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Hailong Hou*, Rui Gu, and Mengdi Tong

Pretty good state transfer on 1-sum of star graphs

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Abstract: Let A be the adjacency matrix of a graph G and suppose U(t) = exp(itA). We say that we have perfect state transfer in G from the vertex u to the vertex v at time t if there is a scalar γ of unit modulus such that $U(t)e_u = \gamma e_v$. It is known that perfect state transfer is rare. So C.Godsil gave a relaxation of this definition: we say that we have pretty good state transfer from u to v if there exists a complex number γ of unit modulus and, for each positive real ϵ there is a time t such that $\|U(t)e_u - \gamma e_v\| < \epsilon$. In this paper, the quantum state transfer on 1-sum of star graphs $F_{k,l}$ is explored. We show that there is no perfect state transfer on $F_{k,l}$, but there is pretty good state transfer on $F_{k,l}$ if and only if k=l.

Keywords: Perfect state transfer, Pretty good state transfer, Kronecker approximation theorem

MSC: 05C50

1 Introduction

Quantum walks on graphs are a natural generalization of classical random walks. In recent years much attention has been paid to quantum walks of graphs and many interesting results concerning graphs and their continuous quantum walks have been obtained (see [1, 2, 5, 11-13] and references therein). It has been applied to key distributions in commercial cryptosystems, and it seems likely that further applications will be found.

Let G be a graph with adjacency matrix A and let U(t) denote the matrix-valued function exp(itA). It is known that U(t) is both symmetric and unitary, and that it determines a continuous quantum walk on G. If $u \in V(X)$, we use e_u to denote the standard basis vector indexed by u. If u and v are distinct vertices in G, we say that we have perfect state transfer from u to v at time t if there exists a complex number γ of unit modulus such that

$$U(t)e_u = \gamma e_v$$
.

There is considerable literature on perfect state transfer. In [5, 6], Christandl et al showed that there is perfect state transfer between the end vertices of the paths P_2 and P_3 , but perfect state transfer does not occur on a path on four or more vertices. Some results on perfect state transfer in gcd-graph can be found in [14]. From [8], Godsil showed that, for any integer k, there are only finite many connected graphs with valency at most k on which perfect state transfer occurs.

^{*}Corresponding Author: Hailong Hou: School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan, 471023, China, E-mail: hailonghou@163.com

Rui Gu: School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan, 471023, China Mengdi Tong: School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan, 471023, China

Perfect state transfer is rare and we consider a relaxation of it. We say that we have pretty good state transfer from u to v if there exists a complex number γ of unit modulus and, for each positive real ϵ there is a time t such that

$$\|U(t)e_u - \gamma e_v\| < \epsilon$$
.

The notion of pretty good state transfer was first introduced by Godsil in [7], since this, there have been multiple published papers which have enhanced his work. In [11], Godsil et al showed that there is pretty good state transfer between the end-vertices on P_n if and only if n + 1 equals to 2^m or p or 2p, where p is an odd prime. Fan and Godsil [4] have studied pretty good state transfer on double star. They showed that a double star $S_{k,k}$ admits pretty good state transfer if and only if 4k + 1 is not a perfect square. Pal [13] showed that a cycle C_n exhibits pretty good state transfer if and only if $n = 2^k$ for some $k \ge 2$.

Let X and Y be two graphs. We say Z is a 1-sum of X and Y at u if Z is obtained by merging the vertex u of X with the vertex u of Y and no edge joins a vertex of $X \setminus u$ with a vertex of $Y \setminus u$. Denoted by $F_{k,l}$, the 1-sum of two star at a vertex u of degree 1. In this paper, the state transfer in quantum walk of these graphs is explored. We show that there is no perfect state transfer in these graphs, but there is pretty good state transfer when k = l.

2 Perfect State transfer

We would like to introduce a useful tool. For more expansive treatment, the refer is referred to [10].

Let G be a graph and $\pi = \{C_1, C_2, \cdots, C_r\}$ be a partition of V(X). We say π is equitable if every vertex in C_i has the same numbers b_{ij} of neighbors in C_j . The quotient graph of G induced by π , denoted by G/π , is a directed graph with vertex set π and b_{ij} arcs from the ith to jth cells of π . The entries of the adjacency matrix of G/π are given by $A(G/\pi)_{ij} = b_{ij}$. We can symmetrize $A(G/\pi)$ to B by letting $B_{ij} = \sqrt{b_{ij}b_{ji}}$. We call the weighted graph with adjacency matrix B, the symmetrized quotient graph. In the following, we always use B to denote the symmetrized form of the matrix $A(G/\pi)$. We list some known results which will be used within this paper.

Lemma 2.1 ([3]). Let G be a graph and let u, v be two vertices of G. Then there is perfect state transfer between u and v at time t with phase γ if and only if all of the following conditions hold.

- i) Vertices u and v are strongly cospectral.
- ii) There are integers a, \triangle where \triangle is square-free so that for each eigenvalue λ in supp_G(u):
- (a) $\lambda = \frac{1}{2}(a + b_{\lambda}\sqrt{\triangle})$, for some integer b_{λ} .
- (b) $e_u^T E_{\lambda}(G) e_v$ is positive if and only if $(\rho(G) \lambda)/g\sqrt{\triangle}$ is even, where

$$g := gcd(\{\frac{\rho(G) - \lambda}{\sqrt{\triangle}}\} : \lambda \in Supp_G(u))$$

Moreover, if the above conditions hold, then the following also hold.

i) There is a minimum time of perfect state transfer between u and v given by

$$t_0 \coloneqq \frac{\pi}{g\sqrt{\triangle}}$$

- ii) The time of perfect state transfer t is an odd multiple of t_0 .
- iii) The phase of perfect state transfer is given by $\gamma = e^{-it\rho(G)}$.

Lemma 2.2 ([8]). Let G be a graph with perfect state transfer between vertices u and v. Then, for each automorphism $\tau \in Aut(G)$, $\tau(u) = u$ if and only if $\tau(v) = v$.

Lemma 2.3 ([10]). Suppose G has pretty good state transfer between vertices u and v. Then u and v are strongly cospectral, and each automorphism fixing u must fix v.

Lemma 2.4 ([2]). Let X be a graph with an equitable π and assume $\{a\}$ and $\{b\}$ are singleton cells of π . Let B denote the adjacency matrix of the symmetrized quotient graph relative π . Then for any time t,

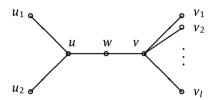
$$(e^{-itA(X)})_{a,b} = (e^{-itB})_{\{a\},\{b\}}$$

and therefore X has perfect state transfer from a to b at time t if and only if the symmetrized quotient graph has perfect state transfer from $\{a\}$ to $\{b\}$.

Let S_{k+1} and S_{l+1} be the two star graphs with k+1 and l+1 edges respectively and w be a vertex of degree one in S_{k+1} and S_{l+1} . Then the 1-sum of S_{k+1} and S_{l+1} , denoted by $F_{k,l}$, is obtained by merging the vertex wof S_{k+1} with the vertex w of S_{l+1} and no edge joins a vertex of $S_{k+1} \setminus w$ with a vertex of $S_{l+1} \setminus w$.

Lemma 2.5. There is no perfect state transfer from u_1 to u_2 on $F_{2,1}$ (Fig. 1).

Fig. 1. Graph F_{2l}



Proof. Let $F_{2,l}$ be a graph shown in Fig. 1. Let π be the equitable partition with cells

$$\{\{u_1\}, \{u_2\}, \{u\}, \{w\}, \{v\}, N(v)\setminus \{w\}\}.$$

Then the adjacency matrix B of the corresponding symmetrized quotient graph induced by π is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & \sqrt{l} \\ 0 & 0 & 0 & 0 & \sqrt{l} & 0 \end{pmatrix}.$$

The eigenvalue of *B* are

$$\theta_1 = 0$$
 with multiplicity 2,

$$\theta_2 = \frac{\sqrt{2}}{2} \sqrt{l + 4 + \sqrt{l^2 - 4l + 8}}$$

$$\theta_2 = \frac{\sqrt{2}}{2} \sqrt{l + 4 + \sqrt{l^2 - 4l + 8}},$$

$$\theta_3 = -\frac{\sqrt{2}}{2} \sqrt{l + 4 + \sqrt{l^2 - 4l + 8}},$$

$$\theta_4 = \frac{\sqrt{2}}{2} \sqrt{l + 4 - \sqrt{l^2 - 4l + 8}},$$

$$\theta_4 = \frac{\sqrt{2}}{2} \sqrt{l + 4 - \sqrt{l^2 - 4l + 8}},$$

$$\theta_5 = -\frac{\sqrt{2}}{2} \sqrt{l + 4 - \sqrt{l^2 - 4l + 8}}.$$

If there is perfect state transfer from $\{u_1\}$ to $\{u_2\}$ on $F_{2,l}/\pi$, by Lemma 2.1,

$$\frac{\theta_2 - \theta_3}{\theta_4 - \theta_5}$$

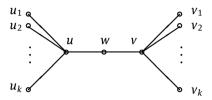
is rational. Therefore

$$\frac{(\theta_2 - \theta_3)^2}{(\theta_4 - \theta_5)^2} = \frac{l + 4 + \sqrt{l^2 - 4l + 8}}{l + 4 - \sqrt{l^2 - 4l + 8}} = \frac{l^2 - 2l + 12}{6l + 4} + \frac{(l + 4)\sqrt{l^2 - 4l + 8}}{6l + 4}$$

is rational, and hence l^2-4l+8 is a perfect square. This means that there exists a integer p such that $l^2-4l+8=(l-2)^2+4=p^2$. Note that both l-2 and p are integers. This can occur only if l=2. But in this case $(\theta_2-\theta_3)/(\theta_4-\theta_5)=\sqrt{2}$ is irrational. This is a contradiction. Hence there is no perfect state transfer from $\{u_1\}$ to $\{u_2\}$ on $F_{2,l}/\pi$ and by Lemma 2.4, there is no perfect state transfer from u_1 to u_2 on v_2 .

Next we start to study perfect state transfer between two vertices u and v on $F_{k,k}$ (see Fig. 2).

Fig. 2. Graph $F_{k,k}$



Lemma 2.6. There is no perfect state transfer from u to v on $F_{k,k}$.

Proof. Suppose $F_{k,k}$ be a graph shown in Fig.2. Let π be the equitable partition with cells

$$\{N(u) \setminus \{w\}, \{u\}, \{w\}, \{v\}, N(v) \setminus \{w\}\}.$$

Then the adjacency matrix B of the corresponding symmetrized quotient graph induced by π is

$$\begin{pmatrix} 0 & \sqrt{k} & 0 & 0 & 0 \\ \sqrt{k} & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \sqrt{k} \\ 0 & 0 & 0 & \sqrt{k} & 0 \end{pmatrix}.$$

The eigenvalue of *B* are

$$\theta_1 = 0$$
, $\theta_2 = \sqrt{k}$, $\theta_3 = -\sqrt{k}$, $\theta_4 = \sqrt{k+2}$, $\theta_5 = -\sqrt{k+2}$.

The corresponding eigenvectors are the columns of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & \sqrt{k} & \sqrt{k} \\ 0 & 1 & -1 & \sqrt{k+2} & -\sqrt{k+2} \\ -\sqrt{k} & 0 & 0 & 2 & 2 \\ 0 & -1 & 1 & \sqrt{k+2} & -\sqrt{k+2} \\ 1 & -1 & -1 & \sqrt{k} & \sqrt{k} \end{pmatrix}.$$

It is easy to check that $\theta_i \in Supp_G(\{u\})$ for i=2,3,4,5. If there is perfect state transfer from $\{u\}$ to $\{v\}$ on $F_{k,k}/\pi$, by Lemma 2.1, there exist integers m, n and a square-free integer p such that $\theta_2 - \theta_3 = m\sqrt{p}$ and $\theta_4 - \theta_5 = n\sqrt{p}$. Then $n > m \ge 1$. Now

$$(n^2 - m^2)p = n^2p - m^2p = (k+2) - k = 2.$$

This is impossible. Hence there is no perfect state transfer from $\{u\}$ to $\{v\}$ on $F_{k,k}/\pi$ and by Lemma 2.4, there is no perfect state transfer from u to v on $F_{k,k}$.

Lemma 2.7. w is not strongly cospectral with the other vertices.

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Proof. If w is strongly cospectral with the vertex a, then they must have the same degree. Thus $a \neq u_i$ (i = $1, 2, \dots, k$) and $a \neq v_i$ $(i = 1, 2, \dots, k)$. Without loss of generality, suppose a = u. Then k = 1. Note that $F_{1,l}\setminus\{u\}=K_1\cup S_{l+1}$ and $F_{1,l}\setminus\{w\}=K_2\cup S_l$. These are not cospectral, hence it follows they are not strongly cospectral which in itself provides a contradiction.

Theorem 2.8. There is no perfect state transfer on $F_{k,l}$.

Proof. We divide into four cases to discuss:

- (1) Note that if there exists perfect state transfer from a to b, then a and b must be strongly cospectral. Since w is not strongly cospectral with the other vertices, there is no perfect state transfer from w to other vertices.
- (2) If there exists perfect state transfer from u to v, then u and v must have the same degree. This means that k = l. By Lemma 2.7, there is no perfect state transfer from u to v on $F_{k,k}$.
- (3) If there exists perfect state transfer from u_i to u_i , by Lemma 2.2, k = 2. By Lemma 2.5, there is no perfect state transfer from u_1 to u_2 on $F_{2,l}$. A similar argument will show that there is no perfect state transfer from v_i to v_i .
- (4) If there exists perfect state transfer from u_i to v_i , by Lemma 2.2, k = l = 1. Then the graph is P_5 . We know that there is no perfect state transfer in P_5 from [5].

Pretty Good State transfer

In this section, we will investigate pretty good state transfer on 1-sum of star graphs. We first introduce Kronecker approximation theorem on simultaneous approximation of numbers, which will be used later.

Lemma 3.1 ([12]). Let $1, \lambda_1, \dots, \lambda_m$ be linearly independent over Q. Let $\alpha_1, \dots, \alpha_m$ be arbitrary real numbers, and let N, ϵ be positive real numbers. Then there are integers l > N and q_1, \dots, q_m so that

$$|l\lambda_k - q_k - \alpha_k| < \epsilon,$$

for each $k = 1, \dots, m$.

Lemma 3.2. There is no pretty good state transfer from u_1 to u_2 on $F_{2,1}$.

Proof. Suppose θ_i ($i=1,2,\cdots,5$) is the eigenvalue of B given in Lemma 2.6. Then the corresponding eigenvectors of θ_1 are

$$x_{11} = (-1, 1, 0, 0, 0, 0)^T$$
, $x_{12} = (\frac{\sqrt{l}}{2}, \frac{\sqrt{l}}{2}, 0, -\sqrt{l}, 0, 1)^T$.

Then $\theta_1 \in Supp(\{u_1\})$. Denote by E_1 the orthogonal projection onto the eigenvector belonging to $\theta_1 = 0$. Note that x_{11} and x_{12} are orthogonal. Then

$$E_1e_{\{u_1\}} = \frac{l}{2}(1, -1, 0, 0, 0, 0)^T + \frac{2}{3l+2}(\frac{l}{4}, \frac{l}{4}, 0, -\sqrt{l}, 0, 1)^T,$$

$$E_1e_{\{u_2\}} = \frac{l}{2}(-1, 1, 0, 0, 0, 0)^T + \frac{2}{3l+2}(\frac{l}{4}, \frac{l}{4}, 0, -\sqrt{l}, 0, 1)^T.$$

Therefore $E_1e_{\{u_1\}} \neq \pm E_1e_{\{u_2\}}$. This means that $\{u_1\}$ and $\{u_2\}$ are not strongly cospectral. By Lemma 2.3, there is no pretty good state transfer between $\{u_1\}$ and $\{u_2\}$. Therefore, by Lemma 2.4, there is no pretty good state transfer between u_1 and u_2 .

Lemma 3.3. There is pretty good state transfer from u to v on $F_{k,k}$ for any positive integer k.

Proof. Let π be the equitable partition with cells $\{N(u) \setminus \{w\}, \{u\}, \{w\}, \{v\}, N(v) \setminus \{w\}\}\}$ and B be the adjacency matrix of the corresponding symmetrized quotient graph induced by π . The eigenvalues and its corresponding eigenvectors are given in Lemma 2.6. Hence

$$exp(itB)_{\{u\},\{u\}} = \sum_{r=1}^{5} exp(it\theta_r)(E_r)_{\{u\},\{u\}}$$

= $-\frac{1}{2}\cos(t\theta_2) + \frac{1}{2}\cos(t\theta_4)$.

Hence there is pretty good state transfer from $\{u\}$ to $\{v\}$ if and only if there exists a sequence of times $(t_i)_{i\geq 0}$ such that

$$\lim_{i\to\infty}\cos(t_i\theta_2)=-\lim_{i\to\infty}\cos(t_i\theta_4)=\pm 1.$$

In the following, we will show that there exists a sequence of times $(t_i)_{i\geq 0}$ such that $\lim_{i\to\infty}\cos(t_i\theta_2)=-\lim_{i\to\infty}\cos(t_i\theta_4)=-1$, which holds if there exist $m,n\in Z$ such that $t_i\theta_2\approx (2m+1)\pi$ and $t_i\theta_4\approx 2n\pi$. The question becomes whether we can choose m,n such that

$$n\sqrt{\frac{k}{k+2}}-m\approx\frac{1}{2}.$$

If $\sqrt{\frac{k}{k+2}}$ is rational, then there exist integers p,q and a square-free integer \triangle such that $\sqrt{k}=p\sqrt{\triangle}$ and $\sqrt{k+2}=q\sqrt{\triangle}$. Then $q>p\geq 1$. Now

$$(q^2 - p^2) \triangle = q^2 \triangle - p^2 \triangle = (k+2) - k = 2,$$

which is impossible. Hence $\sqrt{\frac{k}{k+2}}$ is irrational and so 1 and $\sqrt{\frac{k}{k+2}}$ are linearly independent. By Kronecker approximation theorem there exist $m, n \in \mathbb{Z}$ such that for any positive real number ϵ ,

$$\left|n\sqrt{\frac{k}{k+2}}-m-\frac{1}{2}\right|<\epsilon$$

This means that we can choose m, n such that $n\sqrt{\frac{k}{k+2}}-m\approx\frac{1}{2}$ and so there exists a sequence of times $(t_i)_{i\geq 0}$ such that $\lim_{i\to\infty}\cos(t_i\theta_2)=-\lim_{i\to\infty}\cos(t_i\theta_4)=-1$.

Theorem 3.4. There is pretty good state transfer on $F_{k,l}$ if and only if

- (1) k = l and pretty good state transfer occurs between u and v;
- (2) k = l = 1 and pretty good state transfer occurs between its end vertices.

Proof. We divide into four cases to discuss, which is similar to Theorem 2.8:

- (1) By Lemma 2.7 *w* is not strongly cospectral with the other vertices. By Lemma 2.3 there is no pretty good state transfer from *w* to other vertices.
- (2) If there exists pretty good state transfer from u to v, then u and v must have the same degree. This means that k = l. By Lemma 3.3, pretty good transfer occurs between u and v in this case.
- (3) If there exists pretty good state transfer from u_i to u_j , by Lemma 2.3, k = 2. But by Lemma 3.1, pretty good state transfer does not occur between u_1 and u_2 on $F_{2,l}$ in this case. A similar argument will show that there is no pretty good state transfer between v_i and v_j .
- (4) If there exists perfect state transfer from u_i to v_j , by Lemma 2.3, k = l = 1. Thus the graph is P_5 . We know that there is pretty good state transfer on P_5 between its end vertices.

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