z}^2$ and the topological space $(\mathbb{Z}^2, f_1(\mathcal{B} \otimes \mathcal{B}))$ is called the *Khalimsky plane* (cf. [1]). The connectedness graph of the Khalimsky topology is demonstrated in Figure 4.

It is a basic problem of digital geometry (cf. [10]) to find a structure on the digital plane $\mathbb{Z}^2$ convenient for the study of digital images. The convenience means that such a structure satisfies parallels of some basic geometric and topological properties of the Euclidean topology on the real plane $\mathbb{R}^2$. Of these parallels, the validity of an analogue of the Jordan curve theorem plays a crucial role because digital Jordan curves represent the borders of objects in digital images - see [11, 12]. It is well known that the Khalimsky topology provides such a structure on $\mathbb{Z}^2$ (cf. [1]). But it was shown in [13, 14] that there are some other topologies and closure operators on $\mathbb{Z}^2$ with this property.

According to [1], a circle C in the connectedness graph of the Khalimsky topology $f_1(\mathcal{B} \otimes \mathcal{B})$ is said to be a *Jordan curve* in the Khalimsky plane if the following two conditions are satisfied:

(1) with each of its points, C contains precisely two points adjacent to it,
(2) C separates the Khalimsky plane into exactly two components (i.e., the subspace $\mathbb{Z}^2 - C$ of $(\mathbb{Z}^2, f_1(\mathcal{B} \otimes \mathcal{B}))$ consists of exactly two components)..

Figure 5: A digital image of M.



As an immediate consequence of Theorem 5.6 proved in [1], we get the following digital Jordan curve theorem for the Khalimsky space $(\mathbb{Z}^2, f_1(\mathcal{B} \otimes \mathcal{B}))$: A circle C in the graph H_1 is a Jordan curve in the Khalimsky plane if and only if, at none of its points, it turns at an acute angle of $\frac{\pi}{4}$.

For every $n > 1$, we define (*digital*) *Jordan curves* in $(\mathbb{Z}^2, f_n(\mathcal{B} \otimes \mathcal{B}))$ to be the circles C in H_n that separate $(\mathbb{Z}^2, f_n(\mathcal{B} \otimes \mathcal{B}))$ into exactly two components.

Now the problem arises to determine, for every $n > 1$, those circles in H_n that are Jordan curves in $(\mathbb{Z}^2, f_n(\mathcal{B} \otimes \mathcal{B}))$. The following solution of the problem results from Theorem 3.19 proved in [15] (in quite a laborious way based on using quotient closure spaces and the Jordan curve theorem for the Khalimsky plane, namely, Theorem 5.6 in [1]):

Theorem 3.1. *Every circle in H_n that does not turn at any point $((2k+1)n, (2l+1)n)$, $k, l \in \mathbb{Z}$, is a Jordan curve in $(\mathbb{Z}^2, f_n(\mathcal{B} \otimes \mathcal{B}))$ whenever $n > 1$.*

Thus, the closure operators $f_n(\mathcal{B} \otimes \mathcal{B})$, $n > 1$, may be used as background structures on $\mathbb{Z}^2$ for the study of digital images. The advantage of using these closure operators rather than the Khalimsky topology $f_1(\mathcal{B} \otimes \mathcal{B})$ is that the Jordan curves with respect to them, i.e., circles in H_n, may turn at an acute angle of $\frac{\pi}{4}$ at some points - see the following example:

Example 3.2. Consider the set of points of $\mathbb{Z}^2$ demonstrated in Figure 5, which represents the (border of) letter M (the points may be regarded as centers of pixels in a computer screen). This set is not a Jordan curve in the Khalimsky plane $(\mathbb{Z}^2, f_1(\mathcal{B} \otimes \mathcal{B}))$. For it to be a Jordan curve in the Khalimsky plane, the eight points that are ringed have to be deleted. But this would lead to a certain deformation (loss of sharpness) of the letter - the eight pixels will belong to the white background of the black image of M. On the other hand, the set is a circle in the graph H_2 satisfying the assumption of Theorem 4 and, therefore, it is a Jordan curve in $(\mathbb{Z}^2, f_2(\mathcal{B} \otimes \mathcal{B}))$.

Acknowledgement: This work was supported by the Ministry of Education, Youth and Sports of the Czech Republic from the National Programme of Sustainability (NPU II) project "IT4Innovations Excellence in Science - LQ1602".

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