## Open Mathematics

## Research Article

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# Dunkl analogue of Szász-Mirakjan operators of blending type

**Abstract:** In the present work, we construct a Dunkl generalization of the modified Szász-Mirakjan operators of integral form defined by Păltanea [1]. We study the approximation properties of these operators including weighted Korovkin theorem, the rate of convergence in terms of the modulus of continuity, second order modulus of continuity via Steklov-mean, the degree of approximation for Lipschitz class of functions and the weighted space. Furthermore, we obtain the rate of convergence of the considered operators with the aid of the unified Ditzian-Totik modulus of smoothness and for functions having derivatives of bounded variation.

**Keywords:** Linear positive operators, Szász-Mirakjan operators, unified Ditzian-Totik modulus of smoothness, weighted spaces, Dunkl operator

**MSC:** 41A10, 41A25, 41A28, 41A35, 41A36

## 1 Introduction

The theory of approximation deals with finding out functions which are easy to evaluate, like polynomials, and using them in order to approximate complicated functions. In this direction, Weierstrass (1885) was the first who gave a result for functions in $C[a, b]$, known as the Weierstrass approximation theorem which had a valuable impact on the growth of many branches of mathematics. Many researchers like Picard, Fejer, Landau, De la Vallee Poussin proved the Weierstrass theorem by using singular integrals. In 1912, Bernstein [2] established the Weierstrass theorem for $\mathfrak{h} \in C[0, 1]$, by constructing a sequence of linear positive operators as

$$B_n(\mathfrak{h}; x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \mathfrak{h}\left(\frac{k}{n}\right), \quad n \in \mathbb{N}, \; x \in [0, 1].$$

In 1950, for any $\mathfrak{h} \in C[0, \infty)$, Szász [3] demonstrated that the sequence

$$S_n(\mathfrak{h}; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \mathfrak{h}\left(\frac{k}{n}\right), \quad x \in [0, \infty),$$

converges to $\mathfrak{h}(x)$, provided the infinite series on the right side converges. In [4], Rosenblum defined an expression for the generalized exponential function as

$$e_\nu(x) := \sum_{r=0}^{\infty} \frac{x^r}{\gamma_\nu(r)},$$

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where the coefficients $\gamma_\nu(r)$ are defined as follows:

$$\gamma_\nu(2r) = \frac{2^{2r}r!\Gamma(r+\nu+1/2)}{\Gamma(\nu+1/2)} = (2m)!\frac{\Gamma(r+\nu+1/2)}{\Gamma(\nu+1/2)}\frac{\Gamma(1/2)}{\Gamma(\nu+1/2)}$$

and

$$\gamma_\nu(2r+1) = \frac{2^{2r+1}r!\Gamma(r+\nu+3/2)}{\Gamma(\nu+1/2)} = (2m+1)!\frac{\Gamma(r+\nu+3/2)}{\Gamma(\nu+1/2)}\frac{\Gamma(1/2)}{\Gamma(\nu+3/2)}$$

for $r \in \mathbb{N}_0$ and $\nu > -1/2$.

The generalized factorial $\gamma_\nu$ satisfies the following recurrence relation

$$\gamma_\nu(r+1) = (r+1+2\nu\theta_{r+1})\gamma_\nu(r), \quad r \in \mathbb{N}_0, \tag{1}$$

where $\theta_r$ is defined to be 0 if r is a positive even integer and 1 if r is a positive odd integer.

In 2014, Sucu [5] established a relation of the generalized exponential function with a positive approximation process for continuous functions. For $\nu \geq 0$, $n \in \mathbb{N}$, $x \geq 0$ and $f \in C[0,\infty)$, Sucu defined the following operator generated by extended exponential function as

$$S_n^*(f;x) = \frac{1}{e_\nu(nx)}\sum_{r=0}^\infty \frac{(nx)^r}{\gamma_\nu(r)}f\left(\frac{r+2\nu\theta_r}{n}\right). \tag{2}$$

The operator defined by (2) is known as Dunkl generalization of Szász operators. The author studied qualitative and weighted approximation results for these operators. In 2015, İçöz and Çekim [6], introduced a Dunkl modification of Szász operators defined by Sucu [5] via $q$–calculus and studied the rate of convergence of these operators. In the same year, İçöz and Çekim [7], also studied the approximation properties of Stancu type generalization of Dunkl analogue of Szász-Kantorovich operators.

In 2016, Mursaleen et al. [8], introduced Dunkl generalization of Szász operators involving a sequence $r_n(x) = x - \frac{1}{2n}$, $n \in \mathbb{N}$ and established some direct results. Subsequently, Mursaleen and Nasiruzzaman [9] studied $q$-Dunkl generalization of Kantorovich type Szász-Mirakjan operators.Very recently Wafi and Rao [10] constructed Szász-Durrmeyer type operators based on Dunkl analogue and investigated the rate of convergence by means of classical modulus of continuity, uniform approximation using Korovkin type theorem on a compact interval.

In 2008, Pǎltanea [1] introduced the following family of modified Szász-Mirakjan operators of integral form:

$$L_n^\rho(f;x) = e^{-nx}\left(f(0) + \sum_{k=1}^\infty \frac{(nx)^k}{k!}\int_0^\infty \frac{n\rho}{\Gamma(k\rho)}e^{-n\rho t}(n\rho t)^{k\rho-1}f(t)dt\right), \tag{3}$$

where $n > 0, \rho > 0, x \geq 0$ and $f : [0,\infty) \to \mathbb{R}$ is taken such that the above formula is well defined. In our present work, we define the Dunkl analogue of the operator (3). For $\rho > 0$, $\nu > 0$ and $\mathfrak{h} \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : |f(t)| \leq \mathcal{M}_f(1+t^\gamma)$, ($\mathcal{M}_f$ is a constant depending only on function f) $\forall\, t \geq 0\}$, we introduce

$$\mathcal{L}_{n,\rho}(\mathfrak{h};x) = \frac{1}{e_\nu(nx)}\sum_{k=1}^\infty \frac{(nx)^k}{\gamma_\nu(k)}\int_0^\infty \frac{n\rho}{\Gamma(k\rho)}e^{-n\rho u}(n\rho u)^{k\rho-1}\,\mathfrak{h}(u)du + \frac{f(0)}{e_\nu(nx)} \tag{4}$$

or, equivalently,

$$\mathcal{L}_{n,\rho}(\mathfrak{h};x) = \int_0^\infty \mathcal{W}(x,n,\rho,u)\mathfrak{h}(u)du, \tag{5}$$

where

$$\mathcal{W}(x,n,\rho,u) = \sum_{k=1}^\infty \frac{1}{e_\nu(nx)}\frac{(nx)^k}{\gamma_\nu(k)}\frac{n\rho}{\Gamma(k\rho)}e^{-n\rho t}(n\rho t)^{k\rho-1} + \frac{1}{e_\nu(nx)}\delta(0),$$

$\delta(t)$ being the Dirac-delta function.

We define the operator given by (4) as a Dunkl generalization of modified Szász-Mirakjan operators of integral form. In the present paper, our focus is to study the approximation properties of these operators via weighted Korovkin theorem, Steklov mean, the Lipschitz class functions, and the moduli of continuity (classical and weighted). We also establish the degree of approximation in terms of the unified Ditzian-Totik modulus of smoothness and for functions having derivatives of bounded variation.

## 2 Preliminaries

In the following lemma we obtain the estimates of the moments for the operators $\mathcal{L}_{n,\rho}(.;x)$.

**Lemma 2.1.** *The operators $\mathcal{L}_{n,\rho}$ satisfy the following inequalities:*

(i) $\mathcal{L}_{n,\rho}(1;x) = 1$,

(ii) $|\mathcal{L}_{n,\rho}(u;x) - x| \leq \dfrac{2\nu}{n}$,

(iii) $|\mathcal{L}_{n,\rho}(u^2;x) - x^2| \leq \dfrac{x(1 + 6\nu\rho + \rho)}{n\rho} + \dfrac{4\nu^2\rho - 2\nu}{n^2\rho}$,

(iv) $|\mathcal{L}_{n,\rho}(u^3;x) - x^3| \leq \dfrac{3x^2(1 + \rho)}{n\rho} + \dfrac{x}{n^2\rho^2}(2 + (3 - 6\nu)\rho + (1 + 4\nu^2 - 2\nu)\rho^2)$,

(v) $|\mathcal{L}_{n,\rho}(u^4;x) - x^4| \leq \dfrac{x^3}{n\rho} + \dfrac{x^2}{n^2\rho^2}(11 + 18\rho + (7 + 16\nu^2 + 10\nu)\rho^2) + \dfrac{x}{n^3\rho^3}(6 + (11 + 18\nu)\rho) + (6 + 24\nu^2 +$

$18\nu)\rho^2 + (1 - 34\nu + 4\nu^2 + 17\nu^3)\rho^3 + \dfrac{1}{n^4\rho^3}(32\nu^4\rho^3 + 48\nu^3\rho^2 + 4\nu^2\rho - 12\nu)$.

*Consequently, for the operator $\mathcal{L}_{n,\rho}(.;x)$, we have the following inequalities:*

(a) $|\mathcal{L}_{n,\rho}(u - x;x)| \leq \dfrac{2\nu}{n}$,

(b) $\mathcal{L}_{n,\rho}((u - x)^2;x) \leq \dfrac{x(1 + \rho + 10\nu\rho)}{n\rho} + \dfrac{4\nu^2\rho - 2\nu}{n^2\rho}$,

(c) $\mathcal{L}_{n,\rho}((u - x)^4;x) \leq \dfrac{24x^3(1 + \rho + 2\nu\rho)}{n\rho} + \dfrac{x^2}{n^2\rho^2}(19 + (30 - 36\nu)\rho + (11 + 2\nu + 56\nu^2)\rho^2) + \dfrac{x}{n^3\rho^3}(6 +$

$(11 + 34\nu)\rho + (6 + 18\nu + 72\nu^2)\rho^2 + (1 - 34\nu + 4\nu^2 + 49\nu^3)\rho^3) + \dfrac{1}{n^4\rho^3}(32\nu^4\rho^3 + 48\nu^3\rho^2 + 4\nu^2\rho - 12\nu)$.

*Proof.* (i) $\mathcal{L}_{n,\rho}(1;x) = \dfrac{1}{e_\nu(nx)} \displaystyle\sum_{k=1}^{\infty} \dfrac{(nx)^k}{\gamma_\nu(k)} \int_0^\infty \dfrac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho - 1} . 1 \, dt + \dfrac{1}{e_\nu(nx)}$,

$= \dfrac{1}{e_\nu(nx)} \displaystyle\sum_{k=1}^{\infty} \dfrac{(nx)^k}{\gamma_\nu(k)\Gamma(k\rho)} \int_0^\infty e^{-t} t^{k\rho - 1} \, dt + \dfrac{1}{e_\nu(nx)}$

$= \dfrac{1}{e_\nu(nx)} \displaystyle\sum_{k=0}^{\infty} \dfrac{(nx)^k}{\gamma_\nu(k)}$

$= 1$.

(ii) Using (1), we have

$\mathcal{L}_{n,\rho}(u;x) = \dfrac{1}{e_\nu(nx)} \displaystyle\sum_{k=1}^{\infty} \dfrac{(nx)^k}{\gamma_\nu(k)} \int_0^\infty \dfrac{n\rho}{\Gamma(k\rho)} e^{-n\rho t}(n\rho t)^{k\rho - 1} u \, dt$,

$= \dfrac{1}{n\, e_\nu(nx)} \left[ \displaystyle\sum_{k=0}^{\infty} \dfrac{(nx)^{k+1}}{\gamma_\nu(k)} - 2\nu \sum_{k=1}^{\infty} \theta_k \dfrac{(nx)^k}{\gamma_\nu(k)} \right]$

$= \dfrac{1}{n\, e_\nu(nx)} \left[ nx\, e_\nu(nx) - 2\nu \displaystyle\sum_{k=1}^{\infty} \theta_k \dfrac{(nx)^k}{\gamma_\nu(k)} \right]$

$= x - \dfrac{2\nu}{n\, e_\nu(nx)} \displaystyle\sum_{k=1}^{\infty} \theta_k \dfrac{(nx)^k}{\gamma_\nu(k)}$

Hence

$|\mathcal{L}_{n,\rho}(u;x) - x| \leq \dfrac{2\nu}{n\, e_\nu(nx)} \displaystyle\sum_{k=0}^{\infty} \dfrac{(nx)^k}{\gamma_\nu(k)} = \dfrac{2\nu}{n}$.

Using similar calculations, one can easily prove $(iii) - (v)$, therefore the details are omitted. The consequences $(a) - (c)$ are straightforward, hence we skip the proofs. □

**Remark 2.2.** *Using Lemma 2.1 and choosing $C(\nu, \rho) = Max\left( \dfrac{(1 + \rho + 10\nu\rho)}{\rho}, \dfrac{4\nu^2\rho - 2\nu}{\rho} \right)$, we obtain*

$$\mathcal{L}_{n,\rho}((u - x)^2;x) \leq \dfrac{C(\nu, \rho)}{n}(1 + x) = \psi_{n,\nu,\rho}^2(x), \ say.$$

# 3 Main results

**Theorem 3.1.** *Let* $\mathfrak{h} \in C_\gamma(\mathbb{R}^+)$. *Then,*

$$\lim_{n \to \infty} \mathcal{L}_{n,\rho}(\mathfrak{h}; x) = \mathfrak{h}(x),$$

*uniformly on each compact subset* $\mathcal{A}$ *of* $[0, \infty)$.

*Proof.* In view of Lemma 2.1,

$$\mathcal{L}_{n,\rho}(u^i; x) \to x^i, \text{ as } n \to \infty, \text{ uniformly on } \mathcal{A}, \text{ for } i = 0, 1, 2.$$

Hence, the required result follows from applying the Bohman-Korovkin criterion [11]. □

Let $C_B[0, \infty)$ denote the space of bounded and uniformly continuous functions on $[0, \infty)$ endowed with the sup norm, $\|f\| = \sup_{x \in [0,\infty)} |f(x)|$.

The first and second order modulus of continuity are respectively defined as

$$\omega(f, \delta) = \sup_{x,u,v \in [0,\infty), |u-v| \le \delta} |f(x+u) - f(x+v)|$$

and

$$\omega_2(f, \delta) = \sup_{x,u,v \in [0,\infty), |u-v| \le \delta} |f(x+2u) - 2f(x+u+v) + f(x+2v)|, \quad \delta > 0.$$

**Theorem 3.2.** *Let* $\mathfrak{h} \in C_B[0, \infty)$ *and* $\omega(\mathfrak{h}; \delta)$, $\delta > 0$, *be its first order modulus of continuity. Then the operator* $\mathcal{L}_{n,\rho}(.;)$ *satisfies the inequality*

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \le \left(1 + \sqrt{\frac{x(1 + 10\nu\rho + \rho)}{\rho} + \frac{4\nu^2\rho - 2\nu}{n\rho}}\right) \omega\left(\mathfrak{h}, \frac{1}{\sqrt{n}}\right).$$

*Proof.* By definition of $\omega(\mathfrak{h}; \delta)$, Lemma 2.1 and Cauchy-Schwarz inequality, we may get

$$\begin{aligned}
|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| &\le \mathcal{L}_{n,\rho}(|\mathfrak{h}(u) - \mathfrak{h}(x)|; x) \\
&\le \left(1 + \frac{1}{\delta}\mathcal{L}_{n,\rho}(|u - x|; x)\right)\omega(\mathfrak{h}; \delta) \\
&\le \left(1 + \frac{1}{\delta}\sqrt{\mathcal{L}_{n,\rho}((u-x)^2; x)}\right)\omega(\mathfrak{h}; \delta) \\
&\le \left(1 + \frac{1}{\delta}\sqrt{\frac{x(1 + 10\nu\rho + \rho)}{n\rho} + \frac{4\nu^2\rho - 2\nu}{n^2\rho}}\right)\omega(\mathfrak{h}; \delta).
\end{aligned}$$

Now, choosing $\delta = n^{-1/2}$, we immediately obtain the result. □

**Corollary 3.3.** *If* $\mathfrak{h} \in Lip_M(\alpha)$, $0 < \alpha \le 1$, *then*

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}| \le \frac{M}{n^{\alpha/2}}\left(1 + \sqrt{\frac{x(1 + 10\nu\rho + \rho)}{\rho} + \frac{4\nu^2\rho - 2\nu}{n\rho}}\right).$$

For $f \in C_B[0, \infty)$, the Steklov mean is defined as

$$f_h(x) = \frac{4}{h^2} \int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} [2f(x+u+v) - f(x+2(u+v))]\,du\,dv. \tag{6}$$

**Lemma 3.4** ([12])**.** *The Steklov mean* $f_h(x)$ *satisfies the following properties:*
 1. $\|f_h - f\| \le \omega_2(f, h)$,

2. $f_h', f_h'' \in C_B[0, \infty)$ *and*

$$\|f_h'\| \le \frac{5}{h}\omega(f, h), \quad \|f_h''\| \le \frac{9}{h^2}\omega_2(f, h).$$

**Theorem 3.5.** *For* $\mathfrak{h}' \in C_B[0, \infty)$, *we have*

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \le M\frac{2\nu}{n} + 2\omega(\mathfrak{h}'; \psi_{n,\nu,\rho}(x))\psi_{n,\nu,\rho}(x),$$

*where M is some positive constant, $\psi_{n,\nu,\rho}(x)$ is as defined in Remark 2.2 and $\omega(\mathfrak{h}'; \delta)$ denotes the modulus of continuity of* $\mathfrak{h}'$.

*Proof.* Since $\mathfrak{h}' \in C_B[0, \infty) \; \exists \; M > 0$ such that $|\mathfrak{h}'(x)| \le M, \; \forall \; x \ge 0$.

Using mean value theorem, one may write

$$\mathfrak{h}(u) = \mathfrak{h}(x) + (u - x)\mathfrak{h}'(\zeta)$$
$$= \mathfrak{h}(x) + (u - x)\mathfrak{h}'(x) + (u - x)(\mathfrak{h}'(\zeta) - \mathfrak{h}'(x)),$$

where $\zeta$ lies between u and x.

Now, applying the operator $\mathcal{L}_{n,\rho}(.; x)$ on both sides of the above equality and using Lemma 2.1, we get

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \le |\mathfrak{h}'(x)||\mathcal{L}_{n,\rho}(u - x; x)| + \mathcal{L}_{n,\rho}(|u - x||\mathfrak{h}'(\zeta) - \mathfrak{h}'(x)|; x)$$

$$\le M\frac{2\nu}{n} + \mathcal{L}_{n,\rho}(|u - x||\mathfrak{h}'(\zeta) - \mathfrak{h}'(x)|; x). \tag{7}$$

Now, from (5), and Cauchy-Schwarz inequality, we can get

$$\mathcal{L}_{n,\rho}(|u - x||\mathfrak{h}'(\zeta) - \mathfrak{h}'(x)|; x) = \int_0^\infty \mathcal{W}(x, n, \rho, u)|u - x||\mathfrak{h}'(\zeta) - \mathfrak{h}'(x)|du$$

$$\le \int_0^\infty \mathcal{W}(x, n, \rho, u)|u - x|\omega(\mathfrak{h}', \delta)\left(1 + \frac{|u - x|}{\delta}\right)du$$

$$\le \int_0^\infty \mathcal{W}(x, n, \rho, u)\omega(\mathfrak{h}', \delta)\left(|u - x| + \frac{(u - x)^2}{\delta}\right)du$$

$$\le \omega(\mathfrak{h}', \delta)(\psi_{n,\nu,\rho}(x)) + \frac{\omega(\mathfrak{h}', \delta)}{\delta}\psi_{n,\nu,\rho}^2(x). \tag{8}$$

Choosing $\delta = \psi_{n,\nu,\rho}(x)$ and combining (7)-(8), we arrive to conclusion. □

**Theorem 3.6.** *Let* $\mathfrak{h} \in C_B[0, \infty)$. *Then for each* $x \in [0, \infty)$, *we have*

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \le \frac{10\nu}{\sqrt{n}}\omega(\mathfrak{h}; n^{-1/2}) + \left(2 + \frac{9}{2}\left\{\frac{x(1 + 10\nu\rho + \rho)}{\rho} + \frac{4\nu^\rho - 2\nu}{n\rho}\right\}\right)\omega_2(\mathfrak{h}; n^{-1/2}).$$

*Proof.* Applying Lemma 2.1 and Lemma 3.4, one has

$$|\mathcal{L}_{n,\rho}(\mathfrak{h} - f_h; x)| \le \|\mathfrak{h} - f_h\| \le \omega_2(\mathfrak{h}; h).$$

Since $f_h'' \in C_B[0, \infty)$, by Taylor's expansion,

$$f_h(u) = f_h(x) + (u - x)f_h'(x) + \int_x^u (u - s)f_h''(s)ds.$$

Applying operator $\mathcal{L}_{n,\rho}(.; x)$ on the above equality, we get

$$|\mathcal{L}_{n,\rho}(f_h(u) - f_h(x); x)| \le \|f_h'\| \, |\mathcal{L}_{n,\rho}(u - x; x)| + \frac{\|f_h''\|}{2}\mathcal{L}_{n,\rho}((u - x)^2; x).$$

Hence, using Lemma 2.1, we have

$$|\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x)| \le |\mathcal{L}_{n,\rho}(\mathfrak{h} - f_h;x)| + |\mathcal{L}_{n,\rho}(f_h - f_h(x);x)| + |f_h(x) - \mathfrak{h}(x)|$$

$$\le \omega_2(\mathfrak{h};h) + \|f_h'\|\frac{2\nu}{n} + \frac{\|f_h''\|}{2}\left(\frac{x(1 + 10\nu\rho + \rho)}{n\rho} + \frac{4\nu^2\rho - 2\nu}{n^2\rho}\right) + \|f_h - \mathfrak{h}\|$$

$$\le \frac{10\nu}{nh}\omega(\mathfrak{h};h) + \left(2 + \frac{9}{2h^2}\left\{\frac{x(1 + 10\nu\rho + \rho)}{n\rho} + \frac{4\nu^2\rho - 2\nu}{n^2\rho}\right\}\right)\omega_2(\mathfrak{h};h).$$

Finally, choosing $h = n^{-1/2}$, the required result is obtained. □

Next, we define some weighted spaces on $[0, \infty)$ to obtain the weighted approximation results for the operators defined by (4).

$$B_\sigma(\mathbb{R}^+) := \{f : |f(x)| \le \mathcal{M}_f\sigma(x)\},$$

$$C_\sigma(\mathbb{R}^+) := \{f : f \in B_\sigma(\mathbb{R}^+) \cap C[0, \infty))\},$$

and

$$C_\sigma^k(\mathbb{R}^+) := \left\{f : f \in C_\sigma(\mathbb{R}^+) \text{ and } \lim_{x\to\infty} \frac{f(x)}{\sigma(x)} = k \text{ (some constant))}\right\},$$

where $\sigma(x) = 1 + x^2$ is a weight function and $\mathcal{M}_f$ is a constant depending only on the function $f$. From [13], it is noted that $C_\sigma(\mathbb{R}^+)$ is a normed linear space endowed with the norm $\|f\|_\sigma := \sup_{x\ge0} \frac{|f(x)|}{\sigma(x)}$.

It is well known that the classical modulus of continuity $\omega(f;\delta)$ does not tend to zero if f is continuous on an infinite interval. Therefore, in order to study the approximation of functions in the weighted space $C_\sigma^k(\mathbb{R}^+)$, Ispir and Atakut [13] introduced the following weighted modulus of continuity

$$\Omega(f;\delta) = \sup_{x\in[0,\infty),|h|\le\delta} \frac{|f(x + h) - f(x)|}{(1 + h^2)(1 + x^2)},$$

and proved that $\lim_{\delta\to0^+} \Omega(f;\delta) = 0$ and

$$|f(u) - f(x)| \le 2\left(1 + \frac{|u - x|}{\delta}\right)(1 + \delta^2)(1 + x^2)(1 + (u - x)^2)\Omega(f;\delta), \quad u, x \in [0, \infty).$$

**Theorem 3.7.** *For each* $\mathfrak{h} \in C_\sigma^k(\mathbb{R}^+)$, *the sequence of linear positive operators* $\{\mathcal{L}_{n,\rho}\}$, *satisfies the following equality*

$$\lim_{n\to\infty} \|\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x)\|_\sigma = 0.$$

*Proof.* From Lemma 2.1, clearly $\lim_{n\to\infty} \|\mathcal{L}_{n,\rho}(1;x) - 1\|_\sigma = 0$.

Now,

$$\sup_{x\ge0} \frac{|\mathcal{L}_{n,\rho}(u;x) - x|}{1 + x^2} \le \frac{2\nu}{n}\sup_{x\ge0}\frac{1}{1 + x^2}$$

$$\le \frac{2\nu}{n}.$$

Therefore, $\lim_{n\to\infty} \|\mathcal{L}_{n,\rho}(u;x) - x\|_\sigma = 0$. Again,

$$\sup_{x\ge0} \frac{|\mathcal{L}_{n,\rho}(u^2;x) - x^2|}{1 + x^2} \le \frac{(1 + \rho + 6\nu\rho)}{n\rho}\sup_{x\ge0}\frac{x}{1 + x^2} + \frac{(4\nu^2\rho - 2\nu)}{n^2\rho}\sup_{x\ge0}\frac{1}{1 + x^2}$$

$$\le \frac{(1 + \rho + 6\nu\rho)}{2n\rho} + \frac{(4\nu^2\rho - 2\nu)}{n^2\rho},$$

we obtain $\lim_{n\to\infty} \|\mathcal{L}_{n,\rho}(u^2;x) - x^2\|_\sigma = 0$. Hence, applying weighted Korovkin-type theorem given by Gadzhiev [14], we reach the desired result. □

**Theorem 3.8.** *Let* $\mathfrak{h} \in C_\sigma^k(\mathbb{R}^+)$. *Then the following inequality is verified*

$$\sup_{x \in [0,\infty)} \frac{|\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x)|}{(1+x^2)^{5/2}} \leq \mathcal{K}\Omega\left(\mathfrak{h}; \frac{1}{\sqrt{n}}\right),$$

*where* $\mathcal{K}$ *is a constant not dependent on* $\mathfrak{h}$ *and* $n$.

*Proof.* Using (5), definition of $\Omega(f;\delta)$, Lemma 2.1 and Cauchy-Schwarz inequality, one can easily see that

$$|\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x)| \leq \int_0^\infty \mathcal{W}(x,n,\rho,u)|\mathfrak{h}(u) - \mathfrak{h}(x)|du$$

$$\leq 2(1+\delta^2)(1+x^2)\Omega(\mathfrak{h};\delta)\int_0^\infty \mathcal{W}(x,n,\rho,u)\left(1 + \frac{|u-x|}{\delta}\right)(1+(u-x)^2)du$$

$$\leq 2(1+\delta^2)(1+x^2)\Omega(\mathfrak{h};\delta)\Bigg(\mathcal{L}_{n,\rho}(1;x) + \mathcal{L}_{n,\rho}((u-x)^2;x)$$

$$+ \frac{1}{\delta}(\mathcal{L}_{n,\rho}((u-x)^2;x))^{1/2} + \frac{1}{\delta}(\mathcal{L}_{n,\rho}((u-x)^2;x))^{1/2} \times (\mathcal{L}_{n,\rho}((u-x)^4;x))^{1/2}\Bigg)$$

Now, choosing $\delta = \frac{1}{\sqrt{n}}$, we arrive to conclusion immediately. □

In our next result, we shall discuss a direct result with the help of unified Ditzian-Totik modulus of smoothness $\omega_{\phi^\lambda}(\mathfrak{h};t)$, $0 \leq \lambda \leq 1$. In 2007, Guo et al. [15] discussed the direct, inverse and equivalence approximation results by means of unified modulus. We consider $\phi^2(x) = 1+x$ and $f \in C_B[0,\infty)$. The modulus $\omega_{\phi^\lambda}(\mathfrak{h},t)$, $0 \leq \lambda \leq 1$, is defined as

$$\omega_{\phi^\lambda}(\mathfrak{h},t) = \sup_{0 \leq h \leq t} \sup_{x \pm \frac{h\phi^\lambda(x)}{2} \in [0,\infty)} \left|\mathfrak{h}\left(x + \frac{h\phi^\lambda(x)}{2}\right) - f\left(x - \frac{h\phi^\lambda(x)}{2}\right)\right|,$$

and the corresponding *K*-functional is given by

$$K_{\phi^\lambda}(\mathfrak{h},t) = \inf_{g \in W_\lambda}\left\{\|\mathfrak{h} - g\| + t\|\phi^\lambda g'\|\right\},$$

where $W_\lambda = \{g : g \in A\,C_{loc}[0,\infty), \|\phi^\lambda g'\| < \infty\}$, $A\,C_{loc}$ is defined as the space of locally absolutely continuous functions on $[0,\infty)$.

From [16], there exists a constant $\mathcal{C} > 0$ such that

$$\mathcal{C}^{-1}\omega_{\phi^\lambda}(\mathfrak{h},t) \leq K_{\phi^\lambda}(\mathfrak{h},t) \leq \mathcal{C}\omega_{\phi^\lambda}(\mathfrak{h},t). \tag{9}$$

**Theorem 3.9.** *For each* $\mathfrak{h} \in C_B[0,\infty)$ *and sufficiently large* $n$, *we have*

$$|\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x)| \leq \mathcal{A}\,\omega_{\phi^\lambda}\left(\mathfrak{h}, \frac{\phi^{1-\lambda}}{\sqrt{n}}\right),$$

*where* $\mathcal{A}$ *is some constant not dependent on* $\mathfrak{h}$ *and* $n$.

*Proof.* By the definition of $K_{\phi^\lambda}(\mathfrak{h};t)$, for fixed $\lambda$, $n$ and $x \in [0,\infty)$, we can find a $g = g_{n,x,\lambda} \in W_\lambda$ such that

$$\|\mathfrak{h} - g\| + \frac{\phi^{1-\lambda}(x)}{\sqrt{n}}\|\phi_\lambda g'\| \leq 2K_{\phi^\lambda}\left(\mathfrak{h}; \frac{\phi^{1-\lambda}}{\sqrt{n}}\right). \tag{10}$$

From the representation of $g$ as $g(u) = g(x) + \int_x^u g'(s)ds$, it follows that

$$|\mathcal{L}_{n,\rho}(g;x) - g(x)| \leq \mathcal{L}_{n,\rho}\left(\left|\int_x^u g'(s)ds\right|;x\right). \tag{11}$$

Applying Hölder's inequality,

$$\left| \int_x^u g'(s)ds \right| \le \|\phi^\lambda g'\| \left| \int_x^u \frac{1}{\phi^\lambda(s)}ds \right|$$

$$\le \|\phi^\lambda g'\| |u - x|^{1-\lambda} \left| \int_x^u \frac{1}{\phi(s)}ds \right|^\lambda$$

$$\le \|\phi^\lambda g'\| |u - x|^{1-\lambda} \{2(\sqrt{1+u} - \sqrt{1+x})\}^\lambda$$

$$\le 2^\lambda \|\phi^\lambda g'\| |u - x| \left| \frac{1}{\sqrt{1+u} + \sqrt{1+x}} \right|^\lambda$$

$$\le \frac{2^\lambda \|\phi^\lambda g'\| |u - x|}{(1+x)^{\lambda/2}}.$$

Hence, in view of Cauchy-Schwarz inequality and Remark 2.2,

$$\mathcal{L}_{n,\rho}\left( \left| \int_x^u g'(s)ds \right|; x \right) \le \frac{2^\lambda \|\phi^\lambda g'\|}{(1+x)^{\lambda/2}} \mathcal{L}_{n,\rho}(|u - x|; x)$$

$$\le \frac{2^\lambda \|\phi^\lambda g'\|}{(1+x)^{\lambda/2}} \{\mathcal{L}_{n,\rho}((u-x)^2; x)\}^{1/2}$$

$$\le \frac{2^\lambda \|\phi^\lambda g'\|}{(1+x)^{\lambda/2}} \frac{\{C(\nu,\rho)\}^{1/2}}{\sqrt{n}} (1+x)^{1/2}$$

$$\le \frac{\phi^{(1-\lambda)}(x)}{\sqrt{n}} \|\phi^\lambda g'\| 2^\lambda \{C(\nu,\rho)\}^{1/2}. \tag{12}$$

We may write,

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \le |\mathcal{L}_{n,\rho}(\mathfrak{h} - g; x)| + |\mathcal{L}_{n,\rho}(g; x) - g(x)| + |g(x) - \mathfrak{h}(x)|$$
$$\le 2\|\mathfrak{h} - g\| + |\mathcal{L}_{n,\rho}(g; x) - g(x)|. \tag{13}$$

Now, using (10)-(13), one can easily obtain

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \le 2\|\mathfrak{h} - g\| + \frac{\phi^{(1-\lambda)/2}(x)}{\sqrt{n}} \|\phi^\lambda g'\| 2^\lambda \{C(\nu,\rho)\}^{1/2}.$$

Choosing $\mathcal{A} = Max(2, 2^\lambda(C(\nu,\rho)^{1/2})$, using (10) and the relation given in (9), we arrive at the required result.

□

Next, we discuss the degree of approximation for functions having derivatives of bounded variation. Let $H[0, \infty)$ denote the space of all $f \in C_2[0, \infty)$ such that $f'$ is equivalent to a function locally of bounded variation.

If $f \in H[0, \infty)$, we may write

$$f(x) = \int_0^x g(u)du + f(0),$$

where $g$ is locally of bounded variation on $[0, \infty)$.

**Lemma 3.10.** *Let* $x \in (0, \infty)$. *Then for sufficiently large n, we have*

*(i)* $\vartheta_{n,\rho}(x, t) = \int_0^t \mathcal{W}(x, n, \rho, u)du \le \frac{C(\nu,\rho)}{(x-t)^2} \frac{(1+x)}{n}, \quad 0 \le t < x,$

*(ii)* $1 - \vartheta_{n,\rho}(x, t) = \int_t^\infty \mathcal{W}(x, n, \rho, u)du \le \frac{C(\nu,\rho)}{(x-t)^2} \frac{(1+x)}{n}, \quad x \le t < \infty,$

*where $C(\nu, \rho)$ is a positive constant depending on $\nu$ and $\rho$.*

*Proof.* (i) Using Remark 2.2, for sufficiently large n, we have $\vartheta_{n,\rho}(x, t) = \int\limits_0^t \mathcal{W}(x, n, \rho, u)du \leq$

$$\int\limits_0^t \left(\frac{x-u}{x-t}\right)^2 \mathcal{W}(x, n, \rho, u)du \leq \frac{1}{(x-t)^2}\mathcal{L}_{n,\rho}((u-x)^2; x) \leq \frac{C(\nu, \rho)}{(x-t)^2}\frac{(1+x)}{n}.$$

(ii) By a similar reasoning, one can easily prove (ii). Hence the details are omitted.

$\square$

In the following theorem, it is shown that the points $x \in (0, \infty)$, where the left hand and right hand derivatives of $f'$ exist, $\mathcal{L}_{n,\rho}(f; x) \to f(x)$, as $n \to \infty$.

**Theorem 3.11.** *Let $\mathfrak{h} \in H[0, \infty)$. Then, for each $x \in (0, \infty)$ and sufficiently large n, we have*

$$|\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x)| \leq \left|\frac{\mathfrak{h}'(x+) + \mathfrak{h}'(x-)}{2}\right|\frac{2\nu}{n} + \sqrt{C(\nu, \rho)\frac{1+x}{n}}\left|\frac{\mathfrak{h}'(x+) - \mathfrak{h}'(x-)}{2}\right|$$

$$+ C(\nu, \rho)\left(\frac{1+x}{n}\right)\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x-\frac{x}{m}}^{x}\mathfrak{h}'_x\right) + \frac{x}{\sqrt{n}}\left(\bigvee_{x-\frac{x}{\sqrt{n}}}^{x}f'_x\right)$$

$$+ \left(4M_{\mathfrak{h}} + \frac{\mathcal{M}_{\mathfrak{h}} + |\mathfrak{h}(x)|}{x^2}\right)C(\nu, \rho)\frac{(1+x^2)}{n} + |\mathfrak{h}'(x+)|\sqrt{C(\nu, \rho)\frac{(1+x^2)}{n}}$$

$$+ C(\nu, \rho)\frac{(1+x^2)}{nx^2}|\mathfrak{h}(2x) - \mathfrak{h}(x) - x\mathfrak{h}'(x+)| + \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\mathfrak{h}'_x\right) + C(\nu, \rho)\frac{(1+x^2)}{nx}\sum_{k=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\mathfrak{h}'_x\right),$$

*where $C(\nu, \rho) > 0$, is a constant and $\bigvee_c^d f$ is the total variation of $f$ on $[c, d]$ and $f'_x$ is defined by*

$$f'_x(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x \\ 0, & t = x, \\ f'(t) - f'(x+) & x < t < \infty. \end{cases} \tag{14}$$

*Proof.* Let $f \in H[0, \infty)$. Then from (14), we can easily write

$$\mathfrak{h}'(u) = \frac{1}{2}\left(\mathfrak{h}'(x+) + \mathfrak{h}'(x-)\right) + \mathfrak{h}'_x(u) + \frac{1}{2}\left(\mathfrak{h}'(x+) - \mathfrak{h}'(x-)\right)\text{sgn}(u-x)$$

$$+\delta_x(u)\left(\mathfrak{h}'(u) - \frac{1}{2}(\mathfrak{h}'(x+) + \mathfrak{h}'(x-))\right), \tag{15}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

Using (5), for each $x \in (0, \infty)$, we have

$$\mathcal{L}_{n,\rho}(\mathfrak{h}; x) - \mathfrak{h}(x) = \int\limits_0^\infty \mathcal{W}(x, n, \rho, u)(\mathfrak{h}(u) - \mathfrak{h}(x))du$$

$$= \int\limits_0^\infty \mathcal{W}(x, n, \rho, u)\left(\int\limits_x^u \mathfrak{h}'(t)dt\right)du \tag{16}$$

$$= -\int\limits_0^x \left(\int\limits_u^x \mathfrak{h}'(t)dt\right)\mathcal{W}(x, n, \rho, u)du + \int\limits_x^\infty \left(\int\limits_x^u \mathfrak{h}'(t)dt\right)\mathcal{W}(x, n, \rho, u)du.$$

Let

$$A_1 := \int\limits_0^x \left(\int\limits_u^x \mathfrak{h}'(t)dt\right)\mathcal{W}(x, n, \rho, u)du, \quad A_2 := \int\limits_x^\infty \left(\int\limits_x^u \mathfrak{h}'(t)dt\right)\mathcal{W}(x, n, \rho, u)du.$$

Using $\int_x^u \delta_x(t)dt = 0$, from (15), we obtain

$$A_1 = \int_0^x \left\{ \int_u^x \left( \frac{1}{2}\left(\mathfrak{h}'(x+) + \mathfrak{h}'(x-)\right) + \mathfrak{h}'_x(t) + \frac{1}{2}\left(\mathfrak{h}'(x+) - \mathfrak{h}'(x-)\right)\operatorname{sgn}(t-x) \right)dt \right\}\mathcal{W}(x, n, \rho, u)du$$

$$= \frac{1}{2}\left(\mathfrak{h}'(x+) + \mathfrak{h}'(x-)\right)\int_0^x (x-u)\mathcal{W}(x, n, \rho, u)du + \int_0^x \left( \int_u^x \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du$$

$$- \frac{1}{2}\left(\mathfrak{h}'(x+) - \mathfrak{h}'(x-)\right)\int_0^x (x-u)\mathcal{W}(x, n, \rho, u)du. \tag{17}$$

Similarly,

$$A_2 = \int_x^\infty \left\{ \int_x^u \left( \frac{1}{2}\left(\mathfrak{h}'(x+) + \mathfrak{h}'(x-)\right) + \mathfrak{h}'_x(t) + \frac{1}{2}\left(\mathfrak{h}'(x+) - \mathfrak{h}'(x-)\right)\operatorname{sgn}(t-x) \right)du \right\}\mathcal{W}(x, n, \rho, u)du$$

$$= \frac{1}{2}\left(\mathfrak{h}'(x+) + \mathfrak{h}'(x-)\right)\int_x^\infty (u-x)\mathcal{W}(x, n, \rho, u)du + \int_x^\infty \left( \int_x^u \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du$$

$$+ \frac{1}{2}\left(\mathfrak{h}'(x+) - \mathfrak{h}'(x-)\right)\int_x^\infty (u-x)\mathcal{W}(x, n, \rho, u)du. \tag{18}$$

Combining the equations (16)-(18), we have

$$\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x) = \frac{1}{2}\left(\mathfrak{h}'(x+) + \mathfrak{h}'(x-)\right)\int_0^\infty (u-x)\mathcal{W}(x, n, \rho, u)du$$

$$+ \frac{1}{2}\left(\mathfrak{h}'(x+) - \mathfrak{h}'(x-)\right)\int_0^\infty |u-x|\mathcal{W}(x, n, \rho, u)du$$

$$- \int_0^x \left( \int_u^x \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du + \int_x^\infty \left( \int_x^u \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du.$$

Therefore,

$$|\mathcal{L}_{n,\rho}(\mathfrak{h};x) - \mathfrak{h}(x)| \le \left| \frac{\mathfrak{h}'(x+) + \mathfrak{h}'(x-)}{2} \right| |\mathcal{L}_{n,\rho}(u-x;x)| + \left| \frac{\mathfrak{h}'(x+) - \mathfrak{h}'(x-)}{2} \right| \mathcal{L}_{n,\rho}(|u-x|;x)$$

$$+ \left| \int_0^x \left( \int_u^x \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du \right| + \left| \int_x^\infty \left( \int_x^u \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du \right|. \tag{19}$$

Next, we assume that

$$E_{n,1,\rho}(\mathfrak{h}'_x, x) = \int_0^x \left( \int_u^x \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du,$$

and

$$E_{n,2,\rho}(\mathfrak{h}'_x, x) = \int_x^\infty \left( \int_x^u \mathfrak{h}'_x(t)dt \right)\mathcal{W}(x, n, \rho, u)du.$$

Now, we need to only estimate $E_{n,1,\rho}(\mathfrak{h}'_x, x)$ and $E_{n,2,\rho}(\mathfrak{h}'_x, x)$.

Using the definition of $\vartheta_{n,\rho}$ given in Lemma 3.10 and integrating by parts, we have

$$E_{n,1,\rho}(\mathfrak{h}'_x, x) = \int_0^x \left( \int_u^x \mathfrak{h}'_x(t)dt \right)\frac{\partial \vartheta_{n,\rho}(x, u)}{\partial u}du = \int_0^x \mathfrak{h}'_x(u)\vartheta_{n,\rho}(x, u)du.$$

Therefore,

$$|E_{n,1,\rho}(\mathfrak{h}'_x, x)| \leq \int\limits_0^{x-\frac{x}{\sqrt{n}}} |\mathfrak{h}'_x(u)| \vartheta_{n,\rho}(x, u) du + \int\limits_{x-\frac{x}{\sqrt{n}}}^x |\mathfrak{h}'_x(u)| \vartheta_{n,\rho}(x, u) du.$$

As we have $f'_x(x) = 0$ and $\vartheta_{n,\rho}(x, u) \leq 1$, we obtain

$$\int\limits_{x-\frac{x}{\sqrt{n}}}^x |\mathfrak{h}'_x(u)| \vartheta_{n,\rho}(x, u) du = \int\limits_{x-\frac{x}{\sqrt{n}}}^x |\mathfrak{h}'_x(u) - \mathfrak{h}'_x(x)| \vartheta_{n,\rho}(x, u) du$$

$$\leq \frac{x}{\sqrt{n}} \left( \bigvee_{x-\frac{x}{\sqrt{n}}}^x f'_x \right).$$

Using Lemma 3.10, Remark 2.2, and considering $u = x - \frac{x}{t}$,

$$\int\limits_0^{x-\frac{x}{\sqrt{n}}} |\mathfrak{h}'_x(u)| \vartheta_{n,\rho}(x, u) du \leq C(\nu, \rho) \left( \frac{1+x}{n} \right) \int\limits_0^{x-\frac{x}{\sqrt{n}}} \frac{|\mathfrak{h}'_x(u)|}{(x-u)^2} du$$

$$\leq C(\nu, \rho) \left( \frac{1+x}{nx} \right) \int\limits_1^{\sqrt{n}} \left( \bigvee_{x-\frac{x}{t}}^x \mathfrak{h}'_x \right) dt \leq C(\nu, \rho) \left( \frac{(1+x)}{nx} \right) \sum_{m=1}^{[\sqrt{n}]} \left( \bigvee_{x-\frac{x}{m}}^x \mathfrak{h}'_x \right).$$

Hence,

$$|E_{n,1,\rho}(\mathfrak{h}'_x, x)| \leq C(\nu, \rho) \left( \frac{1+x}{nx} \right) \sum_{m=1}^{[\sqrt{n}]} \left( \bigvee_{x-\frac{x}{m}}^x \mathfrak{h}'_x \right) + \frac{x}{\sqrt{n}} \left( \bigvee_{x-\frac{x}{\sqrt{n}}}^x \mathfrak{h}'_x \right). \tag{20}$$

Again in order to estimate $E_{n,2,\rho}(\mathfrak{h}'_x, x)$, using integration by parts and Lemma 3.10,

$$|E_{n,2,\rho}(\mathfrak{h}'_x, x)| \leq \left| \int\limits_x^{2x} \left( \int\limits_x^u \mathfrak{h}'_x(t) dt \right) \frac{\partial}{\partial u} (1 - \vartheta_{n,\rho}(x, u)) du \right| + \left| \int\limits_{2x}^\infty \left( \int\limits_x^u \mathfrak{h}'_x(t) dt \right) \mathcal{W}(x, n, \rho, u) du \right|$$

$$\leq \left| \int\limits_x^{2x} \mathfrak{h}'_x(t) dt \right| |1 - \vartheta_{n,\rho}(x, 2x)| + \int\limits_x^{2x} |\mathfrak{h}'_x(u)| (1 - \vartheta_{n,\rho}(x, u)) du$$

$$+ \left| \int\limits_{2x}^\infty (\mathfrak{h}(u) - \mathfrak{h}(x)) \mathcal{W}(x, n, \rho, u)(x, u) du \right| + |\mathfrak{h}'(x+)| \left| \int\limits_{2x}^\infty (u - x) \mathcal{W}(x, n, \rho, u) du \right|.$$

We may write

$$\int\limits_x^{2x} |\mathfrak{h}'_x(u)| (1 - \vartheta_{n,\rho}(x, u)) du = \int\limits_x^{x+\frac{x}{\sqrt{n}}} |\mathfrak{h}'_x(u)| (1 - \vartheta_{n,\rho}(x, u)) du + \int\limits_{x+\frac{x}{\sqrt{n}}}^{2x} |\mathfrak{h}'_x(u)| (1 - \vartheta_{n,\rho}(x, u)) du$$

$$= N_1 + N_2, \quad \text{say}$$

Using $f'_x(x) = 0$ and $1 - \vartheta_{n,\rho}(x, u) \leq 1$, we obtain

$$N_1 = \int\limits_x^{x+\frac{x}{\sqrt{n}}} |\mathfrak{h}'_x(u) - \mathfrak{h}'_x(x)| (1 - \vartheta_{n,\rho}(x, u)) du \leq \int\limits_x^{x+\frac{x}{\sqrt{n}}} \left( \bigvee_x^{x+\frac{x}{\sqrt{n}}} \mathfrak{h}'_x \right) du = \frac{x}{\sqrt{n}} \left( \bigvee_x^{x+\frac{x}{\sqrt{n}}} \mathfrak{h}'_x \right).$$

Using Lemma 3.10 and assuming $u = x + \frac{x}{t}$, we obtain

$$N_2 \leq C(\nu, \rho) \left( \frac{1+x}{n} \right) \int\limits_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(u-x)^2} |\mathfrak{h}'_x(u) - \mathfrak{h}'_x(x)| du \leq C(\nu, \rho) \left( \frac{1+x}{n} \right) \int\limits_{x+\frac{x}{\sqrt{n}}}^{2x} \frac{1}{(u-x)^2} \left( \bigvee_x^u \mathfrak{h}'_x \right) du$$

$$= C(\nu, \rho)\left(\frac{1+x}{nx}\right)\int\limits_{1}^{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{t}}\mathfrak{h}'_x\right)dt \le C(\nu, \rho)\left(\frac{1+x}{nx}\right)\sum_{m=1}^{[\sqrt{n}]}\int\limits_{m}^{m+1}\left(\bigvee_{x}^{x+\frac{x}{t}}\mathfrak{h}'_x\right)dt$$

$$\le C(\nu, \rho)\left(\frac{1+x}{nx}\right)\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{m}}\mathfrak{h}'_x\right).$$

Collecting the estimates of $N_1$ and $N_2$, (21), yields us

$$\int\limits_{x}^{2x}|\mathfrak{h}'_x(u)|(1-\vartheta_{n,\rho}(x,u))du \le \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\mathfrak{h}'_x\right) + C(\nu, \rho)\left(\frac{1+x}{nx}\right)\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{m}}\mathfrak{h}'_x\right).$$

Hence, applying Cauchy-Schwarz inequality and Lemma 3.10,

$$|E_{n,2,\rho}(\mathfrak{h}'_x,x)| \le \mathcal{M}_{\mathfrak{h}}\int\limits_{2x}^{\infty}(u^2+1)\mathcal{W}(x,n,\rho,u)du + |\mathfrak{h}(x)|\int\limits_{2x}^{\infty}\mathcal{W}(x,n,\rho,u)du$$

$$+|\mathfrak{h}'(x+)|\sqrt{C(\nu, \rho)\left(\frac{1+x}{n}\right)} + C(\nu, \rho)\left(\frac{1+x}{nx^2}\right)|\mathfrak{h}(2x) - \mathfrak{h}(x) - x\mathfrak{h}'(x+)|$$

$$+\frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\mathfrak{h}'_x\right) + C(\nu, \rho)\left(\frac{1+x}{nx}\right)\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{m}}\mathfrak{h}'_x\right).$$

Now, since $u \le 2(u-x)$ and $x \le u-x$ when $u \ge 2x$, we have

$$\mathcal{M}_{\mathfrak{h}}\int\limits_{2x}^{\infty}(u^2+1)\mathcal{W}(x,n,\rho,u)du + |\mathfrak{h}(x)|\int\limits_{2x}^{\infty}\mathcal{W}(x,n,\rho,u)du$$

$$\le (\mathcal{M}_{\mathfrak{h}}+|\mathfrak{h}(x)|)\int\limits_{2x}^{\infty}\mathcal{W}(x,n,\rho,u)du + 4\mathcal{M}_{\mathfrak{h}}\int\limits_{2x}^{\infty}(u-x)^2\mathcal{W}(x,n,\rho,u)du$$

$$\le \frac{\mathcal{M}_{\mathfrak{h}}+|\mathfrak{h}(x)|}{x^2}\int\limits_{0}^{\infty}(u-x)^2\mathcal{W}(x,n,\rho,u)du + 4\mathcal{M}_{\mathfrak{h}}\int\limits_{0}^{\infty}(u-x)^2\mathcal{W}(x,n,\rho,u)du$$

$$\le \left(4\mathcal{M}_{\mathfrak{h}} + \frac{\mathcal{M}_{\mathfrak{h}}+|\mathfrak{h}(x)|}{x^2}\right)C(\nu, \rho)\left(\frac{1+x}{n}\right).$$

Consequently,

$$|E_{n,2,\rho}(\mathfrak{h}'_x,x)| \le \left(4\mathcal{M}_{\mathfrak{h}} + \frac{\mathcal{M}_{\mathfrak{h}}+|\mathfrak{h}(x)|}{x^2}\right)C(\nu, \rho)\left(\frac{1+x}{n}\right) + |\mathfrak{h}'(x+)|\sqrt{C(\nu, \rho)\left(\frac{1+x}{n}\right)}$$

$$+C(\nu, \rho)\left(\frac{1+x}{nx^2}\right)|\mathfrak{h}(2x) - \mathfrak{h}(x) - x\mathfrak{h}'(x+)| + \frac{x}{\sqrt{n}}\left(\bigvee_{x}^{x+\frac{x}{\sqrt{n}}}\mathfrak{h}'_x\right)$$

$$+C(\nu, \rho)\left(\frac{1+x}{nx}\right)\sum_{m=1}^{[\sqrt{n}]}\left(\bigvee_{x}^{x+\frac{x}{m}}\mathfrak{h}'_x\right). \tag{21}$$

Finally, combining the equations (19)-(21), we arrive at the required result. □

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