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Spectrum of free-form Sudoku graphs

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Abstract: A free-form Sudoku puzzle is a square arrangement of $m \times m$ cells such that the cells are partitioned into m subsets (called blocks) of equal cardinality. The goal of the puzzle is to place integers $1, \ldots, m$ in the cells such that the numbers in every row, column and block are distinct. Represent each cell by a vertex and add edges between two vertices exactly when the corresponding cells, according to the rules, must contain different numbers. This yields the associated free-form Sudoku graph. This article studies the eigenvalues of free-form Sudoku graphs, most notably integrality. Further, we analyze the evolution of eigenvalues and eigenspaces of such graphs when the associated puzzle is subjected to a 'blow up' operation, which scales the cell grid including its block partition.

Keywords: Sudoku, Spectrum, Eigenvectors

MSC: 05C50, 15A18

1 Introduction

The recreational game of Sudoku has been popular for several years now. Its classic variant is played on a board with 9×9 cells, subdivided into a 3×3 grid of square blocks containing 3×3 cells each. Each cell may be empty or contain one of the numbers $1, \ldots, 9$. A number of cells of each puzzle will have been pre-filled by the puzzle creator. The goal of the puzzle solver is to fill the remaining cells with the numbers $1, \ldots, 9$ such that in the completed puzzle the number in each cell occurs only once per row, column and block.

Let us call the classical variant the 3-Sudoku. Its $3^2 \times 3^2$ board contains 3^4 cells. One can readily generalize the game to n-Sudokus played on an $n^2 \times n^2$ board with n^4 cells, subdivided into n^2 square blocks with n^2 cells in each. The permitted numbers in the cells now range from $1, \ldots, n^2$, but the restrictions for a valid solution remain the same. The 2-Sudoku is also known as a Shidoku.

Despite the seemingly recreational character of the game, it offers a surprising number of mathematical facets. This makes Sudoku an interesting topic for mathematics lessons [1, 2]. Among the topics touched by Sudoku are problem solving, latin squares, counting, exhausting symmetry and colouring problems on graphs (see, for example, the introductory book [3]). Due to this, more and more interesting results about Sudoku have been published in the recent past. For instance, different approaches for solvers have been presented in [4–6]. The combinatorial properties of completed Sudoku squares as a family of Latin squares, in particular the search for orthogonal pairs, have been considered in [7–10]. Other researchers focus on algebraic aspects of Sudoku, especially groups and rings associated with Sudoku (cf. [11–13]).

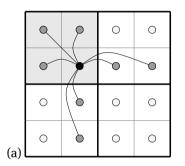
One of the most intuitive links between Sudoku and mathematics is that the processes of solving a Sudoku can be interpreted as completing a given partial vertex colouring of a certain graph (each preassigned colour

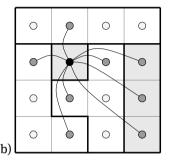
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corresponds to a prefilled number in a cell, a so-called clue). To this end, we represent each cell of the given Sudoku square by a single vertex. Two vertices are adjacent if and only if the associated cells must not contain the same number (according to the rules of the game). Figure 1 (a) depicts the adjacencies in the neighborhood of an exemplary vertex of the 2-Sudoku graph. Using this representation, the valid solutions of a given n-Sudoku can be characterized as proper optimal vertex colourings (using n^2 colors each) of the associated Sudoku graph.

Figure 1: Deriving the graph of a classic and a free-form 2-Sudoku puzzle





Sudoku graphs have also been studied from the perspective of algebraic graph theory. It has been shown that every Sudoku graph is integral, i.e., all eigenvalues of its adjacency matrix are integers. This has been shown algebraically by means of group characters [14], but it also follows from the fact that Sudoku graphs are actually NEPS of complete graphs. Hence they belong to the class of gcd-graphs, a subclass of the integral Cayley graphs over abelian groups [15]. Here, NEPS is the common short form of the non-complete extended *p*-sum, a generalized graph product that includes many known products [16].

There exist many generalizations and variants of the classic Sudoku, the n-Sudokus being the most common one. Besides changing the size of a Sudoku one can also try to vary every other aspect of a Sudoku, e.g. change the rules, introduce additional rules or change the shape of the blocks. The book [3] presents many such variants. Changing the shape of the blocks leads to the notion of a *free-form Sudoku*. Given a square arrangement of $m \times m$ cells (where m need not be a square number any more), we permit the blocks to be an arbitrary partition T of the cells into m subsets of equal cardinality. Note that the blocks are not required to be contiguous arrangements of cells. The Sudoku rules remain unchanged, in the sense that we need to fill in the numbers $1, \ldots, m$ such that each row, column and block contains m distinct numbers. We shall denote the associated graph by FSud(m, T). Figure 1 (b) illustrates the adjacencies in the neighborhood of an exemplary vertex for the graph of a free-form 2-Sudoku. The cells of the (non-contiguous) block containing the considered vertex have been shaded.

It seems that free-form Sudokus have not been studied in the literature so far although they are included in many Sudoku puzzle books that offer challenging variants. Computer experiments readily indicate that integrality of the eigenvalues critically depends on the degree of symmetry exhibited by the chosen cell partition. However, free-form Sudoku graphs are well structured enough to predict the changes of their spectra and eigenspaces exactly, when they are transformed in certain ways.

2 Integrality

In the following, consider a given free-form Sudoku graph FSud(m, T) and let A be its adjacency matrix (with respect to some arbitrary but fixed vertex order). The structure of the adjacency matrix of a free-form Sudoku graph is governed by the rules of Sudoku:

Observation 2.1. Two vertices of a free-form Sudoku graph are adjacent if and only if one of the following mutually exclusive places applies:

- (B) The associated cells belong to the same block of the tiling.
- (H) The associated cells belong to the same row of the puzzle, but not to the same block.
- (V) The associated cells belong to the same column of the puzzle, but not to the same block.

The chosen partition into mutually exclusive cases immediately gives rise to a decomposition of the adjacency matrix according to these three cases:

$$A = L_B + L_H + L_V \in \mathbb{R}^{m^2 \times m^2} \tag{1}$$

In this sense, the graph FSud(m, T) can be interpreted as a composition of three layers of edges, according to the cases (B), (H), (V). These layers can be viewed and studied as graphs in their own right.

Proposition 2.2. For $i, j \in \{1, ..., m\}$ let p_{ij} denote the number of cells in the i-th row (resp. column) of the puzzle that belong to the j-th block of the partition. Then L_H (resp. L_V) is an adjacency matrix of the union of m disjoint complete multipartite graphs $K_{p_{i1},...,p_{im}}$, i = 1,...,m.

Proof. Consider a fixed row i of the given Sudoku puzzle. We group the cells of the considered row according to their respective block membership. In view of rule (H) we see that the vertices corresponding to the grouped cells are adjacent if and only if they belong to different blocks. So the groups of cells form m independent sets of the respective sizes p_{i1}, \ldots, p_{im} , further all edges exist between different groups. The reasoning for L_V is analogous; just consider a fixed column and rule (V).

Proposition 2.3. L_B is an adjacency matrix of the union of m disjoint complete graphs K_m .

Proof. Group the vertices according to block membership of their associated cells. Since each block contains m cells the result follows from rule (B).

Proposition 2.4. The following statements are equivalent:

- (a) L_H (resp. L_V) represents a regular graph.
- (b) L_H (resp. L_V) has constant row sum.
- (c) L_B commutes with L_H (resp. L_V).

Proof. Statements (a) and (b) are clearly equivalent. Next we show that statement (b) implies (c). To this end, note that $(L_H)_{is}(L_B)_{sj}=1$ if and only if $(L_H)_{is}=(L_B)_{sj}=1$. This translates to the requirement that the cells associated with vertices i and s are in the same row (but not the same block) and that the cells associated with vertices s and j belong to the same block. Therefore, $(L_HL_B)_{ij}$ counts the vertices s that meet the mentioned requirement for given i, j. In a similar manner we see that $(L_BL_H)_{ij}$ counts the vertices s such that the cells associated with vertices s and s belong to same block and that the cells associated with vertices s and s are in the same row (but do not belong to the same block). If, however, s has constant row sum, then it follows by rule (H) that each cell has the same common number of cells that are in the same row but not in the same block as the considered cell. Equivalently, since all rows have the same number of cells, each cell has the same common number s of cells that are in the same row and in the same block as the considered cell. Therefore, any block that has cells in a row must have exactly s cells in that row. Hence the two counts mentioned before are identical: s and s if s if s is an area of s in the same row and in that row. Hence the two counts mentioned before are identical: s in s

Conversely, assume that statement (c) is true. Consider two vertices i and j such that their associated cells belong to different rows and different blocks. Then $(L_BL_H)_{ij} = (L_HL_B)_{ij}$ implies that these two blocks have the same number of cells in the two rows. Hence there exists a common number k such that any block that has cells in a row must have exactly k cells in that row. Considering a single row of L_H , it follows that it has row sum m - k since the puzzle row of the cell associated with that matrix row contains m cells and among them k cells that belong to the same block as the considered cell.

Considering the third statement of Proposition 2.4, note that L_H and L_V need not commute even when both L_H and L_V have constant row sum, cf. the graph FSud(4, T) for $T = \{\{1, 8, 9, 16\}, \{3, 6, 10, 15\}, \{2, 7, 11, 14\}, \{4, 5, 12, 13\}\}$ (here we number the cells from left to right, row after row).

Theorem 2.5 (see [17]). The characteristic polynomial of the complete multipartite graph $G = K_{p_1,...,p_k}$ is

$$\chi(G) = x^{n-k} \left(x^k - \sum_{m=2}^k (m-1)\sigma_m x^{k-m} \right),$$

where $\sigma_1 = \sum_{1 \le i \le k} p_i$, $\sigma_2 = \sum_{1 \le i < j \le k} p_i p_j$, $\sigma_3 = \sum_{1 \le i < j < l \le k} p_i p_j p_l$ and so on up to $\sigma_k = \prod_{1 \le i \le k} p_i$.

Corollary 2.6. Let $G = K_{q,...,q}$ be a complete k-partite graph. Then its spectrum is

$$\sigma(G) = \left\{0^{(kq-k)}, (k-1)q^{(1)}, -q^{(k-1)}\right\}.$$

Proof.

$$\chi(G) = x^{kq-k} \left(x^k - \binom{k}{2} q^2 x^{k-2} - 2q^3 \binom{k}{3} x^{k-3} - 3q^4 \binom{k}{4} x^{k-4} - \dots - (k-1)q^k \right)$$

$$= x^{kq-k} \left(x - (k-1)q \right) \left(x^{k-1} + \binom{k-1}{1} x^{k-2} q + \binom{k-1}{2} x^{k-3} q^2 + \dots + q^{k-1} \right)$$

$$= x^{kq-k} (x - (k-1)q)(x+q)^{k-1}.$$

Theorem 2.7. Given a free-form Sudoku puzzle, assume that the following conditions are met:

- (i) There exists a common number q such that for each cell there exist exactly q cells belonging to the same row and block as the considered cell (including itself).
- (ii) A similar condition holds with 'row' replaced by 'column'.
- (iii) For any two cells C_1 , C_2 belonging to different rows and columns, the unique cell lying in the same row as C_1 and the same column as C_2 belongs to the same block as C_1 if and only if the unique cell lying in the same row as C_2 and the same column as C_1 belongs to the same block as C_2 .

Then the associated graph FSud(m, T) is integral.

Proof. Under condition (i) we have $p_{ij} \in \{0, q\}$ for all $i, j \in \{1, ..., m\}$ in Proposition 2.2. Therefore, L_H represents a union of complete multipartite graphs $K_{q,...,q}$. Hence L_H is integral by Corollary 2.6. Likewise, condition (ii) ensures that L_V is integral. L_B is integral by Proposition 2.3 and Corollary 2.6 (for r = 1).

We can conclude from Proposition 2.4 and conditions (i) and (ii) that the pairs L_H , L_B and L_V , L_B commute. Condition (iii) is equivalent to the condition that L_H and L_V commute. Hence it follows that L_H , L_V , L_B are simultaneously diagonizable. Thus their sum $A = L_H + L_V + L_B$ is integral as well.

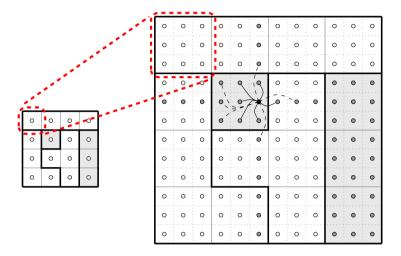
It is easily checked that the classical n-Sudokus fulfill the conditions of Theorem 2.7. Hence the theorem provides a new way of proving integrality of classical Sudoku graphs (e.g. different from the proofs in [14], [15]). Further, Theorem 2.7 helps us to identify many more integral free-form Sudoku graphs besides the classical variants.

3 Transformations

In this section we consider a special kind of transformation of Sudoku puzzles and study its spectral properties. To this end, we define the k-fold blow up of a free-form Sudoku. This is formed by replacing each cell of the original Sudoku by a $k \times k$ arrangement of cells. The block partition T' of the new graph is derived from the original partition T as follows. For each block $B \in T$ we create a block $B' \in T'$ by collecting the replacement cells of all the cells in B. In terms of graphs we see that the k-fold blow up transforms FSud(n, T)

into FSud(kn, T'). Let us denote the latter graph by FSud $^{\uparrow k}(n, T)$. In Figure 2 the 3-fold blow up of a free-form 2-Sudoku is illustrated. It also sketches the neighborhood of an exemplary vertex of the blown up graph.

Figure 2: 3-fold blow up of the free-form 2-Sudoku puzzle from Figure 1 (b)



Let us now investigate how the adjacency matrix of a blown up Sudoku graph can be determined from the adjacency matrix of the original graph. To this end, we a assume that a fixed but otherwise arbitrary numbering of the cells of the original free-form Sudoku is given. The vertices of the associated graph shall be numbered accordingly. Further, we number the cells of a k-fold blow up of the given puzzle as follows. Let S_i denote the $k \times k$ subsquare of the blown up puzzle containing exactly the replacement cells of the original cell number i. We number the $k^2 n^2$ cells by subsequently numbering all vertices in S_1 , then S_2 and so on. Inside each S_i we number the cells by starting in the top left corner and proceeding from left to right, advancing row by row.

Next, note the following facts about the blow up operation:

Observation 3.1.

- If two cells i and j lie in the same row (resp. column) of the original puzzle, then all cells sharing the same relative row (resp. column) index inside S_i and/or S_j lie in the same row (resp. column) of the blown up puzzle.
- If two cells i and j lie in the same block of the original puzzle, then all cells from S_i and S_j lie in the same block of the blown up puzzle.

In view of these facts and due to the chosen cell numbering of the blown up puzzle we can construct the adjacency matrix A^{\uparrow} of FSud $^{\uparrow k}(n, T)$ from the adjacency matrix $A = (a_{ij})$ of FSud(n, T) by replacing each entry a_{ij} of A by a $(k^2 \times k^2)$ -matrix that solely depends on whether (in the original puzzle) cell number j is in the same block as cell number i, or otherwise in the same row or column as j or lies somewhere else.

For this purpose we define the symbolic template adjacency matrix $T(A) = (t_{ij})$ as follows:

- $-t_{ii}=D$,
- $t_{ij} = B$ if i = j and cells i, j are in the same block of the original puzzle,
- t_{ij} = H if cells i, j are not in the same block but in the same row of the original puzzle,
- t_{ij} = V if cells i, j are not in the same block but in the same column of the original puzzle,
- t_{ii} = N else.

The next step is to define the matrices that will be substituted into the template matrix. But first we need some building blocks. Let I_r denote the identity matrix, J_r the all-ones matrix and N_r the zero matrix of size $r \times r$. Further, we will make use of the Kronecker product \otimes of real matrices, cf. [18].

Fixing the given blow up factor k, we define the following $(k^2 \times k^2)$ -matrices:

$$H = I_k \otimes J_k, \quad V = J_k \otimes I_k, \quad B = J_{k^2}, \quad D = J_{k^2} - I_{k^2}, \quad N = N_{k^2}.$$
 (2)

The following result should now be self-evident:

Proposition 3.2. Given an $n \times n$ free-form Sudoku puzzle with block partition T, the adjacency matrix of $FSud^{\uparrow k}(n,T)$ (with respect to the vertex numbering mentioned in the second paragraph of this section) can be obtained from the symbolic template matrix $T(A) = (t_{ij})$ of the adjacency matrix $A = (a_{ij})$ of the graph FSud(n,T) by replacing each entry t_{ij} of T(A) by the contents of the matching matrix from (2) that has the same name as the symbolic value of t_{ij} suggests.

The replacement process can be expressed by means of the Kronecker product. Let us take the template matrix T(A) and use it to partition the non-zero entries a_{ij} of A according to their respective symbols t_{ij} . Setting $L_D = I_{n^2}$, the blown up adjacency matrix A^{\uparrow} can now be expressed as follows:

Proposition 3.3.

$$A^{\uparrow} = L_B \otimes B + L_H \otimes H + L_V \otimes V + L_D \otimes D. \tag{3}$$

Our goal is to study the eigenvalues of blown up free-form Sudokus and express them in terms of the eigenvalues of the original puzzle.

For the rest of this section, we consider an arbitrary but fixed free-form $n \times n$ Sudoku puzzle with tiling T. Let A be the adjacency matrix of its associated graph FSud(n, T) and let A^{\uparrow} be the adjacency matrix of $FSud^{\uparrow k}(n, T)$. We assume that L_V , L_H , L_B etc. denote the matrices appearing in equations (1) and (3). Further, let Eig(M) represent the set of all eigenvectors of a given matrix (or even a graph) M and let ker(M) be the set of all eigenvectors of M associated with eigenvalue 0. Note that neither set contains the null vector.

In the following, we will make use of the following properties of the Kronecker product:

Theorem 3.4 (see [18]).

- 1. $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$ for all $\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{r \times s}$
- 2. $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{r \times s}$
- 3. $(A + B) \otimes C = A \otimes C + B \otimes C$ for all $A, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{r \times s}$
- 4. $A \otimes (B + C) = A \otimes B + A \otimes C$ for all $A \in \mathbb{R}^{p \times q}$, $B, C \in \mathbb{R}^{r \times s}$
- 5. $(A \otimes B)(C \otimes D) = AC \otimes BD$ for all $A \in \mathbb{R}^{p \times q}$, $B \in \mathbb{R}^{r \times s}$, $C \in \mathbb{R}^{q \times k}$, $D \in \mathbb{R}^{s \times l}$

Note that the Kronecker product formally includes the case where one or both factors are vectors.

Lemma 3.5. Suppose that x and y are eigenvectors of L_V and J_k corresponding to the eigenvalues λ and α , respectively. Then, for any $z \in \ker(J_k)$, $x \otimes y \otimes z$ is an eigenvector of A^{\uparrow} corresponding to eigenvalue $\lambda \alpha - 1$.

Proof. We use equations (2), (3) and the facts $L_V x = \lambda x$, $J_k y = \alpha y$, $J_k z = 0$.

$$A^{\uparrow}(x \otimes y \otimes z)$$

$$= (L_B \otimes B + L_H \otimes H + L_V \otimes V + L_D \otimes D)(x \otimes y \otimes z)$$

$$= (L_B \otimes J_{k^2} + L_H \otimes I_k \otimes J_k + L_V \otimes J_k \otimes I_k + L_D \otimes J_{k^2} - L_D \otimes I_{k^2})(x \otimes y \otimes z)$$

$$= ((L_B x) \otimes (J_k y) \otimes (J_k z)) + ((L_H x) \otimes (I_k y) \otimes (J_k z)) + ((L_D x) \otimes (J_k y) \otimes (J_k z)) + ((L_D x) \otimes (J_k y) \otimes (J_k z))$$

$$- ((L_D x) \otimes (I_k y) \otimes (I_k z))$$

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$$= (\lambda x) \otimes (\alpha y) \otimes z - x \otimes y \otimes z$$
$$= (\lambda \alpha - 1)(x \otimes y \otimes z).$$

Lemma 3.6. Suppose that x and z are eigenvectors of L_H and J_k corresponding to the eigenvalues λ and α , respectively. Then, for any $y \in \ker(J_k)$, $x \otimes y \otimes z$ is an eigenvector of A^{\uparrow} corresponding to eigenvalue $\lambda \alpha - 1$.

Proof. Similar to the proof of Lemma 3.5.

The previous two lemmas will play a role in the construction of a basis of eigenvectors of a blown up Sudoku graph. To this end, we need to know the intersection of the spans of the two vector sets mentioned there. In the following, let 1_r denote the all-ones vector of dimension r. Now note the following obvious facts:

Proposition 3.7.

- 1. The spectrum of J_k is $\{k^{(1)}, 0^{(k-1)}\}$.
- 2. The set

$$\mathcal{K}_J := \{(1, -1, 0, 0, \dots, 0, 0)^T, (1, 0, -1, 0, \dots, 0, 0)^T, \dots, (1, 0, 0, 0, \dots, 0, -1)^T\}$$

is a maximal linearly independent subset of $ker(J_k)$.

- 3. The set $\{1_k\} \cup \mathcal{K}_I$ is a maximal linearly independent subset of Eig (J_k) .
- 4. $1_k \perp \ker(J_n)$.

For the next lemma we define the following sets:

$$\begin{split} &X_H = \{x \otimes y \otimes z: \ x \in \operatorname{Eig}(L_H), \ y \in \ker(J_k), \ z \in \operatorname{Eig}(J_k)\}, \\ &X_V = \{x \otimes y \otimes z: \ x \in \operatorname{Eig}(L_V), \ y \in \operatorname{Eig}(J_k), \ z \in \ker(J_k)\}, \\ &\tilde{X}_H = \{x \otimes y \otimes z: \ x \in \operatorname{Eig}(L_H), \ y, z \in \ker(J_k)\}, \\ &\tilde{X}_V = \{x \otimes y \otimes z: \ x \in \operatorname{Eig}(L_V), \ y, z \in \ker(J_k)\}. \end{split}$$

Lemma 3.8.

$$\operatorname{span}(X_H) \cap \operatorname{span}(X_V) = \operatorname{span}(\tilde{X}_H) = \operatorname{span}(\tilde{X}_V).$$

Proof. Since $\ker(J_k) \subseteq \operatorname{Eig}(J_k)$ it is obvious that

$$\operatorname{span}(\tilde{X}_H) \subseteq \operatorname{span}(X_H) \cap \operatorname{span}(X_V)$$
.

Conversely, note that both L_H and L_V are symmetric and therefore diagonizable, i.e. span(Eig(L_H)) = span(Eig(L_V)) = \mathbb{R}^{n^2} . Using this we conclude

$$\operatorname{span}(X_H) \subseteq \mathbb{R}^{n^2} \otimes \operatorname{span}(\ker(J_k)) \otimes \operatorname{span}(\operatorname{Eig}(J_k)),$$

 $\operatorname{span}(X_V) \subseteq \mathbb{R}^{n^2} \otimes \operatorname{span}(\operatorname{Eig}(J_k)) \otimes \operatorname{span}(\ker(J_k)),$
 $\operatorname{span}(\tilde{X}_H) = \mathbb{R}^{n^2} \otimes \operatorname{span}(\ker(J_k)) \otimes \operatorname{span}(\ker(J_k)).$

Once again employing the fact that $ker(J_k) \subseteq Eig(J_k)$, we arrive at

$$\operatorname{span}(X_H) \cap \operatorname{span}(X_V) \subseteq \operatorname{span}(\tilde{X}_H)$$

Lemma 3.9. Let $x \in \mathbb{R}^{n^2}$ and $y, z \in \ker(J_k)$. Then $x \otimes y \otimes z$ is an eigenvector of A^{\uparrow} corresponding to eigenvalue -1.

Proof.

$$A^{\uparrow}(x \otimes y \otimes z) = (L_B \otimes J_{k^2} + L_H \otimes I_k \otimes J_k + L_V \otimes J_k \otimes I_k$$

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$$+ L_D \otimes J_{k^2} - L_D \otimes I_{k^2})(x \otimes y \otimes z)$$

$$= ((L_B x) \otimes (J_k y) \otimes (J_k z)) + ((L_H x) \otimes (I_k y) \otimes (J_k z))$$

$$+ ((L_V x) \otimes (J_k y) \otimes (I_k z)) + ((L_D x) \otimes (J_k y) \otimes (J_k z))$$

$$- ((L_D x) \otimes (I_k y) \otimes (I_k z))$$

$$= -((I_{n^2} x) \otimes (I_k y) \otimes (I_k z))$$

$$= -(x \otimes y \otimes z).$$

Lemma 3.10. For any eigenvector x of the matrix $k^2L_B + kL_H + kL_V$ corresponding to eigenvalue λ , the vector $x \otimes 1_{k^2}$ is an eigenvector of A^{\uparrow} corresponding to eigenvalue $\lambda + k^2 - 1$.

Proof.

$$A^{\uparrow}(x \otimes 1_{k^{2}})$$

$$= (L_{B} \otimes B + L_{H} \otimes H + L_{V} \otimes V + L_{D} \otimes D)(x \otimes 1_{k^{2}})$$

$$= ((L_{B}x) \otimes (J_{k^{2}}1_{k^{2}})) + ((L_{H}x) \otimes (H1_{k^{2}})) + ((L_{V}x) \otimes (V1_{k^{2}}))$$

$$+ ((L_{D}x) \otimes (J_{k^{2}}1_{k^{2}})) - ((L_{D}x) \otimes (I_{k^{2}}1_{k^{2}}))$$

$$= ((L_{B}x) \otimes (k^{2}1_{k^{2}})) + ((L_{H}x) \otimes (k1_{k^{2}})) + ((L_{V}x) \otimes (k1_{k^{2}}))$$

$$+ ((L_{D}x) \otimes (k^{2}1_{k^{2}})) - ((L_{D}x) \otimes 1_{k^{2}})$$

$$= ((k^{2}L_{B}x) \otimes 1_{k^{2}}) + ((kL_{H}x) \otimes 1_{k^{2}}) + ((kL_{V}x) \otimes 1_{k^{2}})$$

$$+ ((k^{2} - 1)(L_{D}x) \otimes 1_{k^{2}})$$

$$= ((k^{2}L_{B} + kL_{H} + kL_{V})x \otimes 1_{k^{2}}) + ((k^{2} - 1)x \otimes 1_{k^{2}})$$

$$= (\lambda + k^{2} - 1)(x \otimes 1_{k^{2}}).$$

Before we present the main result of this section we need one more technical lemma.

Lemma 3.11. Let $\{Y_i\}_{i=1}^n$ be a set of linearly independent vectors. Then, for any set $\{X_i\}_{i=1}^m$ of nonzero vectors and any function

$$\phi: \{1, 2, 3, \ldots, n\} \rightarrow \{1, 2, 3, \ldots, m\},\$$

the set $\{X_{\phi(i)} \otimes Y_i\}_{i=1}^n$ is linearly independent.

Proof. Suppose otherwise that

$$\sum_{i=1}^n c_i(X_{\phi(i)}\otimes Y_i)=0$$

for suitable numbers $c_i \in \mathbb{R}$. Let $X_{\phi(i)} = (x_{1,\phi(i)}, \dots, x_{r,\phi(i)})^T$. By the definition of the Kronecker product we have

$$\sum_{i=1}^n c_i(X_{\phi(i)} \otimes Y_i) = \begin{pmatrix} \sum_{i=1}^n c_i X_{1,\phi(i)} Y_i \\ \vdots \\ \sum_{i=1}^n c_i X_{r,\phi(i)} Y_i \end{pmatrix} = 0.$$

Thus, for each j we have

$$\sum_{i=1}^n c_i x_{j,\phi(i)} Y_i = 0.$$

Due to the linear independence of the vectors Y_i we see that $c_i x_{j,\phi(i)} = 0$ for every $j = 1, \ldots, r$ and $i = 1, \ldots, n$. Now assume that $c_i \not= 0$ for some index i^* . Then $x_{j,\phi(i^*)} = 0$ for all j, therefore $X_{\phi(i^*)} = 0$. But this is impossible since $\{X_i\}_{i=1}^m$ is a set of nonzero vectors.

For what follows, let \mathcal{B}_V , \mathcal{B}_H and \mathcal{B}_M denote arbitrary maximal linearly independent subsets of $\operatorname{Eig}(L_V)$, $\operatorname{Eig}(L_H)$ and $\operatorname{Eig}(k^2L_B+kL_V+kL_H)$, respectively. Further, let $\mathcal{E}=\{e_1,\ldots,e_{n^2}\}$ be the standard basis of \mathbb{R}^{n^2} , where e_i denotes the i-th unit vector.

Theorem 3.12. Given a graph FSud(n, T), define the sets

$$\mathcal{X}_{V} = \{x \otimes 1_{k} \otimes y : x \in \mathcal{B}_{V}, y \in \mathcal{K}_{J}\},$$

$$\mathcal{X}_{H} = \{x \otimes y \otimes 1_{k} : x \in \mathcal{B}_{H}, y \in \mathcal{K}_{J}\},$$

$$\mathcal{X}_{E} = \{x \otimes y \otimes z : x \in \mathcal{E}, y, z \in \mathcal{K}_{J}\},$$

$$\mathcal{X}_{M} = \{x \otimes 1_{k^{2}} : x \in \mathcal{B}_{M}\}.$$

Then, their union $\mathfrak{X}_V \cup \mathfrak{X}_H \cup \mathfrak{X}_E \cup \mathfrak{X}_M$ forms a maximal linearly independent subset of Eig(FSud^{†k}(n, T)).

Proof. By construction and Lemma 3.11, each of the sets \mathcal{X}_V , \mathcal{X}_H , \mathcal{X}_E , \mathcal{X}_M in itself is linearly independent. Further, by construction and Proposition 3.7, the spans of these four sets are mutually disjoint (neglecting the null vector). Moreover, Lemmas 3.5, 3.6, 3.9 and 3.11 guarantee that the union contains only eigenvectors of FSud^{†k}(n, T). Finally, note that

$$|\mathcal{B}_V| = |\mathcal{B}_H| = |\mathcal{E}| = n^2$$
, $|\mathcal{K}_i| = k - 1$

so that

$$|\mathcal{X}_V| = |\mathcal{X}_H| = (k-1)n^2$$
, $|\mathcal{X}_E| = (k-1)^2 n^2$, $|\mathcal{X}_M| = n^2$

and thus

$$|\mathcal{X}_V| + |\mathcal{X}_H| + |\mathcal{X}_E| + |\mathcal{X}_M| = 2(k-1)n^2 + (k-1)^2n^2 + n^2 = k^2n^2$$

Looking closer at Theorem 3.12, we see that if L_V , L_H , $k^2L_B + kL_H + kL_V$ were all integral, then FSud^{†k}(n, T) would be integral as well.

Corollary 3.13. *Under the conditions stated in Theorem 2.7 it follows that the blown up graph* $FSud^{\uparrow k}(n, T)$ *is integral for every k.*

Proof. From the proof of Theorem 2.7 it follows that L_V , L_H and L_B are all integral and commute with each other, hence they are simultaneously diagonizable. Consequently, $k^2L_B + kL_H + kL_V$ is integral and therefore so is FSud^{†k}(n, T).

Owing to Theorem 3.12, we can use Lemmas 3.5, 3.6, 3.9 and 3.11 to establish the spectrum of a k-fold blow up from its original. Interestingly, we can predict explicitly that the largest eigenvalue stems from the set \mathcal{X}_M :

Theorem 3.14. Given a graph FSud(n, T), let λ be the largest eigenvalue of the associated matrix $k^2L_B + kL_V + kL_H$. Then $\lambda + k^2 - 1$ is the largest eigenvalue of FSud^{$\uparrow k$}(n, T).

Proof. For the purposes of this proof we renumber the vertices of FSud(n, T) such that we sequentially number the vertices with one block, then continue with the next block and so on. The order in which the vertices are numbered with a single block is arbitrary. With respect to this vertex order the matrix L_B assumes the form $I_n \otimes J_n - I_{n^2}$. Clearly, this matrix contains $J_n - I_n$ as a principal submatrix. Consequently, the matrix $M := k^2 L_B + k L_V + k L_H$ contains the matrix $k^2 J_n - k^2 I_n$ as a principal submatrix, the latter having maximum eigenvalue $k^2 (n-1)$. By virtue of eigenvalue interlacing (see e.g. [19]) we conclude that $\lambda > (n-1)k^2$. So, according to Lemma 3.10, the largest eigenvalue of FSud $^{\uparrow k}(n, T)$ originating from the set \mathcal{X}_M is at least $(n-1)k^2 + k^2 - 1 = nk^2 - 1$. We will now show that the largest eigenvalues originating from \mathcal{X}_V , \mathcal{X}_H and \mathcal{X}_B are smaller than this bound.

Since the largest eigenvalue of a matrix is bounded from above by the maximum row sum of the matrix, it is clear that the maximum eigenvalue both of L_H and L_V can be at most n-1. Now recall from Proposition

3.7 that k is the maximum eigenvalue of J_k . Combining these findings, it now follows from Lemmas 3.5 and 3.6 that none of the eigenvalues associated with the vectors of the sets \mathcal{X}_V and \mathcal{X}_H exceeds (n-1)k-1, which is less than the lower bound given for \mathcal{X}_M . Finally, Lemma 3.9 tells us that no positive eigenvalue of FSud^{†k}(n, T) originates from \mathcal{X}_B .

4 Conclusion

Up to now, it seems that free-form Sudokus have not been researched at all. Providing a starting point, we have studied integrality of these graphs. Moreover, we have presented the blow up operation and shown how to obtain the eigenvalues of blown up free-form Sudokus from their originals. We would like to inspire more research on this topic, in particular regarding further spectral properties of free-form Sudoku graphs. Let us therefore close with the following open questions:

- 1. Can we find a precise condition on the tiling that allows us to predict exactly when a free-form Sudoku graph is integral or not?
- 2. If a given Sudoku is integral, is its blown up version always integral?
- 3. Can a blown up free-form Sudoku be integral although its original Sudoku is not?

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