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Analyzing a generalized pest-natural enemy model with nonlinear impulsive control

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Abstract: Due to resource limitation, nonlinear impulsive control tactics related to integrated pest management have been proposed in a generalized pest-natural enemy model, which allows us to address the effects of nonlinear pulse control on the dynamics and successful pest control. The threshold conditions for the existence and global stability of pest-free periodic solution are provided by Floquet theorem and analytic methods. The existence of a nontrivial periodic solution is confirmed by showing the existence of nontrivial fixed point of the stroboscopic mapping determined by time snapshot, which equals to the common impulsive period. In order to address the applications of generalized results and to reveal how the nonlinear impulses affect the successful pest control, as an example the model with Holling II functional response function is investigated carefully. The main results reveal that the pest free periodic solution and a stable interior positive periodic solution can coexist for a wide range of parameters, which indicates that the local stability does not imply the global stability of the pest free periodic solution when nonlinear impulsive control is considered, and consequently the resource limitation (i.e. nonlinear control) may result in difficulties for successful pest control.

Keywords: Pest-natural enemy model, Nonlinear impulsive effect, Threshold condition, Bifurcation, Nontrivial periodic solution

MSC: 34A37, 92D25

1 Introduction

Over the past decade, controlling insect pests and other arthropods in agriculture became an increasing important issue. How to reduce losses due to insect pests becomes a great concern for entomologists and the society. To realize this purpose, a wide range of pest control strategies are available to farmers [1,2]. In particular, integrated pest management (IPM) has been proposed and designed which is a long term management strategy with aims to minimize economic, health and environmental risks by combining biological, chemical, cultural and physical tools [3-6].

With the development of the theory and application of impulsive differential equations [7,8], it is possible to depict the control strategies involved in IPM by establishing mathematical models. In particular, impulsive differential equations can accurately depict the dynamic process of spraying insecticides and releasing natural enemies [8-14]. For example, we can assume that the pesticide is applied at each fixed period, and a certain proportion of pests will be killed instantly after each spray. Similarly, the natural enemy is released simultaneously, where a constant amount of a natural enemy is administered at each fixed period. Note that

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the impulsive differential equations with fixed moments can provide a natural description for above assumptions, which can assist in the design and understanding of the ecological systems including the pests, their natural enemies, the surrounding environment and their inter-relationships. Moreover, theoretical analyses can provide valuable information about how to determine the optimal times or application frequencies of spraying pesticides, releasing natural enemies and infected pests [10-15].

Recently, many mathematical models concerning IPM have been developed and investigated [16-23]. In particular, the special prey-predator models with various functional response functions have been employed, and the main assumptions for those models are as follows: a proportion of pest population is killed after a pesticide is applied and simultaneously a natural enemy is released [24-28]. Based on those assumptions, the pest-natural enemy ecological systems with linear IPM measures have been extensively investigated, and almost all of those works focused on the existence and stability of pest free periodic solution. In particular, the threshold conditions under which the pest free periodic solution is locally or globally stable were provided, and this could help us to evaluate the effectiveness of pesticide and its application period on the successful pest control. Moreover, extensively numerical investigations revealed that those models could involve very complex dynamics including the coexistence of multiple attractors, chaotic solutions and period-doubling bifurcations.

The existence and stability of periodic solution for some generalized prey-predator model with linear or constant pulse actions studied so far [29-32] have provided some analytical techniques to deal with the generalized models. However, all of the previous works are basically assuming that the control strategy is linear or constant, which is not consistent with the actual situation. In fact, due to the resources limitation and the saturation effect of pesticide efficiency, the instant killing rate is a monotonic increasing function of pest population which should be a saturation function with a maximal killing rate. Similarly, the number of natural enemies is released depending on current pest density, which means that the larger the natural enemy, the fewer natural enemy is released and vice versa.

Therefore, the main purpose of this paper is to construct and investigate the dynamics of a quite generalized predator-prey model with nonlinear impulsive control due to resource limitation. By using Floquet theorem and qualitative techniques, it is proved that there exists a globally stable pest-free periodic solution under certain threshold conditions. By employing an operator theoretic approach which reduces the existence of the nontrivial periodic solutions to a fixed point and bifurcation problem, we show the existence and stability of positive periodic solution once the pest-free periodic solution loses its stability. In order to apply the main results, we choose the classical pest-natural enemy model with Holling type II functional response function and nonlinear impulsive control, then the exact thresholds for the local and global stability of pest free periodic solution are obtained. The results show that local stability does not imply the global stability which is confirmed by the bi-stability, and this is a novel result comparing with the model under the linear pulse perturbations [19-25].

2 The pest-natural enemy model with nonlinear pulse control

The generalized prey-predator model or pest-natural enemy model employed in the present paper is as follows:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)g(x(t)) - p(x(t))y(t), \\ \frac{dy(t)}{dt} = cp(x(t))y(t) - Dy(t), \end{cases} \tag{1}$$

where $x(t), y(t)$ represent the densities of prey and predator populations, respectively. The function $g(x)$ represents the intrinsic growth rate of the pest in the absence of natural enemy, $p(x)$ denotes the predator response function, c is the efficiency rate, and D is the death rate of the predator population. In order to use our main results for a wide range of biological systems which have been investigated in the literature, we made the following assumptions related to the function $p(x)$ and $g(x)$. Let $g(x)$ and $p(x)$ be locally Lipschitz functions on R^+ such that:

(i) There exists a positive constant $K > 0$ such that $g(x) > 0$ for $0 \le x < K$, $g(K) = 0$ and $g(x) < 0$ for $K < x$.
(ii) The functional response function satisfies $p(0) = 0$, $p'(0) > 0$ and $p(x) > 0$ for all $x > 0$.
(iii) The function $\frac{xg(x)}{p(x)}$ is upper bounded for all $x > 0$.

The first condition means that the pest population follows the density dependent growth in the absence of the natural enemy, and the second condition indicates that the functional response function is positive and monotonically increasing for small pest populations. The last one shows that the pest population can not increase infinitely once the biological control is introduced, and, on the other hand, if the density of pest population is too large then the biological control is impossible.

We assume that the IPM strategy is applied at every time point nT at which the natural enemies are released and pesticides are applied simultaneously, where T denotes the period of control actions and $n \in \mathcal{N}$, which denotes the positive integer set. Moreover, the nonlinear saturation functions or density dependent functions are employed to depict the effects of resource limitation on the pest control, i.e. we choose

$$x(t^+) = \left[1 - \frac{\delta x(t)}{x(t) + h}\right] x(t),\ y(t^+) = y(t) + \frac{\lambda}{1 + \theta y(t)},\ t = nT,$$

where $\delta \ge 0$ and $h \ge 0$ represent the maximal fatality rate and the half saturation constant for the pest with $\delta < 1$, $\lambda \ge 0$ is the release amount of the natural enemy, and $\theta \ge 0$ denotes the shape parameter. We assume that the densities of both the pest and natural enemy populations are updated to $(1 - \frac{\delta x(t)}{x(t)+h})x(t)$ and $y(t) + \frac{\lambda}{1+\theta y(t)}$ at every discrete time point nT and $n \in \mathcal{N}$, respectively.

Taking the control measures shown as the above and model (1) into account, one yields the following differential equation model with IPM strategies:

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= x(t)g(x(t)) - p(x(t))y(t), \\ \frac{dy(t)}{dt} &= cp(x(t))y(t) - Dy(t), \end{aligned}\right\} t \ne nT, \\ \left.\begin{aligned} x(t^+) &= \left[1 - \frac{\delta x(t)}{x(t)+h}\right] x(t), \\ y(t^+) &= y(t) + \frac{\lambda}{1+\theta y(t)}, \end{aligned}\right\} \quad t = nT. \end{cases} \tag{2}$$

The positivity and boundedness of solutions of model (2) are useful for the coming analyses, and we have:

Lemma 2.1. *The solutions of model* (2) *are positive and bounded.*

Proof. The positivity of solutions can be easily shown as the control actions do not influence the positivity of the solutions, thus the positive initial conditions indicate the positivity of the solutions. For the boundedness, it is easy to show that $x(t) < \max\{K, x(0^+)\}$ due to $x(nT^+) < x(nT)$ and assumption (i).

In order to show the boundedness of y-component, we denote $V(t) = cx(t) + y(t)$, then when $t \ne nT$ we have

$$D^+V(t) + DV(t) = cx(t)(g(x(t)) + D) \le M_0$$

with $M_0 = \max\{cx(t)(g(x(t) + D)\}$. When $t = nT$ we have

$$\begin{aligned} V(nT^+) &= c\left[1 - \frac{\delta x(nT)}{x(nT) + h}\right] x(nT) + y(nT) + \frac{\lambda}{1 + \theta y(nT)} \\ &\le cx(nT) + y(nT) + \lambda \\ &= V(nT) + \lambda. \end{aligned}$$

Therefore, for $t \in (nT, (n+1)T]$, we have

$$V(t) \le V(0)\exp(-Dt) + \int_0^t M_0 \exp(-D(t-s))ds + \sum_{0<kT<t} \lambda \exp(-D(t-kT))$$

$$
\begin{aligned}
&= V(0)\exp(-Dt) + \frac{M_0(1-\exp(-Dt))}{D} \\
&\quad + \lambda \frac{\exp(-D(t-T)) - \exp(-D(t-(n+1)T))}{1-\exp(DT)} \\
&\to \frac{M_0}{D} + \frac{\lambda\exp(DT)}{\exp(DT)-1}, (t\to\infty).
\end{aligned}
$$

Thus, $V(t)$ is uniformly ultimately bounded. According to the definition of $V(t)$ we can see that $y(t)$ is bounded. □

One of the main purposes of implementation IPM is to eradicate the pest population when the fixed moments are applied at every period T. To address this, the existence and stability of the pest free periodic solution are crucial for this point. Thus, we first consider the following subsystem

$$
\begin{cases}
\frac{dy(t)}{dt} = -Dy(t), \ t \neq nT, \\
y(t^+) = y(t) + \frac{\lambda}{1+\theta y(t)}, \ t = nT.
\end{cases} \tag{3}
$$

Subsystem (3) is a simple linear growth model with nonlinear control, which can be analytically solved, and we have the following main result.

Theorem 2.2. *If $\theta \neq 0$, then model (3) has a globally stable positive periodic solution*

$$y_p(t) = y^* \exp(-D(t-nT)), t \in (nT, (n+1)T],$$

where $y^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-DT)(1-\exp(-DT))^{-1}}}{2\theta\exp(-DT)}$ is a positive constant and is determined by all the coefficients of model (3).

Proof. Without loss of generality, considering any time interval $(nT, (n+1)T]$ and integrating the first equation of model (3), one has

$$y(t) = y(nT^+)\exp(-D(t-nT)), nT < t \le (n+1)T.$$

At time point $(n+1)T$, the solution experiences one time pulse action resulting in

$$y((n+1)T^+) = y(nT^+)\exp(-DT) + \frac{\lambda}{1+\theta y(nT^+)\exp(-DT)}.$$

Denote $Y_n = y(nT^+)$, it follows from the above equation that we have the following stroboscopic map for system (3)

$$Y_{n+1} = Y_n\exp(-DT) + \frac{\lambda}{1+\theta Y_n\exp(-DT)} \triangleq F(Y_n), \tag{4}$$

and it is easy to see that if $\theta \neq 0$, then equation (4) has a unique positive fixed point

$$y^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-DT)(1-\exp(-DT))^{-1}}}{2\theta\exp(-DT)}.$$

For the local stability of y^*, since $F'(y) = (1 - \frac{\theta\lambda}{(1+\theta y\exp(-DT))^2})\exp(-DT)$, we have

$$
\begin{aligned}
F'(y^*) &= \exp(-DT) - \frac{\theta\lambda\exp(-DT)}{(1+\theta y^*\exp(-DT))^2} \\
&< \exp(-DT) < 1
\end{aligned}
$$

and

$$
\begin{aligned}
F'(y^*) &> -\frac{\theta y^*\exp(-DT)(1-\exp(-DT))}{1+\theta y^*\exp(-DT)} \\
&> -\frac{\theta y^*\exp(-DT)}{1+\theta y^*\exp(-DT)} > -1.
\end{aligned}
$$

All those confirm that $|F'(y^*)| < 1$, i.e. the fixed point y^* of equation (4) is locally stable.

In the following we show the global attractivity of y^*. By simple calculation we have

$$F''(Y) = \frac{2\lambda\theta^2 \exp(-2DT)}{(1+\theta Y \exp(-DT))^3},$$

and solving equation $F'(Y) = 0$ yields two stationary points, denoted by Y_1 and Y_2, i.e.

$$Y_1 = \frac{-\sqrt{\lambda\theta} - 1}{\theta \exp(-DT)} < 0, \ \ Y_2 = \frac{\sqrt{\lambda\theta} - 1}{\theta \exp(-DT)}.$$

Here Y_2 is the local minimum of the function $F(Y)$ with $F(Y_2) = \frac{2\sqrt{\lambda\theta}-1}{\theta}$, $F(0) = \lambda$, and the function $F''(Y) > 0$ for $Y > 0$. Based on the sign of Y_2 and the positional relations between Y_2 and y^*, we consider the following three possible cases for the global attractivity of y^*.

Case (i) $Y_2 < 0 < y^*$.

For this case we have $0 < \lambda\theta < 1$, and $F(Y)$ is a monotonically increasing and concave function for $Y > 0$, and y^* is a unique positive fixed point of function $F(Y) = Y$.

Denote $F^n(Y) = F(F^{n-1}(Y))$ for $n = 2, 3, \cdots$. If $0 < Y_0 < y^*$, it follows from the concavity of the $F(Y)$ that $F^n(Y_0)$ is monotonically increasing as n increases due to $F(Y) > Y$ with $\lim_{n\to+\infty} F^n(Y_0) = y^*$. If $Y_0 > y^*$, then $F^n(Y_0)$ is monotonically decreasing as n increases because of $F(Y) < Y$ with $\lim_{n\to+\infty} F^n(Y_0) = y^*$.

Case (ii) $0 < Y_2 < y^*$.

For this case we have $\lambda\theta > 1$, and the function $F(Y)$ is a monotonically decreasing function for $0 < Y < Y_2$ and increasing concave function for $Y > Y_2$. Thus, similar processes applied to case (i) yield $\lim_{n\to+\infty} F^n(Y_0) = y^*$ for any $Y_2 \le Y_0 < y^*$ or $Y_0 > y^*$.

If $0 < Y_0 < Y_2$, then according to the properties of function $F(Y)$ we can see that there exists a smallest positive integer n_1 such that $Y_2 \le F^{n_1}(Y_0) < y^*$ or $F^{n_1}(Y_0) > y^*$, and consequently we have $\lim_{j\to+\infty} F^{n_1+j}(Y_0) = y^*$.

Case (iii) $0 < y^* < Y_2$.

For this case, it is easy to know that $F(Y)$ is a monotonically decreasing and concave function for $Y \in [0, Y_2]$, and is an increasing function for $Y \in (Y_2, \infty)$. Based on the properties of function $F(Y)$, we conclude that there must exist a positive integer m such that $F^m(Y) \in [y^*, Y_2]$ for any $Y > 0$, which indicates that we only need to show $\lim_{n\to+\infty} F^n(Y_0) = y^*$ for any $Y_0 \in (y^*, Y_2]$.

To do this, we first define $G(Y) = \frac{F(F(Y))}{Y}$ and address its properties. It follows from

$$G'(Y) = \frac{F'(F(Y))F'(Y)Y - F(F(Y))}{Y^2}$$

that $F(Y)$ is a monotonically decreasing and concave function for any $Y \in (y^*, Y_2]$, and $F'(Y)$ is a monotonically increasing function because of $F''(Y) > 0$ for $Y > 0$. Thus, for any $Y \in (y^*, Y_2]$

$$\begin{aligned} F'(Y)Y + F(Y) &= 2Y\exp(-2DT) + \frac{\lambda}{(1+\theta Y \exp(-DT))^2} \\ &> 2Y\exp(-2DT) > 0 \\ &\Rightarrow -F(Y) < F'(Y)Y \le 0 \end{aligned}$$

and

$$F'(F(Y))F(Y) + F(F(Y)) > 2F(Y)\exp(-2DT) > 0.$$

Since $0 < F(Y) < F(y^*) = y^*$ and $F'(Y) < 0$ for $Y \in (y^*, Y_2]$, we have

$$-\frac{F(F(Y))}{F(Y)} < F'(F(Y)) < 0.$$

It is easy to show $F'(F(Y))F'(Y)Y < F(Y)\frac{F(F(Y))}{F(Y)} = F(F(Y))$ for any $Y \in (y^*, Y_2]$, which means that $G'(Y) < 0$ for any $Y \in (y^*, Y_2]$. Therefore, $G(Y)$ is a monotonically decreasing function in interval $Y \in (y^*, Y_2]$. Furthermore, we have $G(y^*) = \frac{F(F(y^*))}{y^*} = 1$ and

$$G(Y) < G(y^*) = 1 \Leftrightarrow F(F(Y)) < Y.$$

Thus, we conclude that $F^{2n}(Y_0)$ is monotonically decreasing as n increases for $Y_0 \in (y^*, Y_2]$, and y^* is a unique positive fixed point, which means that $\lim_{n\to+\infty} F^{2n}(Y_0) = y^*$, and the monotonicity of $F^{2n-1}(Y_0)$ follows as well.

Based on the above relations, there exists a positive integer m such that $F^m(Y_0) \in (y^*, Y_2]$ for any $Y_0 > 0$, which indicates that $\lim_{j\to+\infty} F^{m+j}(Y_0) = y^*$. Thus, the fixed point y^* of equation (4) is globally stable. Further, according to the relations between a fixed point of the stroboscopic map (4) and the periodic solution of model (3), we conclude that model (3) has a globally stable positive periodic solution $y_p(t) = y^* \exp(-D(t-nT))$, $t \in (nT, (n+1)T]$. This completes the proof. □

In particular, if $\theta = 0$ then model (3) has a globally stable positive periodic solution

$$y_p(t) = y^* \exp(-D(t-nT)), t \in (nT, (n+1)T],$$

where $y^* = \frac{\lambda}{1-\exp(-DT)}$.

Therefore, we obtain the general expression of the unique pest-free periodic solution of model (2), i.e.

$$(x_p(t), y_p(t)) = (0, y^* \exp(-D(t-nT))), t \in (nT, (n+1)T],$$

where $y^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-DT)(1-\exp(-DT))^{-1}}}{2\theta\exp(-DT)}$ when $\theta \neq 0$ or $y^* = \frac{\lambda}{1-\exp(-DT)}$ when $\theta = 0$.

As mentioned before, one of the main purposes of IPM is to design suitable control measures such that the pest population dies out eventually, i.e. the pest free periodic solution $(x_p(t), y_p(t))$ is globally stable. Thus, the threshold conditions under which the pest free periodic solution is globally stable are crucial in this work. To do this, we first show the local stability, and consider the behavior of small amplitude perturbations of the solution.

$$\begin{cases} \tilde{x}(t) = x(t) - x_p(t), \\ \tilde{y}(t) = y(t) - y_p(t), \end{cases}$$

where $\tilde{x}(t)$ and $\tilde{y}(t)$ are small perturbations, then model (1) becomes

$$\begin{cases} \dot{\tilde{x}} = \tilde{x}g(\tilde{x}) - p(\tilde{x})(\tilde{y} + y_p(t)), \\ \dot{\tilde{y}} = cp(\tilde{x})(\tilde{y} + y_p(t)) - D\tilde{y}. \end{cases} \tag{5}$$

The impulsive effects on $\tilde{x}$ are unchanged because of $x_p(t) = 0$, so we have

$$\tilde{x}(nT^+) = \left(1 - \frac{\delta\tilde{x}(nT)}{\tilde{x}(nT) + h}\right)\tilde{x}(nT).$$

The impulsive effects on $\tilde{y}$ are defined as follows:

$$\begin{aligned} \tilde{y}(nT^+) &= y(nT) + \frac{\lambda}{1+\theta y(nT)} - y_p(nT) - \frac{\lambda}{1+\theta y_p(nT)} \\ &= \left(1 - \frac{\lambda\theta}{\left(1+\theta(\tilde{y}(nT) + y_p(nT))\right)\left(1+\theta y_p(nT)\right)}\right)\tilde{y}(nT). \end{aligned}$$

The linear approximation of the deviation system of model (5) around the periodic solution $(x_p(t), y_p(t))$ is as follows:

$$\begin{cases} \dot{\tilde{x}} = (g(0) - p'(0)y_p(t))\tilde{x}, \\ \dot{\tilde{y}} = cp'(0)y_p(t)\tilde{x} - D\tilde{y}. \end{cases} \tag{6}$$

In the following, we will show the sufficient condition for the global stability of pest-free periodic solution $(x_p(t), y_p(t))$ of model (2), and we have the following main results for model (2).

Theorem 2.3. *The pest-free periodic solution $(x_p(t), y_p(t))$ is locally stable provided*

$$R_0 = \frac{g(0)DT}{y^*(1-\exp(-DT))p'(0)} < 1 \tag{7}$$

and it is globally attractive if

$$R_1 = \frac{M_s DT}{y^*(1-\exp(-DT))} < 1, \tag{8}$$

where $M_s = \sup_{x\geq 0} \frac{xg(x)}{p(x)}$.

Proof. To prove the local stability of the solution $(x_p(t), y_p(t))$ of model (2), we let $\Phi(t)$ be the fundamental matrix of (2), and then $\Phi(t)$ satisfies

$$\Phi(T) = \begin{pmatrix} \exp(\int_0^T (g(0) - p'(0)y_p(t))dt) & 0 \\ \star & \exp(\int_0^T -Ddt) \end{pmatrix},$$

where $\Phi(0) = I$ represents the identity matrix and the term $*$ is not necessary for the next analyses. The linearization of the impulsive effects of model (2) can be calculated as follows:

$$\begin{pmatrix} \tilde{x}(nT^+) \\ \tilde{y}(nT^+) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{\lambda\theta}{(1+\theta y_p(T))^2} \end{pmatrix} \begin{pmatrix} \tilde{x}(nT) \\ \tilde{y}(nT) \end{pmatrix} = B(T) \begin{pmatrix} \tilde{x}(nT) \\ \tilde{y}(nT) \end{pmatrix}.$$

Therefore, if the module of both eigenvalues of the following matrix

$$\begin{aligned} M &= B(T)\Phi(T) \\ &= \begin{pmatrix} \exp(\int_0^T (g(0) - p'(0)y_p(t))dt) & 0 \\ 0 & (1 - \frac{\lambda\theta}{(1+\theta y_p(T))^2}) \exp(-DT) \end{pmatrix} \end{aligned}$$

is less than one, then the periodic solution $(x_p(t), y_p(t))$ is locally stable. In fact, the eigenvalues of M can be calculated as follows:

$$|\lambda_1| = \exp(\int_0^T (g(0) - p'(0)y_p(t))dt), \ |\lambda_2| = |(1 - \frac{\lambda\theta}{(1+\theta y_p(T))^2})| \exp(-DT)$$

and it is easy to see that $|\lambda_2| = |1 - \frac{\lambda\theta}{(1+\theta y^* \exp(-DT))^2}| \exp(-DT) < 1$ holds true. All those confirm that the pest-free solution $(x_p(t), y_p(t))$ is locally stable if and only if $|\lambda_1| < 1$, i.e.

$$\int_0^T (g(0) - p'(0)y_p(t))dt < 0.$$

It follows from the Theorem 2.2 that $\int_0^T y_p(t)dt = \frac{y^*(1-\exp(-DT))}{D}$, and consequently the periodic solution $(x_p(t), y_p(t))$ is locally stable provided that

$$R_0 = \frac{g(0)DT}{y^*(1-\exp(-DT))p'(0)} < 1.$$

For the global attractivity of the periodic solution $(x_p(t), y_p(t))$, we only need to show that any solution $(x(t), y(t))$ of model (2) tends to $(x_p(t), y_p(t))$ as t goes to infinity. If $R_1 = \frac{M_s DT}{y^*(1-\exp(-DT))} < 1$, then we can choose small enough $\varepsilon > 0$ such that $\int_0^T (M_s - (y_p(s) - \varepsilon))ds < 0$. It follows from model (2) that we have

$$\begin{cases} \frac{dy(t)}{dt} \geq -Dy(t), t \neq nT, \\ y(t^+) = y(t) + \frac{\lambda}{1+\theta y(t)}, t = nT. \end{cases}$$

According to Theorem 2.2 and the impulsive differential comparison theory, we have the following inequality

$$y(t) \geq y_p(t) - \varepsilon$$

for t large enough. To simplify the discussion, we assume, without loss of generality, $y(t) \geq y_p(t) - \varepsilon$ holds true for all $t \geq 0$.

Then, it follows from the first equation of model (2) that

$$\frac{\dot{x}}{p(x)} = \frac{xg(x)}{p(x)} - y$$

and integrating both sides yields

$$\int_{x_0}^{x(t)} \frac{dx}{p(x)} = \int_{t_0}^{t} \left(\frac{x(s)g(x(s))}{p(x(s))} - y \right) ds.$$

Since $p(x) \approx p'(0)x$ with $p'(0) > 0$ for x small enough, the left side of the above equation goes to $-\infty$ if only if x converges to zero. Define

$$G(x) = \int_{\delta}^{x} \frac{1}{p(s)} ds$$

for $\delta > 0$. Thus, it is easy to see that the pest population goes to extinction if $G(x) \to -\infty$ as $t \to \infty$. Therefore, the G function satisfies

$$\frac{dG(x)}{dt} = \frac{1}{p(x)}\dot{x} = \frac{xg(x)}{p(x)} - y.$$

Considering any impulsive interval $(nT, (n+1)T]$, we have

$$\begin{aligned}
G(x(n+1)T^+) &= \int_{\delta}^{x((n+1)T^+)} \frac{1}{p(s)} ds \\
&= \int_{\delta}^{x((n+1)T)} \frac{1}{p(s)} ds + \int_{x((n+1)T)}^{x((n+1)T^+)} \frac{1}{p(s)} ds \\
&= G(x((n+1)T)) + \int_{x((n+1)T)}^{x((n+1)T^+)} \frac{1}{p(s)} ds \\
&\leq G(x(nT^+)) + \int_{nT}^{(n+1)T} \left(\frac{x(s)g(x(s))}{p(x(s))} - (y_p(s) - \varepsilon) \right) ds \\
&\quad + \int_{x((n+1)T)}^{x((n+1)T^+)} \frac{1}{p(s)} ds \\
&\leq G(x(nT^+)) + \int_{nT}^{(n+1)T} (M_s - (y_p(s) - \varepsilon))\, ds.
\end{aligned}$$

For any $t > 0$, there exists an integer l such that for all $t \in (lT, (l+1)T]$ we have

$$\begin{aligned}
G(x(t)) - G(x_0) &\leq \int_0^t (M_s - (y_p(s) - \varepsilon))\, ds \\
&= l \int_0^T (M_s - (y_p(s) - \varepsilon)) ds + \int_{lT}^{t} (M_s - (y_p(s) - \varepsilon)) ds.
\end{aligned}$$

It is clear that the second term of the right-hand side is upper bounded due to the periodicity of $y_p(t)$ with period T. Note that $l \to \infty$ as $t \to \infty$. Thus, if

$$\int_0^T (M_s - (y_p(s) - \varepsilon)) ds < 0,$$

then we have $G(\tilde{x}) \to -\infty$ as $t \to \infty$, i.e. $x(t)$ converges to 0 as $t \to \infty$. That is

$$R_1 = \frac{M_s DT}{y^*(1-\exp(-DT))} < 1.$$

Now we prove that $y(t) \to y_p(t)$ as well. Since $x(t)$ goes to zero, there exists a finite time t_1 such that $p(x) < \varepsilon$ for all $t > t_1$. Therefore, for all $t > t_1$ we have

$$-Dy(t) \le \frac{dy}{dt} \le y(t)(c\varepsilon - D).$$

Considering the following comparison equation

$$\begin{cases} \frac{dz(t)}{dt} = (c\varepsilon - D)z(t), t \ne nT, \\ z(t^+) = z(t) + \frac{\lambda}{1+\theta z(t)}, t = nT, \end{cases} \tag{9}$$

and by employing the same methods as those in proof of Theorem 2.2, we see that model (9) has a positive periodic solution $z_p(t)$, which is globally attractive and

$$z_p(t) = z^* \exp[-(D-c\varepsilon)(t-nT)],\ t \in (nT,(n+1)T],$$

where $z^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-(D-c\varepsilon)T)(1-\exp(-(D-c\varepsilon)T))^{-1}}}{2\theta\exp(-(D-c\varepsilon)T)}$.

It follows from the comparison theorem of impulsive differential equations that

$$y_p(t) \le y(t) \le z(t).$$

Moreover, $z(t) \to z_p(t)$ and $z_p(t) \to y_p(t)$ as $t \to \infty$. Consequently, there exists a t_2 for ε_1 small enough such that $t_2 \ge t_1 > 0$ and

$$y_p(t) - \varepsilon_1 < y(t) < z_p(t) + \varepsilon_1$$

for $t > t_2$. Let $\varepsilon \to 0$, then $y_p(t)-\varepsilon_1 < y(t) < y_p(t)+\varepsilon_1$. Therefore, $y(t) \to y_p(t)$ as $t \to \infty$, which indicates that the pest free periodic solution $(x_p(t), y_p(t))$ of model (2) is globally asymptotically stable. This completes the proof. □

Comparing the formula of both R_0 and R_1 shown in equations (7) and (8) we can see that the conditions for the local and global stability of the pest free periodic solution are different, which depends on the relations between $M_s = \sup \frac{xg(x)}{p(x)}$ and $\frac{g(0)}{f'(0)}$. Note that $\lim_{x\to 0} \frac{xg(x)}{p(x)} = \frac{g(0)}{f'(0)}$ due to l'Hospital rule, and $M_s = \frac{g(0)}{f'(0)} \Leftrightarrow R_0 = R_1$ as $x \to 0$. In general, the globally attractive condition is stronger than the local stability condition due to $M_s \ge \frac{g(0)}{f'(0)} \Leftrightarrow R_0 \le R_1$. The question is whether the local stability implies the global stability of the pest free periodic solution, which will be discussed in more detail in the application section.

3 Threshold condition of bifurcation

For the existence of interior periodic solutions of model (2), we can investigate the bifurcation near the pest free periodic solution, i.e. $(x_p(t), y_p(t))$. To do this, for computation convenience we first exchange the variables $x(t)$ and $y(t)$, and denote $u(t) = y(t)$ and $v(t) = x(t)$, then system (2) becomes as follows:

$$\begin{cases} \left.\begin{aligned} \frac{du(t)}{dt} &= cp(v(t))u(t) - Du(t), \\ \frac{dv(t)}{dt} &= v(t)g(v(t)) - p(v(t))u(t), \end{aligned}\right\} t \ne nT, \\ \left.\begin{aligned} u(t^+) &= u(t) + \frac{\lambda}{1+\theta u(t)}, \\ v(t^+) &= (1 - \frac{\delta v(t)}{v(t)+h})v(t), \end{aligned}\right\} \quad t = nT. \end{cases} \tag{10}$$

We let Φ be the flow associated to the first two equations of (10), and the fundamental solution matrix of (10) is $X(t) = \Phi(t, X_0)$ with $X_0 = X(0) = (u(0^+), v(0^+))$ and $\Phi = (\Phi_1, \Phi_2)$. We employ the notations used in this section as those in [33], then we can define the mapping $\Theta_1, \Theta_2 : R^2 \to R^2$ as follows

$$\Theta_1(u,v) = u + \frac{\lambda}{1+\theta u}, \Theta_2(u,v) = (1 - \frac{\delta v}{v+h})v$$

and the mapping $F_1, F_2 : R^2 \to R^2$ by

$$F_1(u, v) = cp(v)u - Du, F_2(u, v) = vg(v) - p(v)u.$$

Furthermore, we define $\Psi : [0, +\infty) \times R^2 \to R^2$ by

$$\Psi(T, X_0) = \Theta(\Phi(T, X_0)); \Psi(T, X_0) = (\Psi_1(T, X_0), \Psi_2(T, X_0)).$$

Based on the above notations we can see that Ψ is determined by the values of solutions at impulsive points 0 and T, which is called as stroboscopic map of model (10) and T is the stroboscopic time snapshot. We know that $X = (u, v)$ is a periodic solution of (10) with period T if and only if its initial value $X_0 = X(0)$ is a fixed point for map $\Psi(T, \cdot)$. Therefore, in order to establish the existence of nontrivial periodic solutions of (10), we should prove the existence of the nontrivial fixed points of Ψ.

For model (10), it follows from the discussion in the previous section that model (10) has a stable boundary T periodic solution, denoted by

$$\zeta(t) = (u_s(t), 0) = (y^* \exp(-D(t - nT)), 0),\ t \in (nT, (n + 1)T],$$

where $y^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-DT)(1-\exp(-DT))^{-1}}}{2\theta\exp(-DT)}$ is a positive constant. In order to employ the analytical methods developed in [33,34], we now consider the bifurcation of nontrivial periodic solutions near $(u_s(t), 0)$ with initial value $X(0) = (u_s(0^+), 0)$.

In order to obtain a nontrivial periodic solution of period τ with initial value $X(0)$, we have only to find the fixed point problem $X = \Psi(\tau, u)$. Denoting $\tau = T + \tilde{\tau}$, $X = X_0 + \tilde{X}$, and the fixed point problem $X = \Psi(\tau, u)$ equals to

$$X_0 + \tilde{X} = \Psi(T + \tilde{\tau}, X_0 + \tilde{X}).$$

Defining

$$N(\tilde{\tau}, \tilde{X}) = (N_1(\tilde{\tau}, \tilde{X}), N_2(\tilde{\tau}, \tilde{X})) = X_0 + \tilde{X} - \Psi(T + \tilde{\tau}, X_0 + \tilde{X}), \tag{11}$$

so $X_0 + \tilde{X}$ is a fixed point of $\Psi(T, \cdot)$ if $N(\tilde{\tau}, \tilde{X}) = 0$.

According to the variational equations of the first two equations of (10), we have

$$\frac{d}{dt}(\Phi(t, X_0)) = F(\Phi(t, X_0)),$$

which relates to the dynamics of the first two equations in (10). So we obtain that

$$\frac{d}{dt}(D_X(\Phi(t, X_0))) = D_X F(\Phi(t, X_0))(D_X(\Phi(t, X_0))) \tag{12}$$

with the condition $D_X(\Phi(0, X_0)) = I_2$, which is the identity matrix in $M_2(R)$. Thus, it follows from equation (10) that we have the particular form

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix} \frac{\partial\Phi_1}{\partial u} & \frac{\partial\Phi_1}{\partial v} \\ \frac{\partial\Phi_2}{\partial u} & \frac{\partial\Phi_2}{\partial v}\end{pmatrix}(t, X_0) &= \begin{pmatrix} \frac{\partial F_1(\zeta(t))}{\partial u} & \frac{\partial F_1(\zeta(t))}{\partial v} \\ \frac{\partial F_2(\zeta(t))}{\partial u} & \frac{\partial F_2(\zeta(t))}{\partial v}\end{pmatrix}\begin{pmatrix} \frac{\partial\Phi_1}{\partial u} & \frac{\partial\Phi_1}{\partial v} \\ \frac{\partial\Phi_2}{\partial u} & \frac{\partial\Phi_2}{\partial v}\end{pmatrix}(t, X_0) \\ &= \begin{pmatrix} -D & cp'(0)u_s(t) \\ 0 & g(0) - p'(0)u_s(t)\end{pmatrix}\begin{pmatrix} \frac{\partial\Phi_1}{\partial u} & \frac{\partial\Phi_1}{\partial v} \\ \frac{\partial\Phi_2}{\partial u} & \frac{\partial\Phi_2}{\partial v}\end{pmatrix}(t, X_0)\end{aligned}$$

with initial value $D_X(\Phi(0, X_0)) = I_2$.

According to the initial value $\frac{\partial\Phi_2(0,X_0)}{\partial u} = 0$, we obtain the following

$$\frac{\partial\Phi_2(t, X_0)}{\partial u} = \exp\left(\int_0^t (g(0) - p'(0)u_s(v))dv\right)\frac{\partial\Phi_2(0, X_0)}{\partial u},$$

i.e. $\frac{\partial\Phi_2(t,u_0)}{\partial u} = 0$ for all $t > 0$. Further, we obtain

$$\frac{d}{dt}\frac{\partial\Phi_1(t, u_0)}{\partial u} = -D\frac{\partial\Phi_1(t, u_0)}{\partial u},$$

$$\begin{aligned}\frac{d}{dt}\frac{\partial\Phi_1(t,u_0)}{\partial v}&=-D\frac{\partial\Phi_1(t,u_0)}{\partial v}+cp'(0)u_s(t)\frac{\partial\Phi_2(t,u_0)}{\partial v}\\ \frac{d}{dt}\frac{\partial\Phi_2(t,u_0)}{\partial v}&=(g(0)-p'(0)u_s(t))\frac{\partial\Phi_2(t,u_0)}{\partial v}.\end{aligned}$$

According to the initial condition $D_X(\Phi(0,X_0))=I_2$, we obtain that

$$\begin{aligned}\frac{\partial\Phi_1(t,u_0)}{\partial u}&=\exp(-Dt),\\ \frac{\partial\Phi_1(t,u_0)}{\partial v}&=cp'(0)\int_0^t\exp(-D(t-v))u_s(v)\rho(v))dv,\\ \frac{\partial\Phi_2(t,u_0)}{\partial v}&=\rho(t)\end{aligned}$$

for all $0\le t\le T$, where $\rho(t)=\exp\int_0^t(g(0)-p'(0)u_s(v))dv$.

We then compute the derivation of N according to (11), and observe the following matrix

$$\begin{aligned}D_XN(\tilde{\tau},\tilde{X})&=\begin{pmatrix}a' & b'\\ c' & d'\end{pmatrix}\\ &=\begin{pmatrix}1-\left(\frac{\partial\Theta_1}{\partial u}\frac{\partial\Phi_1}{\partial u}+\frac{\partial\Theta_1}{\partial v}\frac{\partial\Phi_2}{\partial u}\right) & -\left(\frac{\partial\Theta_1}{\partial u}\frac{\partial\Phi_1}{\partial v}+\frac{\partial\Theta_1}{\partial v}\frac{\partial\Phi_2}{\partial v}\right)\\ -\left(\frac{\partial\Theta_2}{\partial u}\frac{\partial\Phi_1}{\partial u}+\frac{\partial\Theta_2}{\partial v}\frac{\partial\Phi_2}{\partial u}\right) & 1-\left(\frac{\partial\Theta_2}{\partial u}\frac{\partial\Phi_1}{\partial v}+\frac{\partial\Theta_2}{\partial v}\frac{\partial\Phi_2}{\partial v}\right)\end{pmatrix}_{(T+\tilde{\tau},X_0+\tilde{X})}.\end{aligned}$$

Letting $a'=a'_0$, $b'=b'_0$, $c'=c'_0$ and $d'=d'_0$ when $(\tilde{\tau},\tilde{X})=(0,(0,0))$, at which $\frac{\partial\Theta_1}{\partial v}=\frac{\partial\Theta_2}{\partial u}=\frac{\partial\Phi_2}{\partial u}=0$, then we have

$$D_XN(0,(0,0))=\begin{pmatrix}1-\frac{\partial\Theta_1}{\partial u}\frac{\partial\Phi_1}{\partial u} & -\frac{\partial\Theta_1}{\partial u}\frac{\partial\Phi_1}{\partial v}\\ 0 & 1-\frac{\partial\Theta_2}{\partial v}\frac{\partial\Phi_2}{\partial v}\end{pmatrix}_{(T,X_0)}. \tag{13}$$

Thus, by simple calculations we have

$$\begin{aligned}a'_0&=1-\left(1-\frac{\theta\lambda}{(1+\theta y^*\exp(-DT))^2}\right)\exp(-DT)>0,\\ b'_0&=-\left(1-\frac{\theta\lambda}{(1+\theta y^*\exp(-DT))^2}\right)cp'(0)y^*\exp(-DT)\int_0^T\rho(v)dv,\\ c'_0&=0\end{aligned}$$

and

$$\begin{aligned}d'_0&=1-\exp\left(\int_0^T(g(0)-p'(0)u_s(t))dt\right)\\ &=1-\rho(T).\end{aligned}$$

Based on the above equations, we can see that $D_XN(0,(0,0))$ is an upper triangular matrix with $a'_0>0$. Thus, the necessary condition for the bifurcation of nontrivial solution is

$$\det[D_XN(0,(0,0))]=0,$$

which reduces to $d'_0=0$. By simple calculation, we can see that $d'_0=0$ is equivalent to $R_0=1$. Therefore, in the following we focus on $d'_0=0$ and address the sufficient condition for the bifurcation of nontrivial solution.

Note that $\dim(\ker(D_XN(0,(0,0))))=1$ and a basis of $\ker(D_XN(0,(0,0)))$ is $(-\frac{b'_0}{a'_0},1)$. Thus, the equation $N(\tilde{\tau},\tilde{X})=0$ is equivalent to

$$N_1(\tilde{\tau},aY_0+zE_0)=0,\ N_2(\tilde{\tau},aY_0+zE_0)=0,$$

where $E_0 = (1, 0)$, $Y_0 = (-\frac{b_0'}{a_0'}, 1)$ and $\tilde{X} = aY_0 + zE_0$ represents the direct summation decomposition of $\tilde{X}$ using the projections onto $\ker(D_X N(0, (0, 0)))$ (i.e. the central manifold) and $Im(D_X N(0, (0, 0)))$ (i.e. the stable manifold).

Now, we define

$$f_1(\tilde{\tau}, a, z) = N_1(\tilde{\tau}, aY_0 + zE_0), f_2(\tilde{\tau}, a, z) = N_2(\tilde{\tau}, aY_0 + zE_0),$$

and consequently we have

$$\frac{\partial f_1}{\partial z}(0, 0, 0) = \frac{\partial N_1}{\partial u}(0, (0, 0))\frac{\partial u}{\partial z} = a_0' > 0.$$

Therefore, based on the implicit function theorem, we confirm that there exists a unique continuous z as a function of $\tilde{\tau}$ and a such that $z = z(\tilde{\tau}, a)$ and $z(0, 0) = 0$, which can be solved from the equation $f_1(\tilde{\tau}, a, z) = 0$ near $(0, (0, 0))$. Furthermore, we have

$$f_1(\tilde{\tau}, a, z(\tilde{\tau}, a)) = N_1(\tilde{\tau}, aY_0 + z(\tilde{\tau}, a)E_0) = 0$$

and

$$\frac{\partial N_1(0, 0)}{\partial u}(-\frac{b_0'}{a_0'}) + \frac{\partial N_1(0, 0)}{\partial u}\frac{\partial z}{\partial a}(0, 0) + \frac{\partial N_2(0, 0)}{\partial v} = 0.$$

Thus, we have

$$\frac{\partial z}{\partial a}(0, 0) = -\left(\frac{\partial N_1(0, 0)}{\partial u}\right)^{-1}\frac{\partial N_1(0, 0)}{\partial v} + \frac{b_0'}{a_0'} = 0.$$

It follows from (11) that

$$\begin{aligned}\frac{\partial z}{\partial \tilde{\tau}}(0, 0) &= \frac{1}{a_0'}\frac{\partial \Theta_1}{\partial u}\frac{\partial \Phi_1(T, X_0)}{\partial \tilde{\tau}} \\ &= \frac{1}{a_0'}\left(1 - \frac{\theta\lambda}{(1 + \theta u_s(T))^2}\right)\dot{u}_s(T).\end{aligned}$$

Therefore, we conclude that $N(\tilde{\tau}, \tilde{X}) = 0$ if and only if

$$f_2(\tilde{\tau}, a) = N_2(\tilde{\tau}, aY_0 + z(\tilde{\tau}, a)E_0) = 0, \tag{14}$$

and the number of its roots equals the number of periodic solutions of model (10).

For convenience, we denote

$$f(\tilde{\tau}, a) = f_2(\tilde{\tau}, a)$$

with $f(0, 0) = N_2(0, (0, 0)) = 0$. In order to study the properties of the function f, we first compute the derivatives of f around $(0, 0)$. To do this, we compute the first order partial derivatives $\frac{\partial f}{\partial \tilde{\tau}}(0, 0)$ and $\frac{\partial f}{\partial a}(0, 0)$.

Denote $\eta(\tilde{\tau}) = T + \tilde{\tau}$, $\eta_1(\tilde{\tau}, a) = x_0 - \frac{b_0'}{a_0'}a + z(\tilde{\tau}, a)$ and $\eta_2(\tilde{\tau}, a) = a$, then we have

$$\begin{aligned}\frac{\partial f(\tilde{\tau}, a)}{\partial a} &= \frac{\partial}{\partial a}\left(\eta_2 - \Theta_2\left(\Phi(\eta, \eta_1, \eta_2)\right)\right) \\ &= 1 - \frac{\partial \Theta_2}{\partial v}\left(\frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial u}\left(-\frac{b_0'}{a_0'} + \frac{\partial z(\tilde{\tau}, a)}{\partial a}\right) + \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial v}\right) \\ &= 1 - \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial v}.\end{aligned}$$

So, we have $\frac{\partial f(0,0)}{\partial a} = 1 - \rho(T)$. It follows from $d_0' = 1 - \rho(T)$ that $d_0' = 0$ indicates that $\frac{\partial f}{\partial a}(0, 0) = 0$.

Similarly, we obtain that

$$\begin{aligned}\frac{\partial f}{\partial \tilde{\tau}}(\tilde{\tau}, a) &= \frac{\partial}{\partial \tilde{\tau}}\left(\eta_2 - \Theta_2\left(\Phi(\eta, \eta_1, \eta_2)\right)\right)(\tilde{\tau}, a) \\ &= -\frac{\partial \Theta_2}{\partial v}\left(\frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial \tilde{\tau}} + \frac{\partial \Phi_2(\eta, \eta_1, \eta_2)}{\partial u}\frac{\partial z}{\partial \tilde{\tau}}\right).\end{aligned}$$

It follows from $\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau}} = 0$ at $(\tilde{\tau}, a) = (0,0)$ that

$$\frac{\partial f}{\partial \tilde{\tau}}(0,0) = \frac{\partial f}{\partial a}(0,0) = 0.$$

Furthermore, denote $A = \frac{\partial^2 f(0,0)}{\partial \tilde{\tau}^2}$, $B = \frac{\partial^2 f(0,0)}{\partial \tilde{\tau}\partial a}$ and $C = \frac{\partial^2 f(0,0)}{\partial a^2}$. In the following, we should calculate the second-order partial derivatives in term of the parameters of the equation.

$$\begin{aligned}\frac{\partial^2 f(\tilde{\tau},a)}{\partial \tilde{\tau}^2} &= \frac{\partial}{\partial \tilde{\tau}}\left(-\frac{\partial \Theta_2}{\partial v}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau}} + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial z}{\partial \tilde{\tau}}\right)\right)\\
&= -\frac{\partial^2 \Theta_2}{\partial v^2}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau}} + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial z}{\partial \tilde{\tau}}\right)^2\\
&\quad -\frac{\partial \Theta_2}{\partial v}\left(\frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau}^2} + 2\frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial u\partial \tilde{\tau}}\frac{\partial z}{\partial \tilde{\tau}}\right.\\
&\quad \left. + \frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial u^2}\left(\frac{\partial z}{\partial \tilde{\tau}}\right)^2 + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial^2 z}{\partial \tilde{\tau}^2}\right).\end{aligned}$$

Since $\frac{\partial^2 \Phi_2}{\partial u^2} = \frac{\partial \Phi_2}{\partial \tilde{\tau}} = \frac{\partial \Phi_2}{\partial u} = \frac{\partial^2 \Phi_2}{\partial u \partial \tilde{\tau}} = 0$ for $(\tilde{\tau}, a) = (0,0)$, we have

$$A = \frac{\partial^2 f(0,0)}{\partial \tilde{\tau}^2} = -\frac{\partial^2 \Phi_2(T, X_0)}{\partial \tilde{\tau}^2}.$$

According to $\frac{\partial^2 \Phi_2(t,X_0)}{\partial t^2} = 0$ for all $0 \le t \le T$, we have

$$\frac{\partial^2 \Phi_2(T, X_0)}{\partial \tilde{\tau}^2} = 0,$$

which indicates that $A = 0$. By the same methods as shown above, we have

$$\begin{aligned}\frac{\partial^2 f(\tilde{\tau},a)}{\partial a^2} &= \frac{\partial}{\partial a}\left(1 - \frac{\partial \Theta_2}{\partial v}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\left(-\frac{b_0'}{a_0'} + \frac{\partial z(\tilde{\tau},a)}{\partial a}\right) + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial v}\right)\right)\\
&= -\frac{\partial^2 \Theta_2}{\partial v^2}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial a}\right) + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial v}\right)^2\\
&\quad -\frac{\partial \Theta_2}{\partial v}\left(\frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial u^2}\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial a}\right)^2\right)\\
&\quad -2\frac{\partial \Theta_2}{\partial v}\frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial v \partial u}\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial a}\right)\\
&\quad -\frac{\partial \Theta_2}{\partial v}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial^2 z}{\partial a^2} + \frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial v^2}\right).\end{aligned}$$

According to $\frac{\partial \Theta_2}{\partial v} = 1$, $\frac{\partial^2 \Theta_2}{\partial v^2} = -\frac{2\delta}{h}$ for $(\tilde{\tau}, a) = (0,0)$, in order to calculate C, we only need to calculate two terms, i.e. $\frac{\partial^2 \Phi_2(t,X_0)}{\partial u \partial v}$ and $\frac{\partial^2 \Phi_2(t,X_0)}{\partial^2 v}$.

Then, it follows that

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2 \Phi_2(t,X_0)}{\partial u \partial v}\right) &= \left(\frac{\partial F_2(\zeta(t))}{\partial v} + \frac{\partial F_2(\zeta(t))}{\partial u}\right)\left(\frac{\partial^2 \Phi_2(t,X_0)}{\partial u \partial v}\right)\\
&\quad + \frac{\partial^2 F_2(\zeta(t))}{\partial u \partial v}\frac{\partial \Phi_2(t,X_0)}{\partial v} + \frac{\partial^2 F_2(\zeta(t))}{\partial u^2}\frac{\partial \Phi_1(t,X_0)}{\partial v}\\
&= \left(g(0) - p'(0)u_s(t)\right)\frac{\partial^2 \Phi_2(t,X_0)}{\partial u \partial v} - p'(0)\rho(t)\end{aligned}$$

with initial condition $\frac{\partial^2 \Phi_2(0,X_0)}{\partial u \partial v} = 0$. So, we have

$$\begin{aligned}\frac{\partial^2 \Phi_2(T,X_0)}{\partial u \partial v} &= -p'(0) \int_0^T \rho(u) \exp\left(\int_u^T (g(0) - p'(0)u_s(v))dv \right) du \\ &= -p'(0)\rho(T)T.\end{aligned}$$

In order to obtain the formula for $\frac{\partial^2 \Phi_2(t,X_0)}{\partial v^2}$, we have the following different equation

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial^2 \Phi_2(t,X_0)}{\partial v^2}\right) = & \frac{\partial F_2(\zeta(t))}{\partial v}\left(\frac{\partial^2 \Phi_2(t,X_0)}{\partial v^2}\right) + \frac{\partial^2 F_2(\zeta(t))}{\partial v^2}\frac{\partial \Phi_2(t,X_0)}{\partial v} \\ & + \frac{\partial^2 F_2(\zeta(t))}{\partial v \partial u}\frac{\partial \Phi_1(t,X_0)}{\partial v} + \frac{\partial F_2(\zeta(t))}{\partial u}\frac{\partial^2 \Phi_1(t,X_0)}{\partial v^2} \\ = & (g(0) - p'(0)u_s(t))\frac{\partial^2 \Phi_2(t,X_0)}{\partial v^2} \\ & + (2g'(0) - p''(0)u_s(t))\rho(t) \\ & - p'(0) \int_0^t \exp(-D(t-v))cp'(0)u_s(v)\rho(v))dv\end{aligned}$$

with initial condition $\frac{\partial^2 \Phi_2(0,X_0)}{\partial v^2} = 0$. Integrating the above equation one yields

$$\begin{aligned}\frac{\partial^2 \Phi_2(t,x_0)}{\partial v^2} = & \int_0^t \exp\left(\int_v^t (g(0) - p'(0)u_s(\xi))d\xi\right)\left(2g'(0) - p''(0)u_s(v)\right)\rho(v)dv \\ & - cp'^2(0)\int_0^t \left\{\exp\left(\int_v^t (g(0) - p'(0)x_s(\xi))d\xi\right)\right\} \\ & \cdot\left\{\int_0^v \exp(-D(v-\theta))u_s(\theta)\rho(\theta)d\theta\right\}dv.\end{aligned}$$

Therefore, we already deduce that

$$\begin{aligned}C = & \frac{2\delta}{h}\left(\frac{\partial \Phi_2(T,X_0)}{\partial v}\right)^2 + 2\frac{b_0'}{a_0'}\frac{\partial^2 \Phi_2(T,X_0)}{\partial u \partial v} - \frac{\partial^2 \Phi_2(T,X_0)}{\partial v^2} \\ = & \rho(T)\left(\frac{2\delta\rho(T)}{h} - \frac{2b_0'p'(0)T}{a_0'} - 2g'(0)T + \frac{y^* p''(0)(1-\exp(-DT))}{D}\right) \\ & + cp'^2(0)y^* \int_0^T \left\{\exp\left(\int_v^T (g(0) - u_s(t))dt\right)e^{-Dv}\int_0^v \rho(\theta)d\theta\right\}dv.\end{aligned}$$

For calculation of B, once again as above we have

$$\begin{aligned}\frac{\partial^2 f(\tilde{\tau},a)}{\partial \tilde{\tau} \partial a} = & \frac{\partial}{\partial a}\left(-\frac{\partial \Theta_2}{\partial v}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau}} + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial z}{\partial \tilde{\tau}}\right)\right) \\ = & -\frac{\partial^2 \Theta_2}{\partial v^2}\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau}} + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial z}{\partial \tilde{\tau}}\right) \\ & \times\left(\frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial a}\right) + \frac{\partial \Phi_2(\eta,\eta_1,\eta_2)}{\partial v}\right) \\ & - \frac{\partial \Theta_2}{\partial v}\left(\frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau} \partial u} + \frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial u^2}\frac{\partial z}{\partial \tilde{\tau}}\right)\left(-\frac{b_0'}{a_0'} + \frac{\partial z}{\partial a}\right) \\ & - \frac{\partial \Theta_2}{\partial v}\left(\frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial \tilde{\tau} \partial v} + \frac{\partial^2 \Phi_2(\eta,\eta_1,\eta_2)}{\partial u \partial v}\frac{\partial z}{\partial \tilde{\tau}}\right)\end{aligned}$$

$$-\frac{\partial\Theta_2}{\partial v}\frac{\partial\Phi_2(\eta,\eta_1,\eta_2)}{\partial u}\frac{\partial^2 z}{\partial\tilde{\tau}\partial a}.$$

It follows from

$$\begin{aligned}\frac{\partial^2\Phi_2(t,X_0)}{\partial v\partial\tilde{\tau}} &= \frac{\partial F(\zeta(t))}{\partial v}\exp\left(\int_0^t \frac{\partial F(\zeta(t))}{\partial v}dt\right),\\ &= (g(0)-p'(0)u_s(t))\rho(t)\end{aligned}$$

and

$$\frac{\partial z(0,0)}{\partial\tilde{\tau}} = -\frac{D}{a_0'}\left(1-\frac{\theta\lambda}{(1+\theta u_s(T))^2}\right)u_s(T)$$

that

$$\begin{aligned}B &= -\frac{\partial^2\Phi_2(T,X_0)}{\partial\tilde{\tau}\partial v} - \frac{\partial^2\Phi_2(T,X_0)}{\partial u\partial v}\frac{\partial z(0,0)}{\partial\tilde{\tau}}\\ &= -\rho(T)\left(g(0)-p'(0)u_s(T)+\frac{p'(0)TDu_s(T)}{a_0'}\left(1-\frac{\lambda\theta}{(1+\theta u_s(T))^2}\right)\right).\end{aligned}$$

Furthermore, we study the Taylor expansion of $f(\tilde{\tau},a)$ near $(\tilde{\tau},a)=(0,0)$. Since $A=\frac{\partial^2 f(0,0)}{\partial\tilde{\tau}^2}$, $B=\frac{\partial^2 f(0,0)}{\partial\tilde{\tau}\partial a}$, and $C=\frac{\partial^2 f(0,0)}{\partial a^2}$, we have

$$f(\tilde{\tau},a)=Ba\tilde{\tau}+C\frac{a^2}{2}+o(\tilde{\tau},a)(\tilde{\tau}^2+a^2)=\frac{a}{2}\tilde{f}(\tilde{\tau},a),$$

where $\tilde{f}(\tilde{\tau},a)=2B\tilde{\tau}+Ca+\frac{1}{a}o(\tilde{\tau},a)(\tilde{\tau}^2+a^2)$, $\frac{\partial\tilde{f}(0,0)}{\partial\tilde{\tau}}=2B$ and $\frac{\partial\tilde{f}(0,0)}{\partial a}=C$.

For $B\neq 0$ $(C\neq 0)$, in order to employ the implicit function theorem about $f(\tilde{\tau},a)=0$, we deduce that there exists a unique function $\tilde{\tau}=\sigma(a)$ $(a=\gamma(\tilde{\tau}))$ near 0, which ensures that for all a $(\tilde{\tau})$ near 0 there exists a $\sigma(a)$ $(\gamma(\tilde{\tau}))$ such that $\tilde{f}(\sigma(a),a)=0$ $(\tilde{f}(\tilde{\tau},\gamma(\tilde{\tau}))=0)$ and $\sigma(0)=0$ $(\gamma(0)=0)$.

Therefore, equation (14) is equivalent to

$$2B\tilde{\tau}+Ca+\frac{1}{a}o(\tilde{\tau},a)(\tilde{\tau}^2+a^2)=0.$$

If $BC\neq 0$, solving the above equation, we have $\frac{a}{\tilde{\tau}}\simeq-\frac{2B}{C}$. If $BC=0$, then the above equation can not be solved with relation to the interesting parameters. It is necessary to expand f to the third or a higher order if $BC=0$, which is challenge for calculations. Finally, we have the following theorem

Theorem 3.1. *Assume that $d_0'=0$. If $BC\neq 0$, then in model (2) there occurs bifurcation at the threshold parameter values which satisfy $d_0'=0$, and the bifurcation is supercritical provided $BC<0$ and it is subcritical if $BC>0$.*

Although Theorem 3.1 reveals the existence and stability of nontrivial periodic solution of model (2), the conditions including the sign BC are quite complex. This indicates that it is hard to clarify the effects of impulsive period and nonlinear pulse on the pest control. Therefore, in order to verify our main results we choose the Holling Type II functional response curve as an example in the coming section.

4 Application of the main results

In order to show the applications of the main results and discuss the biological implications of the threshold conditions, we assume that the pest population follows the logistic growth in the absence of predator, i.e.

$g(x) = r(1-\frac{x}{K})$, and choose the Holling Type II functional response function for $p(x)$, i.e. $p(x) = \frac{\alpha x}{1+\omega x}$. Thus, model (2) becomes as the following special system with nonlinear impulsive control

$$\begin{cases} \left.\begin{aligned} \frac{dx(t)}{dt} &= rx(t)(1-\frac{x(t)}{K}) - \frac{\alpha x(t)}{1+\omega x(t)}y(t), \\ \frac{dy(t)}{dt} &= \frac{c\alpha x(t)}{1+\omega x(t)}y(t) - Dy(t), \end{aligned}\right\} t \neq nT, \\ \left.\begin{aligned} x(t^+) &= (1-\frac{\delta x(t)}{x(t)+h})x(t), \\ y(t^+) &= y(t) + \frac{\lambda}{1+\theta y(t)}, \end{aligned}\right\} \quad t = nT, \end{cases} \tag{15}$$

where r, K, α, ω, c and D are positive constants, respectively.

It follows from Theorem 2.2 that we obtain the pest-free periodic solution of model (15) for $\theta > 0$ as follows

$$(x_p(t), y_p(t)) = (0, y^* \exp(-D(t-nT))), t \in (nt, (n+1T)]$$

with $y^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-DT)(1-\exp(-DT))^{-1}}}{2\theta\exp(-DT)}$.

In particular, if $\theta = 0$, then model (15) has a pest-free periodic solution

$$(x_p(t), y_p(t)) = (0, y^* \exp(-D(t-nT))), t \in (nt, (n+1T)]$$

with $y^* = \frac{\lambda}{1-\exp(-DT)}$.

For the stability of pest free periodic solution we employ the main results shown in Theorem 2.3, from which we can see that the pest-free periodic solution of model (15) is locally stable provided

$$R_0 = \frac{rDT}{\alpha(1-\exp(-DT))y^*} < 1$$

and is globally attractive if

$$R_1 = \frac{rDT}{\alpha y^*(1-\exp(-DT))} \sup_{x\geq 0}\left(1-\frac{x}{K}\right)(1+\omega x) < 1.$$

Furthermore, the relations between R_0 and R_1 are as follows:

$$R_1 = R_0 \sup_{x\geq 0}(1-\frac{x}{K})(1+\omega x).$$

Note that if $Kw < 1$ then $\sup_{x\geq 0}(1-\frac{x}{K})(1+\omega x) = 1$, and if $Kw \geq 1$ then

$$\sup_{x\geq 0}(1-\frac{x}{K})(1+\omega x) = \frac{(K\omega+1)^2}{4K\omega}.$$

Therefore, we conclude that

(i) If $K\omega < 1$ then the pest-free periodic solution $(x_p(t), y_p(t))$ of model (15) is globally stable provided

$$R_0 = R_1 = \frac{rDT}{\alpha(1-\exp(-DT))y^*} < 1,$$

which indicates that the local stability implies the global stability.

(ii) If $K\omega \geq 1$ then the pest-free periodic solution $(x_p(t), y_p(t))$ of model (15) is locally stable provided

$$R_0 = \frac{rDT}{\alpha(1-\exp(-DT))y^*} < 1$$

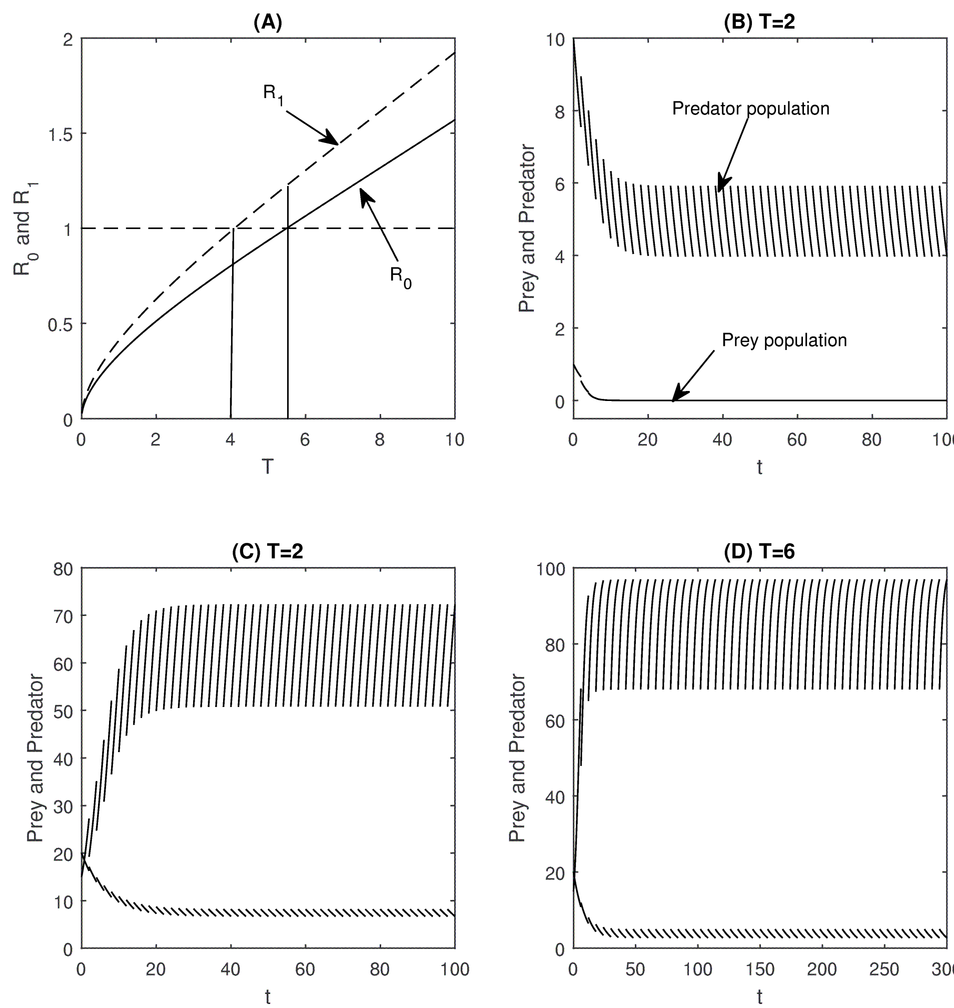
and it is globally attractive if

$$R_1 = \frac{rDT(K\omega+1)^2}{4K\omega\alpha y^*(1-\exp(-DT))} = R_0\frac{(K\omega+1)^2}{4K\omega} < 1.$$

It is easy to see that $(K\omega+1)^2 \geq 4K\omega$ and the equals sign holds true only for $K\omega = 1$, which indicates that $R_1 > R_0$ when $K\omega \neq 1$. Therefore, if $K\omega > 1$, then the local stability can not ensure the global stability of the pest-free periodic solution, and a stronger condition (i.e. $R_1 < 1$) is needed. So the interesting question is whether the condition $R_0 < 1$ can ensure the global stability of the pest free periodic solution or not when $K\omega > 1$. To answer this question numerically, we consider the following three possible cases (as shown in Fig. 1(A)):

$$(i)\ R_0 < R_1 < 1; \quad (ii)\ R_0 < 1 \le R_1; \quad (iii)\ 1 \le R_0 \le R_1.$$

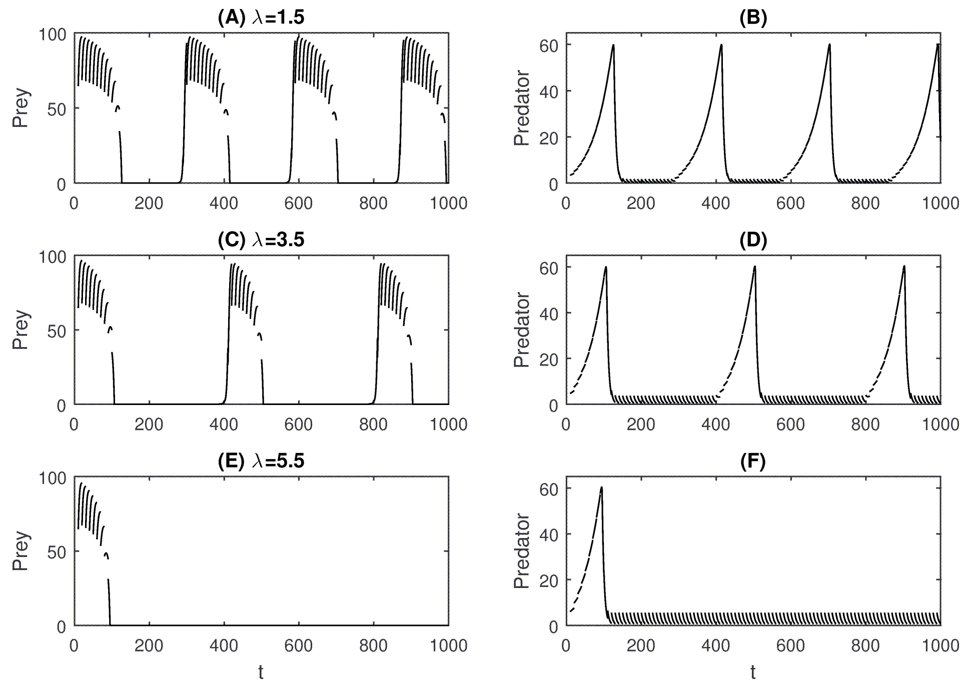
Fig. 1



If we fixed all parameter values as those shown in Fig. 1(B) and (C), then we have $R_0 < 1 < R_1$, which indicates that the pest free periodic solution is locally stable. However, if we chose two different initial values $(1, 10)$ and $(15, 20)$ in (B) and (C), respectively, we can see that the pest population will die out eventually in (B) and oscillates periodically in (C), i.e. the pest free periodic solution and a stable interior positive periodic solution can coexist. All those confirm that the local stability does not imply the global stability, and the condition $R_1 < 1$ is necessary for the global attractivity. If we fixed the parameter values as those shown in Fig. 1(D), then we have $1 \le R_0 \le R_1$, and consequently both the pest and natural enemy populations can oscillate as a positive periodic solution.

The effects of maximal releasing constant λ on oscillations of the both pest and natural enemy populations have been shown in Fig. 2, from which we can see that the oscillation patterns of the pest population can be significantly affected by the variations of biological control (i.e. releasing natural enemies). The results reveal that the larger releasing constant is, the fewer outbreak has, as shown in Fig. 2(A) and (C), although the maximal amplitude of the pest population is quite similar. The pest population will die out once the releasing constant λ exceeds some threshold values such as 5.5, as shown in Fig. 2(E), while the natural enemy population can oscillate with a small size (Fig. 2(F)).

Fig. 2



The numerical bifurcation analyses of model (15) with respect to the bifurcation parameter h for different impulsive period T have been shown in Fig. 3, from which we can see that the parameters related to the IPM strategies can strongly influence on the dynamics of model (15). Comparing the main results shown in Fig. 3(A) and (B), we conclude that a slightly changing the parameters h and T can significantly affect the variations of the pest population, and the pest population could periodically or quasi-periodically oscillate for a wide range of parameters.

Based on the main results shown in Theorem 3.1 we can theoretically address the bifurcations for model (15), and we have the following main results.

Theorem 4.1. *If $R_0 = 1$ (i.e. $d_0' = 0$) and $K\omega < 1$, then the pest-free periodic solution $(x_p(t), y_p(t))$ can bifurcate to another periodic solution as parameter varies, which is supercritical for Case (i) and Case (ii) shown in Theorem 2.2.*

Proof. To discuss the bifurcation of nontrivial periodic solution of system (15), we denote

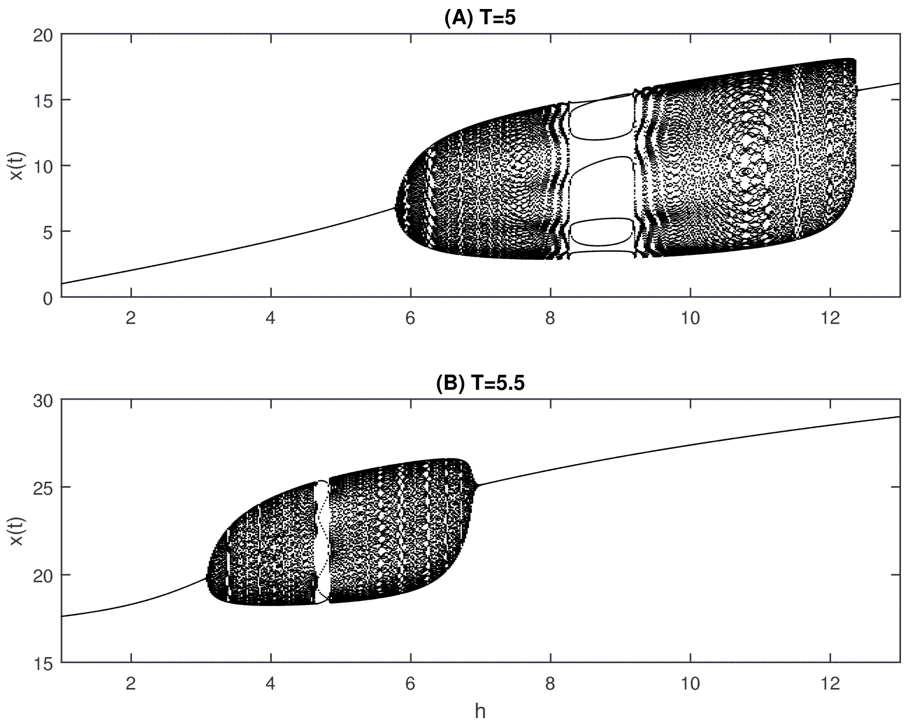
$$F_1(x,y) = \frac{c\alpha y}{1+\omega y}x - Dx,\ F_2(x,y) = ry\left(1-\frac{y}{K}\right) - \frac{\alpha y}{1+\omega y}x,$$
$$\Theta_1(x,y) = x + \frac{\lambda}{1+\theta x},\ \Theta_2(x,y) = \left(1-\frac{\delta y}{y+h}\right)y.$$

Therefore, according to Theorem 3.1, a necessary condition for the bifurcation of the nontrivial periodic solutions near the trivial periodic solution $(x_p(t), y_p(t))$ is $d_0' = 0$, and by simple calculation we have

$$d_0' = 1 - \exp\left(rT - \frac{\alpha y^*(1-\exp(-DT))}{D}\right)$$

with $y^* = \frac{-1+\sqrt{1+4\lambda\theta\exp(-DT)(1-\exp(-DT))^{-1}}}{2\theta\exp(-DT)}$. This indicates that if the parameter space satisfies $d_0' = 0 \Leftrightarrow R_0 = 1$, then the stability of the pest-free periodic solution is lost. Thus, in order to address the bifurcation, without

Fig. 3



loss of generality, we assume that $d_0' = 0$ holds true. For example, if we choose T as a bifurcation parameter, then the critical value of T is the root of $d_0' = 0$.

According to the formula (13), we have that

$$a_0' = 1 - \left(1 - \frac{\theta\lambda}{(1 + \theta y^* \exp(-DT))^2}\right) \exp(-DT) > 0,$$

$$b_0' = -c\alpha y^* \exp(-DT)\left(1 - \frac{\theta\lambda}{(1 + \theta y^* \exp(-DT))^2}\right) \int_0^T \rho(v)dv,$$

where $\rho(t) = \exp\left(rt - \frac{\alpha y^*(1-\exp(-Dt))}{D}\right)$ and $\rho(T) = 1$.

Further, according to the initial condition, we obtain that

$$\begin{aligned}
\frac{\partial \Phi_2(T, X_0)}{\partial y} &= \rho(T), \\
\frac{\partial^2 \Phi_2(T, X_0)}{\partial x \partial y} &= -\alpha T < 0, \\
\frac{\partial^2 \Phi_2(T, X_0)}{\partial y \partial \tau} &= r - \alpha y^* \exp(-DT), \\
\frac{\partial \Phi_1(T, X_0)}{\partial \tilde{\tau}} &= -Dy^* \exp(-DT) < 0, \\
\frac{\partial^2 \Phi_2(T, X_0)}{\partial y^2} &= 2rT(\omega - \frac{1}{K}) - c\alpha^2 y^* \int_0^T \exp(-Dv) \exp\Bigg(r(T - v) \\
&\quad - \frac{\alpha y^*(\exp(-Dv) - \exp(-DT))}{D}\Bigg) \\
&\quad \cdot \left\{ \int_0^v \exp\left(r\theta - \frac{\alpha y^*(1 - \exp(-D\theta))}{D} d\theta\right)\right\} dv.
\end{aligned}$$

Based on Theorem 3.1, we have

$$\begin{aligned} C &= \frac{2\delta}{h}\left(\frac{\partial \Phi_2(T,X_0)}{\partial y}\right)^2 + 2\frac{b_0'}{a_0'}\frac{\partial^2\Phi_2(T,X_0)}{\partial x \partial y} - \frac{\partial^2\Phi_2(T,X_0)}{\partial y^2} \\ &= \frac{2\delta}{h} - \frac{2b_0'\alpha T}{a_0'} - \frac{\partial^2\Phi_2(t,x_0)}{\partial y^2} \end{aligned}$$

and

$$\begin{aligned} B &= -\frac{\partial^2\Phi_2(T,X_0)}{\partial \tilde{\tau} \partial y} - \frac{\partial^2\Phi_2(T,X_0)}{\partial x \partial y}\frac{\partial z(0,0)}{\partial \tilde{\tau}} \\ &= -\left(r - \alpha y^* \exp(-DT) + \frac{\alpha T D y^* \exp(-DT)}{a_0'}\left(1 - \frac{\lambda\theta}{(1+\theta y_p(T))^2}\right)\right). \end{aligned}$$

In order to determine the sign of B, let $g(t) = r - \alpha y^* \exp(-Dt)$, then $g'(t) = \alpha D y^* \exp(-Dt) > 0$, which means that $g(t)$ is strictly increasing. Further, we have

$$\int_0^T g(t)dt = rT - \frac{\alpha y^*(1-\exp(-DT))}{D} = 0,$$

which indicates that $g(T) = r - \alpha y^* \exp(-DT) > 0$. Moreover, for Case (i), we have $1 - \frac{\lambda\theta}{(1+\theta y_p(T))^2} > 0$ because of $\lambda\theta < 1$. For Case (ii), we get $1 - \frac{\lambda\theta}{(1+\theta y_p(T))^2} > 0$ because $0 < Y_2 < y^* \Leftrightarrow \lambda\theta < (1+\theta y_p(T))^2$. Therefore, $B < 0$ and $b_0' < 0$ for Case (i) and Case (ii), and if $K\omega < 1$ then we have $\frac{\partial^2\Phi_2(T,X_0)}{\partial y^2} < 0$, and consequently $C > 0$ holds true. This completes the proof. □

In particular, if $\theta = 0$, then we have the following theorem:

Theorem 4.2. *If $R_0 = 1$ (i.e. $d_0' = 0$) and $K\omega < 1$, then the pest-free periodic solution $(x_p(t), y_p(t))$ can bifurcate to another periodic solution as parameter varies, which is supercritical.*

5 Conclusion

In the present work, a generalized predator-prey model with nonlinear impulsive control strategy is proposed and investigated. The existence and local stability of the pest free periodic solution have been addressed in more detail, and some new methods for the proof of local stability are provided, which depends on the difference equation determined by the impulsive point series. For the global stability of the pest free periodic solution, our main results reveal that the local stability does not imply the global stability, which means that a stronger sufficient condition is needed, i.e. there exists another threshold condition R_1 such that the pest free periodic solution is globally stable provided $R_0 \le R_1 < 1$. Note that if $M_s = \frac{g(0)}{f'(0)}$, then we have $R_0 = R_1$, which reveals that for some classical Lotka-Volterra systems the local stability implies the global stability of the pest free periodic solution.

In order to verify the main results and confirm that the stronger threshold condition $R_1 < 1$ is necessary for global attractivity of the pest free periodic solution, we further consider the Holling type II functional response function in application section. As discussed on Section 4, for the parameter values fixed as those in Fig.1(B) and (C), it is easy to see that the inequalities $R_0 < 1 < R_1$ hold true, and the pest free periodic solution is locally stable. It is interesting to note that for this parameter set, model (15) can have two periodic solutions, one is the pest free periodic solution and the other is the interior periodic solution, which can coexist. This reveals that the local stability does not imply the global stability, and the condition $R_1 < 1$ is necessary for the global attractivity. Therefore, we conclude that in the present work we provide some analytical methods to analyze the generalized models with nonlinear impulsive control, and the threshold conditions are useful for designing the IPM strategy. Furthermore, it is believed that the techniques of investigating the generalized models could be applied to population dynamics in relation to: chemotherapeutic treatment of disease[33] or vaccination strategies in epidemiology [35].

We should emphasize here that the density dependent releasing function considered in this paper only depends on the density of natural enemies. However, a more realistic case is that it should depend on the density of the pest population, i.e. the density dependent releasing function could be a saturation function of the pest population, which will bring difficulties for theoretical analysis. We will work on this in the near future.

Conflict of interest
The authors declare that there is no conflict of interests regarding the publication of this article.

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