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## Research Article

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# Boundary value problems of a discrete generalized beam equation via variational methods

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**Abstract:** The authors explore the boundary value problems of a discrete generalized beam equation. Using the critical point theory, some sufficient conditions for the existence of the solutions are obtained. Several consequences of the main results are also presented. Examples are given to illustrate the theorems.

**Keywords:** Boundary value problems, Discrete problems, Generalized beam equation, Saddle point theorem, Variational methods

**MSC:** 39A10, 34B05, 58E05, 65L10

## 1 Introduction and statement of main results

Beam equations have historical importance, as they have been the focus of attention for prominent scientists such as Leonardo da Vinci (14th Century) and Daniel Bernoulli (18th Century) [9]. In this article, we study the existence of solutions to boundary value problem (BVP) of discrete generalized beam equation

$$\Delta^4 x(t-2) - \Delta(r(t-1)\Delta x(t-1)) = f(t, x(t)), \quad t \in [1, T]_{\mathbb{Z}}, \quad (1.1)$$

with boundary value conditions

$$\Delta^k x(-1) = \Delta^k x(T-1), \quad k = 0, 1, 2, 3, \quad (1.2)$$

where  $1 \leq T \in \mathbb{N}$ ,  $\Delta$  is the forward difference operator defined by  $\Delta x(t) = x(t+1) - x(t)$ ,  $\Delta^k x(t) = \Delta(\Delta^{k-1} x(t))$  ( $2 \leq k \leq 4$ ),  $\Delta^0 x(t) = x(t)$ , define  $[1, T]_{\mathbb{Z}} := [1, T] \cap \mathbb{Z}$ ,  $r(t) \in \mathbb{R}^{T+1}$  with  $r(0) = r(T)$ ,  $f(t, x) \in C([1, T]_{\mathbb{Z}} \times \mathbb{R}, \mathbb{R})$ .

(1.1) and (1.2) can be considered as a discrete analogue of

$$x^{(4)}(s) - [r(s)x'(s)]' = f(s, x(s)), \quad s \in (0, 1), \quad (1.3)$$

with boundary value conditions

$$x^{(k)}(0) = x^{(k)}(1), \quad k = 0, 1, 2, 3. \quad (1.4)$$

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(1.3) is a generalization of the beam equation

$$x^{(4)}(s) = f(s, x(s)), \quad s \in \mathbb{R}.$$

Practical applications of the beam equations [9] are evident in mechanical structures built under the premise of beam theory. In recent years, many researchers [6,10,12,21,22] have paid a lot of attention to equations similar to (1.3).

Difference equations [1-5,7,11,13-15,17-20,23,24,26,27] are widely found in mathematics itself and in its applications to combinatorial analysis, quantum physics, chemical reactions and so on. Many authors were interested in difference equations and obtained many significant conclusions.

Cabada and Dimitrov [4] studied the following nonlinear singular and non-singular fourth-order difference equation

$$x(t+4) + Mx(t) = \lambda g(t)f(x(t)) + c(t), \quad t \in \{0, 1, \dots, T-1\},$$

coupled with periodic boundary value conditions. They obtained some sufficient conditions on existence and nonexistence theorems.

In 2010, He and Su [11] considered boundary value problems of the fourth order nonlinear difference equation

$$\Delta^4 x(t-1) + \eta \Delta^2 x(t-1) - \xi x(t) = \lambda f(t, x(t)), \quad t \in \mathbb{Z}[a+1, b+1],$$

with three parameters by using the critical point theory and monotone operator theory. They obtained some existence, multiplicity, and nonexistence of nontrivial solutions.

By using the Dancer's global bifurcation theorem, Ma and Lu [15] investigated the boundary value problem to the following fourth order nonlinear difference equation

$$\Delta^4 x(t-2) = \lambda h(t)f(x(t)), \quad t \in \{2, 3, \dots, T\},$$

and gave the existence and multiplicity of positive solutions.

Fang and Zhao [7] in 2009 established a sufficient condition for the existence of nontrivial homoclinic orbits for fourth order difference equation

$$\Delta^4 x(t-2) - r(t)x(t) + f(t, x(t+1), x(t), x(t-1)) = 0, \quad t \in \mathbb{Z},$$

by using Mountain Pass Theorem, a weak convergence argument and a discrete version of Lieb's lemma.

There are numerous papers dealing with similar problems to the one that we study (nonlinear difference equation with semidefinite linear parts, periodic boundary value problems) and many of these use similar techniques-matrix formulation in  $\mathbb{R}^N$  with various variational methods (see, e.g., [3, 8, 19, 25]) and in many cases even the saddle point theorem (e.g., [17, 23]). In this article, the boundary value problems of a discrete generalized beam equation are explored. Applying the critical point theory, we establish some criteria for the existence of the solutions for (1.1) and (1.2). The eigenvalues of some symmetric matrix associated with the problem are used in proving main results. The motivation for this article comes from the recent article [11] since it deals with boundary value problem to the fourth order difference equation by using the critical point theory.

Let

$$G(t, x) = \int_0^x f(t, s) ds,$$

for any  $(t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}$ .

The rest of this article is organized as follows. In Section 2, we state some preliminary lemmas and transfer the existence of the BVP of (1.1) and (1.2) into the existence of the critical points of some functionals. Our main results are given in Section 3. In Section 4, we prove our main results by making use of variational methods. Two examples are presented to illustrate our main results in Section 5.

## 2 Preliminary lemmas

Let  $Q$  and  $R$  be Banach spaces, and  $P \subset Q$  be an open subset of  $Q$ . A function  $I : P \rightarrow R$  is called Fréchet differentiable at  $x \in P$  if there exists a bounded linear operator  $L_x : Q \rightarrow R$  such that

$$\lim_{h \rightarrow 0} \frac{\|I(x+h) - I(x) - L_x(h)\|_R}{\|h\|_Q} = 0.$$

We write  $I'(x) = L_x$  and call it the Fréchet derivative of  $I$  at  $x$ .

Let  $X$  be a real Banach space and  $I \in C^1(X, \mathbb{R})$  be a continuously Fréchet differentiable functional defined on  $X$ . As usually,  $I$  is said to satisfy the Palais-Smale condition if any sequence  $\{x_k\}_{k=1}^\infty \subset X$  for which  $\{I(x_k)\}_{k=1}^\infty$  is bounded and  $I'(x_k) \rightarrow 0$  as  $k \rightarrow \infty$  possesses a convergent subsequence. Here, the sequence  $\{x_k\}_{k=1}^\infty$  is called a Palais-Smale sequence.

Let  $X$  be a real Banach space. We denote by the symbol  $B_r$  the open ball in  $X$  about 0 of radius  $r$ ,  $\partial B_r$  its boundary, and  $\bar{B}_r$  its closure.

Denote a space  $X$  as

$$X := \{x : [-1, T+2]_{\mathbb{Z}} \rightarrow \mathbb{R} \mid \Delta^k x(-1) = \Delta^k x(T-1), k = 0, 1, 2, 3\}.$$

For any  $x \in X$ , we define

$$(x, y) := \sum_{t=1}^T x(t)y(t), \quad \forall x, y \in X,$$

and

$$\|x\| := \left( \sum_{t=1}^T x^2(t) \right)^{\frac{1}{2}}, \quad \forall x \in X.$$

Denote the norm  $\|\cdot\|_q$  on  $X$  by

$$\|x\|_q = \left( \sum_{t=1}^T |x(t)|^q \right)^{\frac{1}{q}}, \quad (2.1)$$

for all  $x \in X$  and  $q > 1$ .

As usual, we use  $\|x\| = \|x\|_2$  for the Euclidean norm. Since  $\|x\|_q$  and  $\|x\|$  are equivalent, there are numbers  $\tau_1, \tau_2$  such that  $\tau_2 \geq \tau_1 > 0$ , and

$$\tau_1 \|x\| \leq \|x\|_q \leq \tau_2 \|x\|, \quad \forall x \in X. \quad (2.2)$$

**Remark 2.1.** For any  $x \in X$ , it is obvious that

$$x(-1) = x(T-1), \quad x(0) = x(T), \quad x(1) = x(T+1), \quad x(2) = x(T+2). \quad (2.3)$$

In fact,  $X$  is isomorphic to  $\mathbb{R}^T$ . In the later sections of this article, when we write  $x = (x(1), x(2), \dots, x(T)) \in \mathbb{R}^T$ , we always imply that  $x$  can be extended to a vector in  $X$  so that (2.3) is satisfied.

For any  $x \in X$ , we denote the functional  $I$  by

$$I(x) := -\frac{1}{2} \sum_{t=1}^T \left( \Delta^2 x(t-2) \right)^2 - \frac{1}{2} \sum_{t=1}^T r(t) (\Delta x(t))^2 + \sum_{t=1}^T G(t, x(t)). \quad (2.4)$$

Hence  $I \in C^1(X, \mathbb{R})$ . By computing, we have

$$\frac{\partial I}{\partial x(t)} = -\Delta^4 x(t-2) + \Delta(r(t-1)\Delta x(t-1)) + f(t, x(t)), \quad t \in [1, T]_{\mathbb{Z}}.$$

Accordingly,  $I'(x) = 0$  if and only if

$$\Delta^4 x(t-2) - \Delta(r(t-1)\Delta x(t-1)) = f(t, x(t)), \quad t \in [1, T]_{\mathbb{Z}}.$$

As a result, a function  $x \in X$  is a critical point of the functional  $I$  on  $X$  if and only if  $x$  is a solution of the BVP (1.1) and (1.2). For convenience, we define two  $T \times T$  matrices as follows.

When  $T = 1$ , define  $A = B = (0)$ .

When  $T = 2$ , define

$$A = \begin{pmatrix} 8 & -8 \\ -8 & 8 \end{pmatrix},$$

and

$$B = \begin{pmatrix} r(0) + r(1) & -r(0) - r(1) \\ -r(0) - r(1) & r(0) + r(1) \end{pmatrix}.$$

When  $T = 3$ , define

$$A = \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix}.$$

When  $T = 4$ , define

$$A = \begin{pmatrix} 6 & -4 & 2 & -4 \\ -4 & 6 & -4 & 2 \\ 2 & -4 & 6 & -4 \\ -4 & 2 & -4 & 6 \end{pmatrix}.$$

When  $T \geq 5$ , define

$$A = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & -4 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 6 & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 & -4 \\ -4 & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -4 & 6 \end{pmatrix}.$$

When  $T \geq 3$ , define

$$B = \begin{pmatrix} r(0) + r(1) & -r(1) & 0 & \cdots & -r(0) \\ -r(1) & r(1) + r(2) & -r(2) & \cdots & 0 \\ 0 & -r(2) & r(2) + r(3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -r(T-1) \\ -r(0) & 0 & 0 & \cdots & r(T-1) + r(0) \end{pmatrix}.$$

Let  $\Omega := A + B$ . Consequently, the functional  $I(x)$  can be rewritten as

$$I(x) = -\frac{1}{2}x^* \Omega x + \sum_{t=1}^T G(t, x(t)). \quad (2.5)$$

**Lemma 2.2** (Saddle Point Theorem [16]). *Let  $X$  be a real Banach space,  $X = X_1 \oplus X_2$ , where  $X_1 \neq \{0\}$  and is finite dimensional. Assume that  $I \in C^1(X, \mathbb{R})$  satisfies the Palais-Smale condition and*

*(I<sub>1</sub>) there exist two constants  $\sigma, \rho > 0$  such that  $I|_{\partial B_\rho \cap X_1} \leq \sigma$ ;*

*(I<sub>2</sub>) there exists  $e \in B_\rho \cap X_1$  and a constant  $\omega > \sigma$  such that  $I_{e+X_2} \geq \omega$ .*

*Then  $I$  possesses a critical value  $c \geq \omega$ , where*

$$c = \inf_{h \in \Gamma} \max_{x \in B_\rho \cap X_1} I(h(x)), \quad \Gamma = \{h \in C(\bar{B}_\rho \cap X_1, X) \mid h|_{\partial B_\rho \cap X_1} = id\}$$

*and  $id$  denotes the identity operator.*

### 3 Main results

We shall give our main results in this section.

**Theorem 3.1.** *Suppose that the following conditions are satisfied:*

(r) for any  $t \in [0, T]_{\mathbb{Z}}$ ,  $r(t) \geq 0$ ;

( $G_1$ ) there is a positive constant  $c_1$  such that

$$|f(t, x)| \leq c_1, \quad \forall (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R};$$

( $G_2$ )  $G(t, x) \rightarrow +\infty$  uniformly for  $t \in [1, T]_{\mathbb{Z}}$  as  $|x| \rightarrow +\infty$ .

Then the BVP (1.1), (1.2) possesses at least one solution.

**Remark 3.2.** Condition ( $G_1$ ) means that there is a positive constant  $c_2$  such that

$$(G'_1) |G(t, x)| \leq c_1|x| + c_2, \quad \forall (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}.$$

**Theorem 3.3.** *Suppose that (r) and the following conditions are satisfied:*

( $G_3$ ) there are two constants  $1 < \mu < 2$  and  $\delta > 0$  such that

$$0 < xf(t, x) \leq \mu G(t, x), \quad \forall t \in [1, T]_{\mathbb{Z}}, |x| \geq \delta;$$

( $G_4$ ) there are three constants  $c_3 > 0$ ,  $c_4 > 0$  and  $1 < \nu \leq \mu$  such that

$$G(t, x) \geq c_3|x|^\nu - c_4, \quad \forall (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}.$$

Then the BVP (1.1), (1.2) possesses at least one solution.

**Remark 3.4.** Condition ( $G_3$ ) means that there are two positive constants  $c_5$  and  $c_6$  such that

$$(G'_3) G(t, x) \leq c_5|x|^\mu + c_6, \quad \forall (t, x) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}.$$

In the case that  $f(t, x)$  is independent of  $x$ , we study the autonomous fourth order discrete system

$$\Delta^4 x(t-2) - \Delta(r(t-1)\Delta x(t-1)) = f(x(t)), \quad t \in [1, T]_{\mathbb{Z}}, \quad (3.1)$$

where  $f \in C(\mathbb{R}, \mathbb{R})$ .

**Corollary 3.5.** *Suppose that (r) and the following conditions are satisfied:*

( $H_1$ ) there is a function  $H(x) \in C^1(\mathbb{R}, \mathbb{R})$  such that

$$H'(x) = f(x);$$

( $H_2$ ) there is a positive constant  $\kappa_1$  such that

$$|f(x)| \leq \kappa_1, \quad \forall x \in \mathbb{R};$$

( $H_3$ )  $H(x) \rightarrow +\infty$  as  $|x| \rightarrow +\infty$ .

Then the BVP (3.1), (1.2) possesses at least one solution.

**Corollary 3.6.** *Suppose that (r), ( $H_1$ ) and the following conditions are satisfied:*

( $H_4$ ) there are two constants  $1 < \tilde{\mu} < 2$  and  $\tilde{\delta} > 0$  such that

$$0 < xf(x) \leq \tilde{\mu} H(x), \quad \forall |x| \geq \tilde{\delta};$$

( $H_5$ ) there are three constants  $\kappa_2 > 0$ ,  $\kappa_3 > 0$  and  $\tilde{\nu}$  with  $1 < \tilde{\nu} \leq \tilde{\mu}$  such that

$$H(x) \geq \kappa_2|x|^{\tilde{\nu}} - \kappa_3, \quad \forall x \in \mathbb{R}.$$

Then the BVP (3.1), (1.2) possesses at least one solution.

Let  $\Omega$  satisfy:

(A1)  $\Omega$  is a symmetric and positive semidefinite matrix;

(A2)  $\lambda_1 = 0$  is a simple eigenvalue of  $\Omega$  with multiplicity one and with the eigenvector  $e_1 = [1, 1, \dots, 1]^T$ .

**Remark 3.7.**

(i) Our results could be extended to other problems with matrices satisfying (A1) – (A2). For example, it is obvious that we can also use the discrete beam equation with Neumann initial conditions.

(ii) On the other hand, we can directly use results from other papers for other nonlinearities and obtain, for example, multiplicity results for the beam equation with bistable nonlinearities [17] or nonlinearities satisfying Landesman-Lazer type conditions [23].

## 4 Proofs of the main results

In this section, we shall prove our main results by using variational methods.

Throughout this section,  $\Omega$  is a symmetric and positive semidefinite matrix, 0 is a simple eigenvalue with multiplicity one and with the eigenvector  $(1, 1, \dots, 1)^T$ . We denote the eigenvalues of  $\Omega$  by  $\lambda_1, \lambda_2, \dots, \lambda_T$ .

Denote

$$\underline{\lambda} = \min \{ \lambda_i | \lambda_i \neq 0, i = 1, 2, \dots, T \}, \quad (4.1)$$

and

$$\bar{\lambda} = \max \{ \lambda_i | \lambda_i \neq 0, i = 1, 2, \dots, T \}. \quad (4.2)$$

Set  $X_2 = \{(d, d, \dots, d)^T \in X | d \in \mathbb{R}\}$ . Obviously,  $X_2$  is an invariant subspace of  $X$ . We denote a subspace  $X_1$  of  $X$  by

$$X = X_1 \oplus X_2.$$

*Proof of Theorem 3.1.* Let  $\{x_k\}_{k \in \mathbb{N}} \subset X$  be such that  $\{I(x_k)\}_{k \in \mathbb{N}}$  is bounded and  $I'(x_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Accordingly, for any  $k \in \mathbb{N}$ , there is a number  $c_7 > 0$  such that

$$-c_7 \leq I(x_k) \leq c_7.$$

Let  $x_k = x_k^{(1)} + x_k^{(2)} \in X_1 \oplus X_2$ . On one hand, for  $k$  large enough, since

$$-\|x\| \leq (I'(x_k), x) = -(\Omega x_k, x) + \sum_{t=1}^T f(t, x_k(t)) x(t),$$

combining with  $(G_1)$ , we have

$$\begin{aligned} (\Omega x_k, x_k^{(1)}) &\leq \sum_{t=1}^T f(t, x_k(t)) x_k^{(1)}(t) + \|x_k^{(1)}\| \\ &\leq c_1 \sum_{t=1}^T |x_k^{(1)}| + \|x_k^{(1)}\| \\ &\leq (c_1 \sqrt{T} + 1) \|x_k^{(1)}\|. \end{aligned}$$

On the other hand, we have

$$(\Omega x_k, x_k^{(1)}) = (\Omega x_k^{(1)}, x_k^{(1)}) \geq \underline{\lambda} \|x_k^{(1)}\|^2.$$

Consequently, we have

$$\underline{\lambda} \|x_k^{(1)}\|^2 \leq (c_1 \sqrt{T} + 1) \|x_k^{(1)}\|. \quad (4.3)$$

(4.3) means that  $\{x_k^{(1)}\}_{k=1}^\infty$  is bounded.

Then, we shall prove that  $\{x_k^{(2)}\}_{k=1}^\infty$  is bounded.

As a matter of fact,

$$\begin{aligned} c_7 \geq I(x_k) &= -\frac{1}{2}x_k^\top \Omega x_k + \sum_{t=1}^T G(t, x_k(t)) \\ &= -\frac{1}{2}(x_k^{(1)})^\top \Omega x_k^{(1)} + \sum_{t=1}^T [G(t, x_k(t)) - G(t, x_k^{(2)}(t))] + \sum_{t=1}^T G(t, x_k^{(2)}(t)). \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{t=1}^T G(t, x_k^{(2)}(t)) &\leq c_7 + \frac{1}{2}(x_k^{(1)})^\top \Omega x_k^{(1)} + \sum_{t=1}^T |G(t, x_k(t)) - G(t, x_k^{(2)}(t))| \\ &\leq c_7 + \frac{\bar{\lambda}}{2} \|x_k^{(1)}\|^2 + \sum_{t=1}^T |f(t, x_k^{(1)}(t) + \xi x_k^{(2)}(t))| \cdot |x_k^{(1)}(t)| \\ &\leq c_7 + \frac{\bar{\lambda}}{2} \|x_k^{(1)}\|^2 + c_1 \sqrt{T} \|x_k^{(1)}(t)\|, \end{aligned}$$

which means that  $\left\{ \sum_{t=1}^T G(t, x_k^{(2)}(t)) \right\}$  is bounded. Here  $\xi \in (0, 1)$ .

It comes from condition  $(G_2)$  that  $\{x_k^{(2)}\}_{k=1}^\infty$  is bounded. If not, assume that  $\|x_k^{(2)}\| \rightarrow +\infty$  as  $k \rightarrow \infty$ . In that there are  $c_k \in \mathbb{R}$ ,  $k \in \mathbb{N}$ , such that  $x_k^{(2)} = (c_k, c_k, \dots, c_k)^\top \in X$ , then

$$\|x_k^{(2)}\| = \left( \sum_{t=1}^T |x_k^{(2)}(t)|^2 \right)^{\frac{1}{2}} = \left( \sum_{t=1}^T |c_k|^2 \right)^{\frac{1}{2}} = \sqrt{T} |c_k| \rightarrow +\infty, \quad k \rightarrow \infty.$$

Since  $G(t, x_k^{(2)}(t)) = G(t, c_k)$ , then  $G(t, x_k^{(2)}(t)) \rightarrow +\infty$  as  $k \rightarrow \infty$ . This contradicts the fact  $\left\{ \sum_{t=1}^T G(t, x_k^{(2)}(t)) \right\}$  is bounded. Therefore, the functional  $I(x)$  satisfies the Palais-Smale condition. Therefore, it suffices to prove that  $I(x)$  satisfies the conditions  $(I_1)$  and  $(I_2)$  of Saddle Point Theorem.

First, we shall prove the condition  $(I_2)$ . For any  $x^{(2)} \in X_2$ ,  $x^{(2)} = (x^{(2)}(1), x^{(2)}(2), \dots, x^{(2)}(T))^\top$ , there is  $c \in \mathbb{R}$  such that

$$x^{(2)}(i) = c, \quad \forall i \in [1, T]_{\mathbb{Z}}.$$

It comes from  $(G_2)$  that there is a constant  $c_8 > 0$  such that  $G(t, c) > 0$  for  $t \in \mathbb{Z}$  and  $|c| > c_8$ .

Set

$$c_9 = \min_{n \in [1, T]_{\mathbb{Z}}, |c| \leq c_8} G(t, c), \quad c_{10} = \min\{0, c_9\}.$$

Thus,

$$G(t, c) \geq c_{10}, \quad \forall (t, c) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}.$$

Then

$$I(x^{(2)}) = \sum_{t=1}^T G(t, x^{(2)}(t)) = \sum_{t=1}^T G(t, c) \geq T c_{10}, \quad \forall x^{(2)} \in X_2.$$

Next, we shall prove the condition  $(I_1)$ . For any  $x^{(1)} \in X_1$ , by  $(G'_1)$ , we have

$$\begin{aligned} I(x^{(1)}) &= -\frac{1}{2}(x^{(1)})^\top \Omega x^{(1)} + \sum_{t=1}^T G(t, x^{(1)}(t)) \\ &\leq -\frac{\lambda}{2} \|x^{(1)}\|^2 + c_1 \sum_{t=1}^T |x^{(1)}(t)| + T c_2 \\ &\leq -\frac{\lambda}{2} \|x^{(1)}\|^2 + c_1 \sqrt{T} \|x^{(1)}\| + T c_2. \end{aligned}$$

Take

$$\omega = Tc_{10}.$$

Then, there exists a constant  $\rho > 0$  large enough such that

$$I(x^{(1)}) \leq \omega - 1 = \sigma < \omega, \quad \forall x^{(1)} \in X_1, \quad \|x^{(1)}\| = \rho.$$

The conditions of  $(I_1)$  and  $(I_2)$  of Saddle Point Theorem are satisfied. In the light of Saddle Point Theorem, Theorem 3.1 holds.  $\square$

*Proof of Theorem 3.3.* Let  $\{x_k\}_{k \in \mathbb{N}} \subset X$  be such that  $\{I(x_k)\}_{k \in \mathbb{N}}$  is bounded and  $I'(x_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Accordingly, for any  $k \in \mathbb{N}$ , there is a number  $c_{11} > 0$  such that

$$-c_{11} \leq I(x_k) \leq c_{11}.$$

For  $k$  large enough, it comes from  $\lim_{k \rightarrow \infty} I'(x_k) = 0$  that

$$|(I'(x_k), x_k)| \leq \|x_k\|.$$

Since

$$(I'(x_k), x_k) = -x_k^\top \Omega x_k + \sum_{t=1}^T f(t, x_k(t)) x_k(t).$$

Accordingly, for  $k$  large enough, we have

$$\begin{aligned} c_{11} + \frac{1}{2} \|x_k\| &\geq I(x_k) - \frac{1}{2} (I'(x_k), x_k) \\ &= \sum_{t=1}^T \left[ G(t, x_k(t)) - \frac{1}{2} f(t, x_k(t)) x_k(t) \right]. \end{aligned}$$

Denote

$$\begin{aligned} I_1 &= \{t \in [1, T]_{\mathbb{Z}} : |x_k(t)| \geq \delta\}; \\ I_2 &= \{t \in [1, T]_{\mathbb{Z}} : |x_k(t)| < \delta\}. \end{aligned}$$

Combining with  $(G_3)$ , we have

$$\begin{aligned} c_{11} + \frac{1}{2} \|x_k\| &\geq \sum_{t=1}^T G(t, x_k(t)) - \frac{1}{2} \sum_{t \in I_1} f(t, x_k(t)) x_k(t) - \frac{1}{2} \sum_{t \in I_2} f(t, x_k(t)) x_k(t) \\ &\geq \sum_{t=1}^T G(t, x_k(t)) - \frac{\mu}{2} \sum_{t \in I_1} G(t, x_k(t)) - \frac{1}{2} \sum_{t \in I_2} f(t, x_k(t)) x_k(t) \\ &= \left(1 - \frac{\mu}{2}\right) \sum_{t=1}^T G(t, x_k(t)) + \frac{1}{2} \sum_{t \in I_2} [\mu G(t, x_k(t)) - f(t, x_k(t)) x_k(t)]. \end{aligned}$$

Set

$$L(t, x) = \mu G(t, x) - f(t, x) x.$$

By the continuity of  $L(t, x)$  with respect to the first and second variables, we have that there is a constant  $c_{12} > 0$  such that

$$L(t, x) \geq -c_{12},$$

for all  $t \in [1, T]_{\mathbb{Z}}$  and  $|x| \geq \delta$ . Hence,

$$c_{11} + \frac{1}{2} \|x_k\| \geq \left(1 - \frac{\mu}{2}\right) \sum_{t=1}^T G(t, x_k(t)) - \frac{1}{2} Tc_{12}, \quad |x| \geq \delta.$$



It follows from  $(G_4)$  and (2.2) that

$$\begin{aligned} c_{11} + \frac{1}{2} \|x_k\| &\geq \left(1 - \frac{\mu}{2}\right) c_3 \sum_{t=1}^T |x_k(t)|^\nu - \left(1 - \frac{\mu}{2}\right) c_4 T - \frac{1}{2} T c_{12} \\ &\geq \left(1 - \frac{\mu}{2}\right) c_3 \tau_1^\nu \|x_k\|^\nu - \left(1 - \frac{\mu}{2}\right) c_4 T - \frac{1}{2} T c_{12}. \end{aligned}$$

Let

$$c_{13} = -\left(1 - \frac{\mu}{2}\right) c_4 T - \frac{1}{2} T c_{12}.$$

Then,

$$c_{11} + \frac{1}{2} \|x_k\| \geq \left(1 - \frac{\mu}{2}\right) c_3 \tau_1^\nu \|x_k\|^\nu + c_{13}.$$

Thus,

$$\left(1 - \frac{\mu}{2}\right) c_3 \tau_1^\nu \|x_k\|^\nu - \frac{1}{2} \|x_k\| \leq c_{11} - c_{13},$$

which means that  $\{\|x_k\|\}_{k=1}^\infty$  is bounded. For the reason that  $X$  is a finite dimensional space,  $\{x_k\}_{k=1}^\infty$  possesses a convergent subsequence. Accordingly, the Palais-Smale condition is proved.

To exploit the Saddle Point Theorem, we shall prove that the functional  $I$  satisfies the conditions  $(I_1)$  and  $(I_2)$ .

First, we shall prove that the functional  $I$  satisfies the condition  $(I_2)$ . For any  $x^{(2)} \in X_2$ , in that  $\Omega x^{(2)} = 0$ , we have

$$I(x^{(2)}) = \sum_{t=1}^T G(t, x^{(2)}(t)).$$

On account of  $(G_4)$ ,

$$I(x^{(2)}) \geq c_3 \sum_{t=1}^T |x^{(2)}(t)|^\nu - c_4 T \geq -c_4 T.$$

Then, we shall prove the condition  $(I_1)$ . For any  $x^{(1)} \in X_1$ , combining with  $(G_4)$ , we have

$$\begin{aligned} I(x^{(1)}) &= -\frac{1}{2} (x^{(1)})^\top \Omega x^{(1)} + \sum_{t=1}^T G(t, x^{(1)}(t)) \\ &\leq -\frac{\lambda}{2} \|x^{(1)}\|^2 + c_3 \sum_{t=1}^T |x^{(1)}(t)|^\nu + T c_4 \\ &\leq -\frac{\lambda}{2} \|x^{(1)}\|^2 + c_3 \tau_2^\nu \sqrt{T} \|x^{(1)}\|^\nu + T c_4. \end{aligned}$$

Take

$$\omega = -c_4 T.$$

Owing to  $1 < \nu < 2$ , there exists a constant  $\rho > 0$  large enough such that

$$I(x^{(1)}) \leq \omega - 1 = \sigma < \omega, \quad \forall x^{(1)} \in X_1, \quad \|x^{(1)}\| = \rho.$$

Hence, the condition  $(I_1)$  is satisfied.

As a result of Saddle Point Theorem, the BVP (1.1), (1.2) possesses at least one solution. The proof is complete.  $\square$

**Remark 4.1.** In the light of Theorems 3.1 and 3.3, the results of Corollaries 3.5 and 3.6 are obviously true.

## 5 Example

In this section, an example is given to illustrate our main result.

**Example 5.1.** For  $t \in [1, 3]_{\mathbb{Z}}$ , suppose that

$$\Delta^4 x(t-2) - 2\Delta((t-2)^2 \Delta x(t-1)) = \nu x(t)|x(t)|^{\nu-2} + \mu x(t)|x(t)|^{\mu-2}, \quad (5.1)$$

satisfies the boundary value conditions

$$x(-1) = x(2), \Delta x(-1) = \Delta x(2), \Delta^2 x(-1) = \Delta^2 x(2), \Delta^3 x(-1) = \Delta^3 x(2). \quad (5.2)$$

We have

$$r(t) = 2t^2, \quad t \in [1, 3]_{\mathbb{Z}},$$

with

$$r(0) = 18,$$

and

$$f(t, x) = \nu x|x|^{\nu-2} + \mu x|x|^{\mu-2}, \quad G(t, x) = |x|^{\nu} + |x|^{\mu}.$$

Besides,

$$\Omega = \begin{pmatrix} 26 & -5 & -21 \\ -5 & 16 & -11 \\ -21 & -11 & 32 \end{pmatrix},$$

and the eigenvalues of  $\Omega$  are  $\lambda_1 = 0$ ,  $\lambda_2 = 23$  and  $\lambda_3 = 51$ . It is obvious that all the conditions of Theorem 3.3 are satisfied and then the BVP (5.1), (5.2) possesses at least one nontrivial solution.

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