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## Research Article

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# Univariate approximating schemes and their non-tensor product generalization

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**Abstract:** This article deals with univariate binary approximating subdivision schemes and their generalization to non-tensor product bivariate subdivision schemes. The two algorithms are presented with one tension and two integer parameters which generate families of univariate and bivariate schemes. The tension parameter controls the shape of the limit curve and surface while integer parameters identify the members of the family. It is demonstrated that the proposed schemes preserve monotonicity of initial data. Moreover, continuity, polynomial reproduction and generation of the schemes are also discussed. Comparison with existing schemes is also given.

Keywords: Subdivision scheme, continuity, polynomial reproduction, monotonicity, non-tensor product

**MSC:** 65D17; 65D07; 65D05

# 1 Introduction

One of the important areas of study in Computer Aided Geometric Design is subdivision. Subdivision schemes have become very important for providing smooth curves and surfaces through an iterative process from a finite set of control points. At each step of iteration, a new set of points is created from the old points. In general, approximating subdivision schemes produce smoother curves and surfaces as compared to interpolating subdivision schemes.

Approximating schemes were first developed by Rham [1]. A famous corner cutting linear approximation scheme was introduced by Chaikin [2], which can generate the piecewise continuous  $C^1$  limiting curves. Consequent to this, a lot of work has been done by different authors in the area of binary approximating subdivision schemes. Mustafa et al. [3] presented the m-point binary approximating subdivision scheme. Zheng et al. [4] introduced a general formula to generate a family of integer-point binary approximating subdivision schemes with a parameter. Mustafa et al. [5] presented a family of (2n-1)-point binary approximating subdivision schemes with free parameters for describing curves. Khan and Mustafa [6] introduced a new approach to construct a non-tensor product  $C^1$  subdivision scheme for quadrilateral meshes. Zheng et al. [7] devised a multi-parameter method which generates a class of existing binary subdivision schemes. By using their method continuity of existing schemes can be increased up to  $C^{k+n}$  by multiplying the factor  $\left(\frac{1+z}{2}\right)^k$  with the symbol of the existing scheme.

Lane and Riesenfeld [8] then presented a unified framework to represent the uniform B-spline curves and their tensor product extensions by a subdivision process. This framework consists of two stages, the first

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stage doubles the control point by taking each point twice and the second stage is the midpoint averaging of these points.

Cashman et al. [9] presented the generalized Lane-Riesenfeld algorithm with 4-point variant. A subdivision step *T* is therefore

$$T = S^k R$$
,

where *R* is refine stage and *S* is smoothing stage.

Ashraf et al. [10] applied a six point variant on the Lane-Riesenfeld algorithm to generate a family of subdivision schemes by defining

$$Q_q = S_q^m W_q,$$

where  $W_a$  is refine stage and  $S_a$  is smoothing stage.

Hormann and Sabin [11] proposed a family of subdivision schemes with symbol  $a_k(z)$  by convolution of uniform B-spline with kernel given by

$$a_k(z) = 2\sigma(z)^k K_k(z),$$

where  $\sigma(z)$  is a smoothing operator of the B-spline and  $K_k(z)$  is a convolution of the order-k B-spline with the kernel

Conti and Romani [12] proposed a strategy for constructing dual *m*-ary approximating subdivision schemes of de Rham-type, starting from two primal schemes of arity *m* and 2 respectively. Symbol of their scheme is

$$c(z) = a_{odd}(z)b(z),$$

where  $a_{odd}(z)$  is the odd sub-symbol of a primal binary scheme and b(z) is the symbol of a primal m-ary scheme. Mustafa et al. [13] presented an algorithm that generates a family of binary univariate dual and primal approximating subdivision schemes, starting with two binary schemes, defined as

$$P_l(z) = (m_{even}(z))^l n(z),$$

where  $m_{even}(z)$  is the even sub-symbol of [14] and n(z) is the symbol of [11]. Romani [15] introduced an algorithm which generates the univariate and bivariate non-tensor product subdivision schemes with tension parameter. The symbol of the scheme is defined as

$$a_{n,w}(z) = (s(z))^n r_{n,w}(z),$$

where  $s(z) = \frac{1+z}{2}$  is smoothing stage while  $r_{n,w}(z)$  is refine stage.

#### 1.1 Motivation

All the above algorithms are also called Refine-Smooth algorithms. In these algorithms there is one smoothing operator followed by one refining operator. But in the proposed algorithm there are two smoothing operators followed by one refining operator. That is, we propose an algorithm which uses symbols of well known subdivision schemes, starting with three binary schemes i.e.

$$q_{m,n,\mu}(z) = (\alpha_{odd}(z))^m (\beta_{even}(z))^n \gamma_{\mu}(z),$$

where  $\alpha_{odd}(z)$  is the extracted odd sub-symbol of [4],  $\beta_{even}(z)$  is the extracted even sub-symbol of [14] and  $\gamma_{\mu}(z)$  is the symbol of [5]. The schemes produced by this algorithm are continuous up to  $C^{m+n+2}$ , where m and n are parameters that identify members of the family and play a crucial role in the continuity of the proposed schemes. The parameter  $\mu$  controls the shape of the limit curves of the schemes. Moreover, this algorithm

produces higher order continuous schemes compared with to the existing algorithms. This algorithm can easily be generalized to produce non-tensor product binary approximating schemes for surface generation. Furthermore, monotonicity preservation is also an important shape preserving property of subdivision schemes. In [16–21] the monotonicity of univariate schemes has been discussed. In this paper, we examine monotonicity preservation of univariate schemes and non-tensor product schemes.

The remainder of this article is organized into 3 sections. In Section 2, firstly we present an algorithm which generates a family of univariate binary approximating subdivision schemes with a tension parameter. Secondly, we discuss the smoothness analysis of univariate schemes and finally we discuss the monotonicity, polynomial generation and reproduction of the schemes. Section 3 extends the ideas presented in Section 2 to design a new family of non-tensor product subdivision schemes for quadrilateral meshes. The smoothness analysis of non-tensor product schemes is also discussed in the same section. In Section 3, we also discuss the monotonicity, polynomial generation and reproduction properties of non-tensor product subdivision schemes. Applications and conclusion are also given in this section.

# 2 Algorithm for univariate schemes

In this section, we present an algorithm for the construction of a family of binary approximating subdivision schemes.

For this, we consider the odd sub-symbol of cubic B-spline scheme [4]

$$\alpha_{odd}(z) = \frac{1+z}{2}. (1)$$

Similarly, the even sub-symbol of 4-point binary interpolating scheme [14] is

$$\beta_{even}(z) = \left(\frac{1+z}{2}\right) \left(-\frac{1}{8}z^2 + \frac{10}{8}z - \frac{1}{8}\right). \tag{2}$$

The symbol of the three point scheme [5] is given by

$$\gamma_{\mu}(z) = \left(\frac{1+z}{2}\right)^{3} \left(8\mu z^{2} + (2-16\mu)z + 8\mu\right). \tag{3}$$

Let us denote the family of the binary approximating subdivision scheme by  $P_{q_{m,n,u}}$ , where the general member of the proposed family has the symbol of the form

$$q_{m,n,\mu}(z) = (\alpha_{odd}(z))^m (\beta_{even}(z))^n \gamma_{\mu}(z). \tag{4}$$

Substituting (1), (2) and (3) in (4), we get the symbol of the scheme  $P_{q_{m,n,\mu}}$ 

$$q_{m,n,\mu}(z) = \left(\frac{1+z}{2}\right)^{m+n+3} \left(-\frac{1}{8}z^2 + \frac{10}{8}z - \frac{1}{8}\right)^n \left(8\mu z^2 + (2-16\mu)z + 8\mu\right),\tag{5}$$

where m and n are non-negative integers. As it is apparent that the symbol of the scheme  $P_{q_{m,n,\mu}}$  is dependent on the parameter  $\mu$  and on two other parameters m and n. The parameter  $\mu$  controls the shape of the limit curves of the schemes while m and n characterize the elements of the scheme  $P_{q_{m,n,u}}$ .

# 2.1 Smoothness analysis of univariate schemes

In this section, we discuss the continuity and Hölder continuity of the schemes. We use the theory of generating function [22] for continuity and Rioul's [23] method for Hölder continuity.

In the following theorem, we examine the convergence and smoothness of the scheme  $P_{q_{m,0,\nu}}$ .

**Theorem 2.1.** *The scheme*  $P_{q_{m,0,\mu}}$  *is*  $C^{m+2}$  *for*  $\mu \in (0, 0.125)$ .

*Proof.* Symbol of the scheme  $P_{q_{m,0,\mu}}$  is given by

$$q_{m,0,\mu}(z) = \left(\frac{1+z}{2}\right)^m a(z),$$
 (6)

where

$$a(z) = \left(\frac{1+z}{2}\right)^3 b(z),\tag{7}$$

and

$$b(z) = 8\mu z^2 + (2 - 16\mu)z + 8\mu$$
.

Let  $S_b$  be the scheme corresponding to the symbol b(z). Since

$$\left\|\frac{1}{2}S_b\right\|_{\infty} = \max\left\{\frac{1}{2}\sum_{j\in\mathbb{Z}}|b_{2j}|, \frac{1}{2}\sum_{j\in\mathbb{Z}}|b_{2j+1}|\right\},\,$$

then for  $\mu \in (0, 0.125)$ , we have

$$\left\|\frac{1}{2}S_b\right\|_{\infty} = \max\left\{\left|\frac{8\mu}{2}\right| + \left|\frac{8\mu}{2}\right|, \left|\frac{2-16\mu}{2}\right|\right\} < 1.$$

Hence  $S_b$  is contractive. Therefore, by Corollary 4.11 of [22], the scheme  $S_a$  is  $C^2$  for  $\mu \in (0, 0.125)$ . So by (6) scheme  $P_{q_{m,0,\mu}}$  is  $C^{m+2}$  for  $\mu \in (0, 0.125)$ .

Similarly, we can easily find out the continuity of other schemes  $P_{q_{m,n,\mu}}$  by taking into account the same formalism. The order of continuity of some proposed univariate subdivision schemes  $P_{q_{m,0,\mu}}$ ,  $P_{q_{m,1,\mu}}$ ,  $P_{q_{m,2,\mu}}$  and  $P_{q_{m,3,\mu}}$  for certain ranges of parameter is shown in Table 1. Hölder continuity is an extension to the notion

**Table 1:** The order of continuity O(C) of proposed binary approximating schemes for certain ranges of parameter.

n	Scheme	Ranges	O(C)	n	Scheme	Ranges	<i>O</i> ( <i>C</i> )
0	$P_{q_{m,0,\mu}}$	$-0.375 < \mu < 0.625$	$C^{m+0}$	2	$P_{q_{m,2,\mu}}$	$-0.195 < \mu < 0.445$	$C^{m+0}$
	•••	$-0.125 < \mu < 0.375$	$C^{m+1}$		•••	$-0.194 < \mu < 0.442$	$C^{m+1}$
	•••	$0 < \mu < 0.125$	$C^{m+2}$		•••	$-0.034 < \mu < 0.282$	$C^{m+2}$
					•••	$-0.026 < \mu < 0.235$	$C^{m+3}$
					•••	$0.045 < \mu < 0.09$	$C^{m+4}$
1	$P_{q_{m,1,\mu}}$	$-0.275 < \mu < 0.525$	$C^{m+0}$	3	$P_{q_{m,3,\mu}}$	-0.356 < <i>μ</i> < 0.618	$C^{m+0}$
	•••	$-0.075 < \mu < 0.3$	$C^{m+1}$		•••	$-0.131 < \mu < 0.380$	$C^{m+1}$
	•••	$-0.068 < \mu < 0.295$	$C^{m+2}$		•••	$-0.128 < \mu < 0.375$	$C^{m+2}$
	•••	$0.025 < \mu < 0.104$	$C^{m+3}$		•••	$-0.002 < \mu < 0.235$	$C^{m+3}$
					•••	$0.003 < \mu < 0.191$	$C^{m+4}$
					•••	$0.006 < \mu < 0.081$	$C^{m+5}$

of continuity. In the following theorem, we compute the Hölder continuity of the scheme  $P_{q_{m,0,u}}$ .

**Theorem 2.2.** The Hölder continuity of the scheme  $P_{q_{m,0,u}}$  is 3.

*Proof.* From (7), let  $b_0 = 8\mu$ ,  $b_1 = 2 - 16\mu$ ,  $b_2 = 8\mu$ , then  $M_0$ ,  $M_1$  are the matrices with elements

$$\begin{cases} (M_0)_{ij} = b_{2+i-2j}, \\ (M_1)_{ij} = b_{2+i-2j+1}, \end{cases}$$

where i, j = 1, 2. This implies

$$M_0 = \begin{pmatrix} 2 - 16\mu & 0 \\ 8\mu & 8\mu \end{pmatrix}, M_1 = \begin{pmatrix} 8\mu & 8\mu \\ 0 & 2 - 16\mu \end{pmatrix}.$$
 (8)

From (8) and [23], the spectral radius  $\lambda$  of the metrics  $M_0$  and  $M_1$  can be express as follows

$$\max\{2-16\mu, 2-16\mu\} \le \lambda \le \max\{2-16\mu, 2-16\mu\}$$
.

Since the largest eigenvalue and the max-norm of the metrics is 1 for  $\mu = 0.0625$ , where  $\mu \in (0, 0.125)$ , so the Hölder continuity  $h = 2 - \log_2(1) = 3$ . So by (6), the Hölder continuity of the scheme  $P_{q_{m,0,\mu}}$  is  $C^{m+3}$ .

Similarly, we can compute the Hölder continuity of other members of the family. If the largest eigenvalue

Table 2: Continuity of some members of the family of schemes

n	$\mu$ Continuity		Lower bound on Hölder continuity	Upper bound on Hölder continuity	
0	0.0625	$C^{m+2}$	$C^{m+3}$	$C^{m+3}$	
1	0.0375	$C^{m+3}$	$C^{m+3.255}$	$C^{m+3.2603}$	
2	0.0676	$C^{m+3}$	$C^{m+4.478}$	$C^{m+5}$	

and the max-norm of the metrics are not equal then we calculate the lower and upper bounds of the Hölder continuity. The lower bound of the Hölder continuity is  $h = 2 - \log_2(\|b\|^l)/l$  for some integer l and the upper bound of the Hölder continuity is  $h = 2 - \log_2(\lambda)$ . It is clear from Table 2 that as we increase n, the level of continuity and the Hölder continuity of the schemes  $P_{q_{m,n,\mu}}$  increase.

## 2.2 Response of univariate schemes to polynomial and monotone data

In this section, we examine the response of schemes to polynomial data by taking into account the polynomial generation and reproduction. We also examine the behaviour of the schemes for monotone data. We use the techniques developed in [15] to discuss polynomial generation and polynomial reproduction.

#### 2.2.1 Polynomial generation

The polynomial generation of degree d is the ability of subdivision scheme to generate the full space of polynomials up to degree d denoted by  $\pi_d$ . The generation degree of a subdivision scheme is the maximum degree of a polynomial that can potentially be generated by the scheme.

**Theorem 2.3.** The subdivision scheme  $P_{q_{m,n,\mu}}$  generates  $\pi_{m+n+2}$  for all  $m, n \in \mathbb{N}$ . Moreover, if  $\mu = \frac{1}{16}$ , then  $P_{q_{m,n,\mu}}$  generates  $\pi_{m+n+4}$ .

**Proof.** Since conditions

$$q_{m,n,u}(1) = 2$$
,  $q_{m,n,u}(-1) = 0$ ,  $D^{(k)}q_{m,n,u}(-1) = 0$ ,  $k = 1, 2, ..., m + n + 2$ ,

are verified by  $q_{m,n,\mu}(z)$  for all  $\mu \in \mathbb{R}$  and  $D^{(k)}$  denotes the kth derivative. Thus, in view of Proposition 2.1 of [15] degree of polynomial generation is m+n+2 for all  $\mu \in \mathbb{R}$ . Moreover, by setting  $\mu=\frac{1}{16}$  two more terms (1+z) can be factored out from  $q_{m,n,\mu}(z)$ , then we have  $D^{(k+1)}q_{m,n,\mu}(-1)=D^{(k+2)}q_{m,n,\mu}(-1)=0$ . So the degree of polynomial generation is m+n+4.

#### 2.2.2 Polynomial reproduction

Polynomial reproduction is an attractive property for a subdivision scheme. For a subdivision scheme to reproduce  $\pi_d$  it must be able to generate polynomials of the same degree as the limit functions for some initial data. The degree of polynomial reproduction can never exceed the degree of polynomial generation.

**Theorem 2.4.** If applying the parameter shift  $\tau = \frac{5+m+3n}{2}$ , then the subdivision scheme  $P_{q_{m,n,\mu}}$  reproduces  $\pi_1$  with respect to the parametrization in [15] for all  $m, n \in \mathbb{N}$  and  $\mu \in \mathbb{R}$ . Moreover, if  $\mu = -\frac{3+m}{32}$ , then  $P_{q_{m,n,\mu}}$  reproduces  $\pi_3$  for all  $m, n \in \mathbb{N}$ .

*Proof.* Since the condition  $D^{(1)}q_{m,n,\mu}(1)=5+m+3n$  is verified by the symbol  $q_{m,n,\mu}(z)$  for all  $\mu\in\mathbb{R}$ , so polynomial reproduction of  $P_{q_{m,n,\mu}}$  is  $\pi_1$  with the parameter shift  $\tau=\frac{5+m+3n}{2}$ . We observe that when  $\mu=-\frac{3+m}{32}$ , the following two more conditions

$$D^{(2)}q_{m,n,\mu}(z)|_{z=1} = 2\tau(\tau-1), \quad D^{(3)}q_{m,n,\mu}(z)|_{z=1} = 2\tau(\tau-1)(\tau-2),$$

are satisfied for all m,  $n \in \mathbb{N}$ . Thus reproduction of  $P_{q_{m,n,u}}$  is  $\pi_3$ .

## 2.3 Monotonicity preservation

Monotonicity preserving plays a key role in the shape preserving properties of subdivision schemes.

**Definition 2.1.** [18] Univariate data  $(x_i, f_i)$ , i = 0, 1, 2, ..., n is monotonically increasing if  $f_i < f_{i+1} \ \forall \ i = 0, 1, 2, ..., n$  and the derivative at the data points obey the condition  $d_i > 0 \ \forall \ i = 0, 1, 2, ..., n$ .

In the following, we examine monotonicity preservation of binary scheme  $P_{q_{1,0,\mu}}$  .

**Theorem 2.5.** Let  $\{f_i^0\}_{i\in\mathbb{Z}}$  satisfy

$$\dots f_{-1}^0 < f_0^0 < f_1^0 < \dots < f_{n-1}^0 < f_n^0 < f_{n+1}^0 \dots$$

Denote

$$d_i^k = f_{i+1}^k - f_i^k, \quad r_i^k = \frac{d_{i+1}^k}{d_i^k}, \quad R^k = \max_i \{r_i^k, \frac{1}{r_i^k}\}, \quad k \ge 0, \quad k \in \mathbb{Z}, \quad i \in \mathbb{Z}.$$

Furthermore, let  $0.1 \le \mu \le 0.9$  and  $\xi = -\frac{1}{\mu}$ ,  $\xi \in \mathbb{R}$ . If  $\frac{1}{\xi} \le R^0 \le \xi$ ,  $\{f_i^k\}$  is defined by the subdivision scheme  $P_{q_{1,0,\mu}}$ , then

$$d_i^k > 0, \quad \frac{1}{\xi} \le R^k \le \xi, \quad k \ge 0, \quad k \in \mathbb{Z}, \quad i \in \mathbb{Z}.$$
 (9)

*Proof.* We use mathematical induction to prove (9). When k=0,  $d_i^0=f_{i+1}^0-f_i^0>0$ ,  $\frac{1}{\xi}\leq R^0\leq \xi$ , then (9) is true.

Suppose that (9) holds for k,  $d_i^k = f_{i+1}^k - f_i^k > 0$ ,  $\frac{1}{\overline{\xi}} \le R^k \le \xi$ . Next we will prove that (9) holds for k+1. Consider

$$d_{2i}^{k+1} = \left\{ \left(\frac{1}{8} + \frac{1}{2}\mu\right)d_i^k + \left(\frac{3}{8} - \mu\right)d_{i+1}^k + \left(\frac{1}{2}\mu\right)d_{i+2}^k \right\}.$$

This implies

$$d_{2i}^{k+1} = d_i^k \left\{ \left( \frac{1}{8} + \frac{1}{2} \mu \right) + \left( \frac{3}{8} - \mu \right) r_i^k + \left( \frac{1}{2} \mu \right) r_{i+1}^k r_i^k \right\}.$$

This further implies

$$d_{2i}^{k+1} \geq d_i^k \left\{ \left(\frac{1}{8} + \frac{1}{2}\mu\right) + \left(\frac{3}{8} - \mu\right)\frac{1}{\xi} + \left(\frac{1}{2}\mu\right)\frac{1}{\xi^2} \right\}.$$

We know that  $d_i^k > 0$  and

$$d_i^k \left\{ \left( \frac{1}{8} + \frac{1}{2}\mu \right) + \left( \frac{3}{8} - \mu \right) \frac{1}{\xi} + \left( \frac{1}{2}\mu \right) \frac{1}{\xi^2} \right\} > 0, \text{ for } 0.1 \le \mu \le 0.9 \text{ and } \xi = -\frac{1}{\mu}.$$

This implies that  $d_{2i}^{k+1} > 0$ . Similarly, we see that  $d_{2i+1}^{k+1} > 0$  for  $0.1 \le \mu \le 0.9$  and  $\xi = -\frac{1}{\mu}$ . Now we prove that  $\frac{1}{\xi} \le R^{k+1} \le \xi$ . First we show that  $r_{2i}^{k+1} - \xi \le 0$ . Since

$$r_{2i}^{k+1} = \frac{d_{2i+1}^k}{d_{2i}^k} = \frac{\left\{ \left( \frac{1}{2}\mu \right) d_i^k + \left( \frac{3}{8} - \mu \right) d_{i+1}^k + \left( \frac{1}{8} + \frac{1}{2}\mu \right) d_{i+2}^k \right\}}{\left\{ \left( \frac{1}{8} + \frac{1}{2}\mu \right) d_i^k + \left( \frac{3}{8} - \mu \right) d_{i+1}^k + \left( \frac{1}{2}\mu \right) d_{i+2}^k \right\}},$$

then

$$r_{2i}^{k+1}-\xi \leq \frac{d_i^k\left\{\left(\frac{3}{8}+\frac{1}{2}\mu\right)\xi^2+\left(\frac{1}{4}-\frac{3}{2}\mu\right)\xi+\left(\frac{3}{2}\mu-\frac{3}{8}\right)+\left(-\frac{1}{2}\mu\right)\frac{1}{\xi}\right\}}{d_{i+1}^k\left\{\left(\frac{1}{8}+\mu\right)\xi+\left(\frac{3}{8}\mu\right)\right\}}.$$

The denominator and numerator of the right hand side of the above expression are less than and greater than zero respectively for  $0.1 \le \mu \le 0.9$  and  $\xi = -\frac{1}{\mu}$ .

This implies that

$$r_{2i}^{k+1}-\xi\leq 0.$$

It implies further that  $r_{2i}^{k+1} \le \xi$ . Now we show that  $\frac{1}{r_{2i}^{k+1}} - \xi < 0$ .

$$\frac{1}{r_{2i}^{k+1}} = \frac{d_{2i}^k}{d_{2i+1}^k} = \frac{\left\{ \left( \frac{1}{8} + \frac{1}{2}\mu \right) d_i^k + \left( \frac{3}{8} - \mu \right) d_{i+1}^k + \left( \frac{1}{2}\mu \right) d_{i+2}^k \right\}}{\left\{ \left( \frac{1}{2}\mu \right) d_i^k + \left( \frac{3}{8} - \mu \right) d_{i+1}^k + \left( \frac{1}{8} + \frac{1}{2}\mu \right) d_{i+2}^k \right\}}.$$

This implies

$$\frac{1}{r_{2i}^{k+1}} - \xi \leq \frac{d_i^k \{\frac{1}{2} \mu \xi^2 - \frac{1}{4} \xi - (\frac{1}{2} \mu + \frac{1}{4})\}}{d_{i+1}^k \{(\frac{1}{8} + \mu) \xi + (\frac{3}{8} - \mu)\}} \,.$$

The denominator and numerator of the right hand side of the above expression are less than and greater than zero respectively for  $0.1 \le \mu \le 0.9$  and  $\xi = -\frac{1}{\mu}$ . This implies that

$$\frac{1}{r_{2i}^{k+1}}-\xi\leq 0.$$

It implies further that  $\frac{1}{r_{2i}^{k+1}} \le \xi$ . In the same way, we can get  $r_{2i+1}^{k+1} \le \xi$  and  $\frac{1}{r_{2i+1}^{k+1}} \le \xi$ . So  $R^{k+1} \le \xi$ . Since  $R^{k+1} = \max_i \{r_i^k, \frac{1}{r_i^k}\}$ , it is obvious that  $R^{k+1} \ge \frac{1}{\xi}$ . This completes the proof.

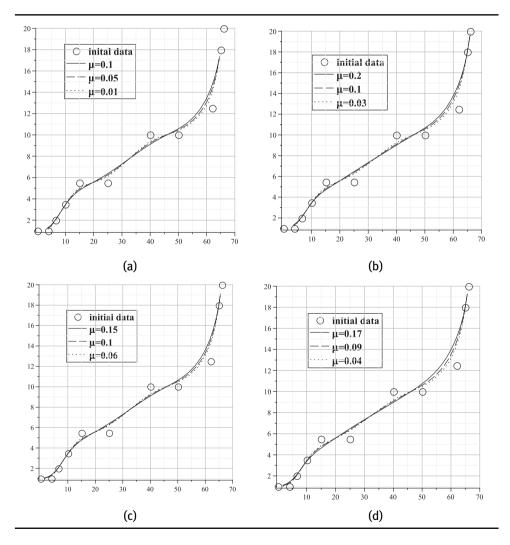
# 2.4 Numerical experiments of univariate schemes

In this section, we present the performance, geometrical behaviour and effect of a parameter on the limit curves of the schemes. We also present the response of the limit curves produced by the schemes towards the initial data.

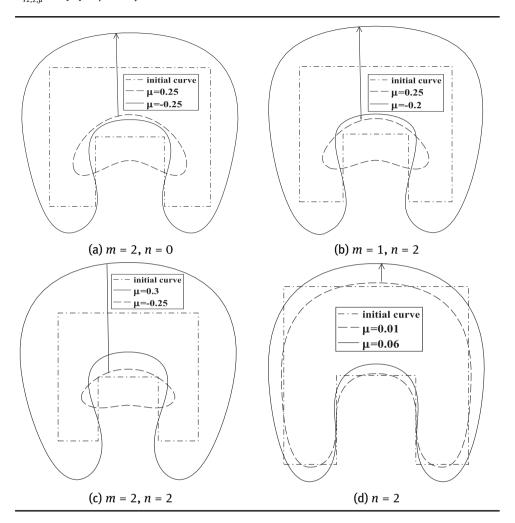
Table 3: Monotone data set [24].

i	1	2	3	4	5	6	7	8	9	10	11
$x_i$	0.1	4	6.5	10	15	25	40	50	62	65	66
$y_i$	1	1	2	3.5	5.5	5.5	10	10	12.5	18	20

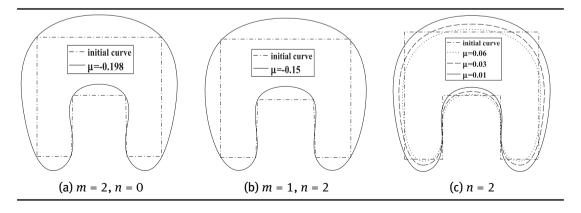
**Figure 1:** The curves (a), (b), (c) and (d) are generated by the schemes  $P_{q_{1,0,\mu}}$ ,  $P_{q_{1,1,\mu}}$ ,  $P_{q_{1,2,\mu}}$  and  $P_{q_{1,3,\mu}}$  respectively, using the monotone data set given below.



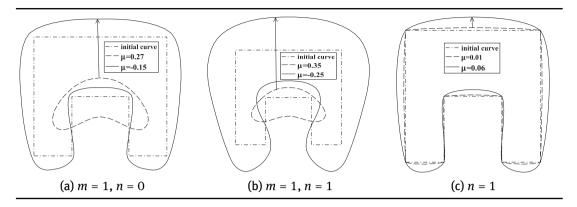
**Figure 2:** Most expanded and most shrinked curves: The curves (a), (b), (c) and (d) are generated by the schemes  $P_{q_{2,0,\mu}}$ ,  $P_{q_{1,2,\mu}}$ , and [15] respectively.



**Figure 3:** Interpolating behaviour: The curves (a) , (b) and (c) are generated by the schemes  $P_{q_{2,0,\mu}}$ ,  $P_{q_{1,2,\mu}}$  and [15] respectively.



**Figure 4:** Most expanded and most shrinked curves: The curves (a), (b) and (c) are generated by the schemes  $P_{q_{1,0,\mu}}$ ,  $P_{q_{1,1,\mu}}$  and [15] respectively.



**Figure 5:** Interpolating behaviour: The curves (a), (b) and (c) are generated by the schemes  $P_{q_{1,0,u}}$ ,  $P_{q_{1,1,u}}$  and [15] respectively.

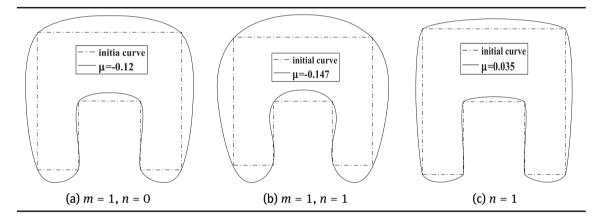


Figure 1 is produced by using the monotone data set given in Table 3. Figures 1(a)-1(d) are monotone curves obtained by the schemes  $P_{q_{1,0,\mu}}$ ,  $P_{q_{1,1,\mu}}$ ,  $P_{q_{1,2,\mu}}$  and  $P_{q_{1,3,\mu}}$  respectively.

The Figure 2-5 shows a comparison of proposed schemes with existing schemes [15]. Dashed dotted lines indicate the initial polygon. Solid lines show the most expanded curves and dashed lines show the most shrinked curves. Arrows show the distance between most expanded and most shrinked curves. Figures 2(a)-2(c) show that the most expanded and most shrinked curves are obtained by the schemes  $P_{q_{2,0,\mu}}$ ,  $P_{q_{1,2,\mu}}$  and  $P_{q_{2,2,\mu}}$  at different parametric values and Figure 2(d) shows the behaviour of existing scheme of [15].

We can see that the Figures 3(a)-3(b) represent the interpolating behaviour of proposed scheme  $P_{q_{2,0,\mu}}$ ,  $P_{q_{1,2,\mu}}$  respectively. Figure 3(c) shows the non-interpolating behaviour of [15] at any parametric value. The proposed schemes  $P_{q_{2,0,\mu}}$  and  $P_{q_{1,2,\mu}}$  show the approximating behaviour as well as the interpolating behaviour at different parametric values.

The Figures 4(a)-4(c) show the most expanded and most shrinked curves that are generated by the schemes  $P_{q_{1,0,\mu}}$ ,  $P_{q_{1,1,\mu}}$  and [15] at different parametric values respectively. The limit curves presented in Figures 5(a)-5(c) show the interpolating behaviour by of schemes  $P_{q_{1,0,\mu}}$ ,  $P_{q_{1,1,\mu}}$  and [15] respectively.

The schemes  $P_{q_{1,0,\mu}}$  and  $P_{q_{1,1,\mu}}$  have both approximating and interpolating behaviour while the scheme in [15] gives only interpolating behaviour.

# 3 Algorithm for non-tensor product schemes

By generalizing the algorithm as devised in Section 2, we get a family of non-tensor product approximating schemes with tension parameter  $\mu$  for quadrilateral meshes. Let  $\mathbf{P}_{\mathbf{q}_{m,n,\mu}}$  be the family of non-tensor product bivariate subdivision schemes then we propose the symbol of this family as

$$\mathbf{q}_{m,n,u}(z_1, z_2) = (\alpha_{odd}(z_1))^m (\beta_{even}(z_2))^n \gamma_{\mu}(z_1) \gamma_{\mu}(z_2). \tag{10}$$

By substituting m = 1 and n = 0 in (10), we get symbol of the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,u}}$  as follows:

$$\mathbf{q}_{1,0,\mu}(z_1, z_2) = \left(\frac{1+z_1}{2}\right)^4 \left(\frac{1+z_2}{2}\right)^3 \left(8\mu z_1^2 + (2-16\mu)z_1 + 8\mu\right) \times$$

$$\left(8\mu z_2^2 + (2-16\mu)z_2 + 8\mu\right).$$
(11)

The bivariate subdivision scheme  $\mathbf{P}_{\mathbf{q}_{1,0,u}}$  has the mask

$$\mathbf{q}_{1,0,\mu}(z_1, z_2) = \begin{pmatrix} \frac{1}{2}\mu^2 & \frac{1}{2}\mu^2 + \frac{1}{8}\mu & -\mu^2 + \frac{3}{8}\mu \\ \mu^2 + \frac{1}{8}\mu & \mu^2 + \frac{3}{8}\mu + \frac{1}{32} & 2\mu^2 + \frac{1}{2}\mu + \frac{3}{32} \\ -\frac{1}{2}\mu^2 + \frac{1}{2}\mu & -\frac{1}{2}\mu^2 + \frac{3}{8}\mu + \frac{1}{8} & \mu^2 - \frac{11}{8}\mu + \frac{3}{8} \\ -2\mu^2 + \frac{3}{4}\mu - 2\mu^2 + \frac{1}{4}\mu + \frac{3}{16} & 4\mu^2 - 3\mu + \frac{9}{16} \\ -\frac{1}{2}\mu^2 + \frac{1}{2}\mu & -\frac{1}{2}\mu^2 + \frac{3}{8}\mu + \frac{1}{8} & \mu^2 - \frac{11}{8}\mu + \frac{3}{8} \\ \mu^2 + \frac{1}{8}\mu & \mu^2 + \frac{3}{8}\mu + \frac{1}{32} - 2\mu^2 + \frac{1}{2}\mu + \frac{3}{32} \\ \frac{1}{2}\mu^2 & \frac{1}{2}\mu^2 + \frac{1}{8}\mu & -\mu^2 + \frac{3}{8}\mu \end{pmatrix}$$

$$-\mu^{2} + \frac{3}{8}\mu \qquad \frac{1}{2}\mu^{2} + \frac{1}{8}\mu \qquad \frac{1}{2}\mu^{2}$$

$$-2\mu^{2} + \frac{1}{2}\mu + \frac{3}{32} \qquad \mu^{2} + \frac{3}{8}\mu + \frac{1}{32} \qquad \mu^{2} + \frac{1}{8}\mu$$

$$\mu^{2} - \frac{11}{8}\mu + \frac{3}{8} \qquad -\frac{1}{2}\mu^{2} + \frac{3}{8}\mu + \frac{1}{8} - \frac{1}{2}\mu^{2} + \frac{1}{2}\mu$$

$$4\mu^{2} - 3\mu + \frac{9}{16} \qquad -2\mu^{2} + \frac{1}{4}\mu + \frac{3}{16} \qquad -2\mu^{2} + \frac{3}{4}\mu$$

$$\mu^{2} - \frac{11}{8}\mu + \frac{3}{8} \qquad -\frac{1}{2}\mu^{2} + \frac{3}{8}\mu + \frac{1}{8} - \frac{1}{2}\mu^{2} + \frac{1}{2}\mu$$

$$-2\mu^{2} + \frac{1}{2}\mu + \frac{3}{32} \qquad \mu^{2} + \frac{3}{8}\mu + \frac{1}{32} \qquad \mu^{2} + \frac{1}{8}\mu$$

$$-\mu^{2} + \frac{3}{8}\mu \qquad \frac{1}{2}\mu^{2} + \frac{1}{8}\mu \qquad \frac{1}{2}\mu^{2}$$

$$(12)$$

## 3.1 Smoothness analysis of bivariate proposed schemes

Here, we use the theory of generating function [22] to derive continuity of non-tensor product schemes.

**Theorem 3.1.** If  $\mu \in (-0.2215, 0.4785)$  then the subdivision scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  converges to a continuous surface when starting from any regular quadrilateral mesh. Moreover, if  $\mu \in (-0.05178, 0.3017)$  and  $\mu \in (-0.0517, 0.25)$ , then the limit surfaces generated by scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  have  $C^1$  and  $C^2$ -continuity respectively.

*Proof.* From (11), we have

$$b_{1,0,\mu}(z_1,z_2) = \left(8\mu z_1^2 + (2-16\mu)z_1 + 8\mu\right) \left(8\mu z_2^2 + (2-16\mu)z_2 + 8\mu\right).$$

In view of [22](Theorem 4.30), we can determine the range of the parameter  $\mu$  which guarantees the convergence of the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  by checking the contractivity of the scheme. Since the scheme with symbol  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^3\left(\frac{1+z_2}{2}\right)^3b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^4\left(\frac{1+z_2}{2}\right)^2b_{1,0,\mu}(z_1,z_2)$  is contractive for  $\mu\in(-0.2215,\ 0.4785)$  and then scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  is convergent for  $\mu\in(-0.2215,\ 0.4785)$ . In the same way, the scheme with symbol  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^3b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^3\left(\frac{1+z_2}{2}\right)^2b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^4\left(\frac{1+z_2}{2}\right)^3b_{1,0,\mu}(z_1,z_2)$  is contractive for  $\mu\in(-0.05178,\ 0.3017)$  therefore the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  is  $C^1$ -continuous. Again since, the scheme with symbol  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)\left(\frac{1+z_2}{2}\right)^3b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^2b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^4\left(\frac{1+z_2}{2}\right)^3b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^4b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^2\left(\frac{1+z_2}{2}\right)^4b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}{2}\right)^4b_{1,0,\mu}(z_1,z_2)$ ,  $\frac{1}{2}\left(\frac{1+z_1}$ 

**Table 4:** The order of continuity O(C) of proposed non-tensor product schemes with some existing non-tensor product schemes.

Scheme	Туре	<i>O</i> ( <i>C</i> )
Binary non-tensor product [15]	Interpolating	$C^1$
Binary non-tensor product [15]	Approximating	$C^1$
Binary non-tensor product [6]	Approximating	$C^1$
Proposed binary non-tensor product $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$	Approximating	$C^2$
Proposed binary non-tensor product $\mathbf{P}_{\mathbf{q}_{1,1,\mu}}$	Approximating	$C^3$

In Table 4, we compare the continuity of proposed non-tensor product schemes with some existing binary non-tensor product schemes. It is observed that the continuity of proposed schemes is better than the continuity of existing schemes.

# 3.2 Response of non-tensor product schemes to polynomial and monotone data

In this section, we investigate the capability of the non-tensor product approximating subdivision schemes  $\mathbf{P}_{\mathbf{q}_{1,0,y}}$  and  $\mathbf{P}_{\mathbf{q}_{1,1,y}}$  of generating and reproducing polynomials as well as monotonicity preservation of the data.

**Theorem 3.2.** The subdivision scheme  $P_{q_{1,0,\mu}}$  generates  $\pi_2$  for all  $\mu \in \mathbb{R}$  and generates  $\pi_4$  for  $\mu = \frac{1}{16}$ .

*Proof.* Let  $w_1 = (1, -1)$ ,  $w_2 = (-1, 1)$ ,  $w_3 = (-1, -1)$  and let  $D^j$  with  $j \in \mathbb{N}^2$ , denote a directional derivative. Since  $\mathbf{q}_{1,0,y}(1, 1) = 4$  and

$$D^{(1,0)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(1,0)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(1,0)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$
  
$$D^{(0,1)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(0,1)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(0,1)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

then scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  generates  $\pi_1$  for all  $\mu \in \mathbb{R}$ . Again since

$$D^{(1,1)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(1,1)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(1,1)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(2,0)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(2,0)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(2,0)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(0,2)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(0,2)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(0,2)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

then the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  generates  $\pi_2$  for all  $\mu \in \mathbb{R}$ . Further

$$D^{(2,1)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(2,1)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(2,1)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(1,2)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(1,2)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(1,2)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(3,0)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(3,0)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(3,0)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(0,3)}\mathbf{q}_{1,0,\mu}(w_1) = 48\mu - 3, \quad D^{(0,3)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(0,3)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

so the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  generates  $\pi_3$  for  $\mu = \frac{1}{16}$ . Further more

$$D^{(2,2)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(2,2)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(2,2)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(3,1)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(3,1)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(3,1)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(1,3)}\mathbf{q}_{1,0,\mu}(w_1) = 144\mu - 9, \quad D^{(1,3)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(1,3)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(4,0)}\mathbf{q}_{1,0,\mu}(w_1) = 0, \quad D^{(4,0)}\mathbf{q}_{1,0,\mu}(w_2) = 96\mu - 6, \quad D^{(4,0)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

$$D^{(0,4)}\mathbf{q}_{1,0,\mu}(w_1) = 48\mu - 3, \quad D^{(0,4)}\mathbf{q}_{1,0,\mu}(w_2) = 0, \quad D^{(0,4)}\mathbf{q}_{1,0,\mu}(w_3) = 0,$$

so the scheme  $\mathbf{P}_{\mathbf{q}_{1,0}\mu}$  generates  $\pi_4$  for  $\mu = \frac{1}{16}$ . This completes the proof.

**Theorem 3.3.** For the parameter shift  $(\tau_1, \tau_2) = (\frac{12}{4}, \frac{10}{4})$ , the subdivision scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  reproduces  $\pi_1$  with respect to the parametrization defined in [15] for all  $\mu \in \mathbb{R}$ .

*Proof.* Let  $D^j$  with  $j \in \mathbb{N}^2$ , denote a directional derivative. Since the symbol  $\mathbf{q}_{1,0,\mu}(z_1,z_2)$  satisfies the conditions in Theorem 3.2. Since  $\mathbf{q}_{1,0,\mu}(1,1)=4$  and

$$D^{(1,0)}\mathbf{q}_{1,0,\mu}(1,1) - 4\tau_1 = 0, \quad D^{(0,1)}\mathbf{q}_{1,0,\mu}(1,1) - 4\tau_2 = 0,$$

then the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  produced  $\pi_1$  for all  $\mu \in \mathbb{R}$ .

Similarly, we can prove the following theorems.

**Theorem 3.4.** The subdivision scheme  $P_{q_{1,1,\mu}}$  generates  $\pi_3$  for all  $\mu \in \mathbb{R}$  and generates  $\pi_4$  for  $\mu = \frac{1}{16}$ .

**Theorem 3.5.** If applying the parameteric shift  $(\tau_1, \tau_2) = (3, 4)$ , the subdivision scheme  $\mathbf{P}_{\mathbf{q}_{1,1,\mu}}$  reproduces  $\pi_1$  with respect to the parametrization in [15] for all  $\mu \in \mathbb{R}$ .

Now, we examine monotonicity preservation of the binary non-tensor product approximating subdivision scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$ .

**Definition 3.1.** [18] Bivariate data  $(x_i, y_j, f_{i,j})$ , i = 0, 1, 2, ..., n and j = 0, 1, 2, ..., m, where  $x_1 < x_2 < ... < x_n$  and  $y_1 < y_2 < ... < y_m$  are said to be monotonically increasing if  $f_{i,j} < f_{i+1,j}$  and  $f_{i,j} < f_{i,j+1} \ \forall i = 0, 1, 2, ..., n$  and j = 0, 1, 2, ..., m, if the derivative at the data points obey the condition  $d_{i,j} > 0 \ \forall i = 0, 1, 2, ..., n$  and j = 0, 1, 2, ..., m.

**Theorem 3.6.** Suppose that the initial data  $\{f_{i,j}^0\} = (x_i^0, y_j^0, f_{i,j}^0)$  are strictly monotonically increasing for all  $i, j \in \mathbb{Z}$ . Denote

$$\begin{split} d^k_{i,j} &= f^k_{i+1,j+1} - f^k_{i+1,j} - f^k_{i,j+1} + f^k_{i,j}, \\ y^k_{i,j+t} &= \frac{d^k_{i+1,j+t}}{d^k_{i,j+t}}, \quad y^k_{i+1,j+t} = \frac{d^k_{i+2t,j+t+1}}{d^k_{i+1,j+t}}, \\ Y^k_{i,j+t} &= \max_{i,j} \{y^k_{i,j+t}, \frac{1}{y^k_{i,j+t}}\}, \qquad Y^k_{i+1,j+t} = \max_{i,j} \{y^k_{i+1,j+t}, \frac{1}{y^k_{i+1,j+t}}\}, \end{split}$$

where t = 0, 1 and  $k \ge 0, k \in \mathbb{Z}, i, j \in \mathbb{Z}$ .

Furthermore, let  $0.1 \le \mu \le 0.9$  and  $\delta = -\frac{1}{\mu}$ ,  $\delta \in \mathbb{R}$ . If  $\frac{1}{\delta} \le Y^0_{i,j+t}$ ,  $Y^0_{i+1,j+t} \le \delta$ ,  $\{f^k_{i,j}\}$  is defined by the subdivision scheme  $\mathbf{P}_{q_{1,0,\mu}}$ , then

$$d_{i,j}^k > 0, \quad \frac{1}{\delta} \le Y_{i,j+t}^k, Y_{i+1,j+t}^k \le \delta, \quad k \ge 0, \quad k \in \mathbb{Z}, \quad i, j \in \mathbb{Z}.$$
 (13)

*Proof.* We use mathematical induction to prove (13). When k=0,  $d_{i,j}^0>0$ ,  $\frac{1}{\delta}\leq Y_{i,j+t}^0$ ,  $Y_{i+1,j+t}^0\leq \delta$ , then (13) is true.

Suppose that (13) holds for k i.e.  $d_{i,j}^k > 0$ ,  $\frac{1}{\delta} \le Y_{i,j+t}^k$ ,  $Y_{i+1,j+t}^k \le \delta$ . Next we will prove that (13) holds for k+1. First we show that  $d_{2i,2j}^{k+1} > 0$ . Consider

$$d_{2i,2j}^{k+1} = f_{2i+1,2j+1}^{k+1} - f_{2i+1,2j}^{k+1} - f_{2i,2j+1}^{k+1} + f_{2i,2j}^{k+1}.$$

After some simplification and substituting  $\delta = -\frac{1}{u}$ , we get

$$\begin{split} d_{2i,2j}^{k+1} &= d_{i,j+3}^k \left\{ -\frac{27}{2} \mu^{11} + \frac{153}{8} \mu^{10} - \frac{363}{16} \mu^9 + \frac{781}{32} \mu^8 - \frac{831}{32} \mu^7 + \frac{881}{32} \mu^6 - \frac{711}{64} \mu^5 \right. \\ &\left. + \frac{377}{64} \mu^4 - \frac{105}{32} \mu^3 + 2 \mu^2 - \frac{19}{32} \mu + \frac{5}{32} \right\} \,. \end{split}$$

We know that  $d_{i,j+3}^k > 0$  and

$$\left\{ -\frac{27}{2}\mu^{11} + \frac{153}{8}\mu^{10} - \frac{363}{16}\mu^{9} + \frac{781}{32}\mu^{8} - \frac{831}{32}\mu^{7} + \frac{881}{32}\mu^{6} - \frac{711}{64}\mu^{5} + \frac{377}{64}\mu^{4} - \frac{105}{32}\mu^{3} + 2\mu^{2} - \frac{19}{32}\mu + \frac{5}{32} \right\} > 0.$$

This implies that  $d_{2i,2j}^{k+1} > 0$ . Similarly, we see that  $d_{2i+1,2j}^{k+1} > 0$ ,  $d_{2i,2j+1}^{k+1} > 0$  and  $d_{2i+1,2j+1}^{k+1} > 0$  for  $0.1 \le \mu \le 0.9$  and  $\delta = -\frac{1}{\mu}$ .

Now we prove that  $\frac{1}{\delta} \le Y_{i,j+t}^k$ ,  $Y_{i+1,j+t}^k \le \delta$ . First we show that  $y_{2i,2j}^{k+1} - \delta \le 0$ . For this, consider

$$y_{2i,2j}^{k+1} - \delta = \frac{d_{2i+1,2j}^{k+1}}{d_{2i,2j}^{k+1}} - \delta.$$

After some simplification and substituting  $\delta = -\frac{1}{\mu}$ , we get

$$y_{2i,2j}^{k+1}-\delta\leq\frac{\psi_1}{\psi_2},$$

where

$$\psi_1 = \left\{ -\frac{9}{8}\mu^3 + \frac{549}{32}\mu^2 - \frac{287}{8}\mu + \frac{2885}{64} + \frac{5}{16\mu^8} - \frac{53}{32\mu^7} + \frac{285}{32\mu^6} - \frac{341}{16\mu^5} + \frac{1543}{32\mu^4} \right.$$
$$\left. -\frac{821}{16\mu^3} + \frac{3461}{64\mu^2} - \frac{1679}{32\mu} \right\},$$

and

$$\psi_2 = \left\{ -\frac{9}{8}\mu^3 + \frac{513}{32}\mu^2 - \frac{635}{32}\mu + \frac{1615}{64} - \frac{5}{32\mu^7} + \frac{3}{4\mu^6} - \frac{69}{16\mu^5} + \frac{41}{4\mu^4} - \frac{783}{32\mu^3} \right\}$$

$$+\frac{859}{32\mu^2}-\frac{1743}{64\mu}$$
.

The denominator and numerator of the right hand side of the above expression are less than and greater than zero respectively for  $0.1 \le \mu \le 0.9$ . This implies that

$$y_{2i,2i}^{k+1}-\delta\leq 0.$$

Further this implies that  $y_{2i,2j}^{k+1} \le \delta$ . Now we show that  $\frac{1}{y_{2i,2j}^{k+1}} - \delta < 0$ . For this, consider

$$\frac{1}{y_{2i,2i}^{k+1}} - \delta = \frac{d_{2i,2j}^k}{d_{2i,2i+1}^k} - \delta.$$

After some simplification and substituting  $\delta = -\frac{1}{u}$ , we get

$$\frac{1}{y_{2i,2i}^{k+1}} - \delta \leq \frac{\chi_1}{\chi_2},$$

where

$$\begin{split} \chi_1 &= \left\{ -\frac{9}{8}\mu^3 + \frac{549}{32}\mu^2 - \frac{287}{8}\mu + \frac{2885}{64}\frac{5}{32\mu^9} + \frac{29}{32\mu^8} - \frac{19}{4\mu^7} + \frac{189}{16\mu^6} - \frac{449}{16\mu^5} + \frac{1111}{32\mu^4} \right. \\ &\left. - \frac{821}{16\mu^3} + \frac{3461}{64\mu^2} - \frac{1679}{32\mu} \right\}, \end{split}$$

and

$$\begin{split} \chi_2 &= \left\{ -\frac{9}{8}\mu^3 + \frac{513}{32}\mu^2 - \frac{635}{32}\mu + \frac{1615}{64} + \frac{5}{32\mu^8} - \frac{29}{32\mu^7} + \frac{147}{32\mu^6} - \frac{177}{16\mu^5} + \frac{95}{4\mu^4} - \frac{783}{32\mu^3} \right. \\ &\left. + \frac{859}{32\mu^2} - \frac{1743}{64\mu} \right\}. \end{split}$$

The denominator and numerator of the right hand side of the above expression are greater than and less than zero respectively for  $0.1 \le \mu \le 0.9$ . This implies that

$$\frac{1}{y_{2i,2j}^{k+1}} - \delta \le 0.$$

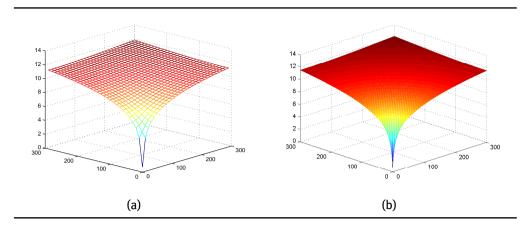
Further this implies that  $\frac{1}{y_{2i+1}^{k+1}} \le \delta$ . In the same way, we can get  $y_{2i,2j+1}^{k+1} \le \delta$ ,  $y_{2i+1,2j}^{k+1} \le \delta$ ,  $y_{2i+1,2j+1}^{k+1} \le \delta$ ,  $y_{2i+1,2j+1}^{k+1} \le \delta$ ,  $y_{2i+1,2j+1}^{k+1} \le \delta$ , and  $y_{2i+1,2j+1}^{k+1} \le \delta$ . Since  $y_{i,j+t}^{k} = \max_{i,j} \{y_{i,j+t}^{k}, \frac{1}{y_{i,j+t}^{k}}\}$  and  $y_{i+1,j+t}^{k} = \max_{i,j} \{y_{i+1,j+t}^{k}, \frac{1}{y_{i+1,j+t}^{k}}\}$ , it is obvious that  $y_{i,j+t}^{k}$ ,  $y_{i+1,j+t}^{k} \ge \frac{1}{\delta}$ . This completes the proof.

## 3.3 Numerical experiments of non-tensor product schemes

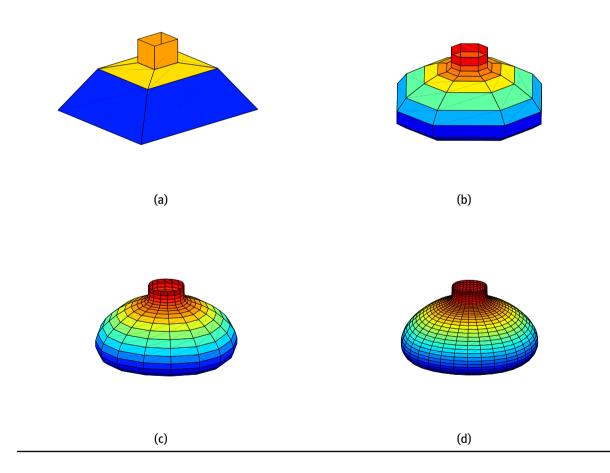
In this section, we show the performance, geometrical behaviour and effect of a parameter on the limit surfaces of the schemes  $P_{q_{1,0,\mu}}$  and  $P_{q_{1,1,\mu}}$ .

The monotone data set given in Table 5 has been used to produce monotone surfaces. Figure 6(a) is the initial mesh of the monotone data. Figure 6(b) is the monotone surface generated by the scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  for  $\mu=0.5$ . Figure 7(a) is the initial control mesh while Figures 7(b)-7(d) are the surfaces produced by the proposed scheme  $\mathbf{P}_{\mathbf{q}_{1,0,\mu}}$  at first, second and third subdivision levels with  $\mu=0.1$  respectively. Figure 8(a) is the initial control mesh while Figures 8(b)-8(d) are the surfaces produced by the proposed scheme  $\mathbf{P}_{\mathbf{q}_{1,1,\mu}}$  at first, second and third subdivision levels with  $\mu=0.15$  respectively.

**Figure 6:** (a) Initial monotone data. (b) A monotonicity preserving surface obtained by the proposed scheme  $P_{q_{1,0,\mu}}$ .



**Figure 7:** (a) Control mesh. (b)-(d) Surfaces obtained by the proposed schemes  $P_{q_{1,0,\mu}}$  at first, second and third subdivision levels respectively.



# 3.4 Conclusion

In this paper, we have proposed two algorithms to generate the families of univariate and bivariate approximating subdivision schemes with one tension and two integer parameters. The integer parameters identify

**Figure 8:** (a) Control mesh. (b)-(d) Surfaces obtained by the proposed schemes  $P_{q_{1,1,\mu}}$  at first, second and third subdivision levels respectively.

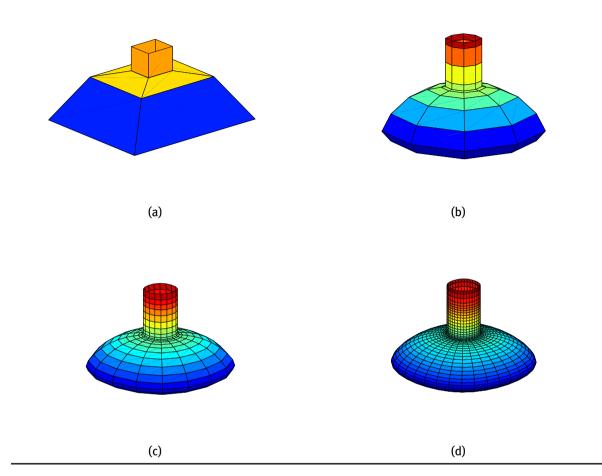


Table 5: Monotone data set [25].

x/y	1	100	200	300
1	0.6931	9.2104	10.5967	11.4076
100	9.2104	9.9035	10.8198	11.5129
200	10.5967	10.8198	11.2898	11.7753
300	11.4076	11.5129	11.7753	12.1007

members of the proposed family. It has been shown that the proposed schemes have higher continuity and Hölder continuity compared with existing schemes. Comparison of the continuity of proposed non-tensor product schemes with some of the existing non-tensor schemes has also been given. It has been demonstrated through several examples that geometrical behaviour of the univariate and bivariate subdivision schemes depends on the tension parameter. Monotonicity preservation of proposed univariate and bivariate schemes has been proved. Moreover, polynomial reproduction and generation of the proposed schemes have also been discussed.

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