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On the algebraicity of coefficients of half-integral weight mock modular forms

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Abstract: Extending works of Ono and Boylan to the half-integral weight case, we relate the algebraicity of Fourier coefficients of half-integral weight mock modular forms to the vanishing of Fourier coefficients of their shadows.

Keywords: Weakly holomorphic modular forms

MSC: 11F11, 11F67, 11F37

1 Introduction and statement of results

Let k be an integer greater than 1 and let N be a positive integer. The space of cusp forms of weight $2k$ for $\Gamma_0(N)$ is denoted by $S_{2k}(N)$. Throughout this paper let $p = 1$ or a prime number. For $\kappa \in \mathbb{Z} + \frac{1}{2}$ we denote by $M^!_\kappa(\Gamma_0(4p))$ the space of weakly holomorphic modular forms of weight κ on $\Gamma_0(4p)$. As usual, $M_\kappa(\Gamma_0(4p))$ (resp. $S_\kappa(\Gamma_0(4p))$) stands for the space of weight κ modular forms (resp. cusp forms) on $\Gamma_0(4p)$. Let $H_\kappa(\Gamma_0(4p))$ be the space of weight κ harmonic weak Maass forms on $\Gamma_0(4p)$. Let $\mathbb{M}^!_\kappa(p)$ (resp. $\mathbb{H}_{2-\kappa}(p)$) denote the subspace of $M^!_\kappa(\Gamma_0(4p))$ (resp. $H_{2-\kappa}(\Gamma_0(4p))$), in which each form satisfies Kohnen's plus space condition, that is, its Fourier expansion is supported only on those $n \in \mathbb{Z}$ for which $(-1)^{\kappa-\frac{1}{2}} n \equiv \square \pmod{4p}$. Let $\kappa = k + \frac{1}{2}$ and $\mathbb{M}_\kappa(p)$ (resp. $\mathbb{S}_\kappa(p)$) denote the subspace of $M_\kappa(\Gamma_0(4p))$ (resp. $S_\kappa(\Gamma_0(4p))$), in which each form satisfies Kohnen's plus space condition.

Let $\Delta(\tau) \in S_{12}(1)$ be the Ramanujan's Delta function. The famous Lehmer's conjecture states that the Fourier coefficients of $\Delta(\tau)$ never vanish. Concerning this conjecture, Ono [1] related the algebraicity of Fourier coefficients of weight -10 mock modular form whose shadow is $\Delta(\tau)$ to the vanishing of Fourier coefficients of $\Delta(\tau)$. Generalizing Ono's results, Boylan [2] related the algebraicity of Fourier coefficients of weight $2-2k$ mock modular forms to the vanishing of Fourier coefficients of their shadows when $\dim S_{2k}(1) = 1$. In this paper we will extend their works to the half-integral weight case. In the following we recall some known facts.

Fact 1. *By Shimura correspondence [3, Proposition 1] we have*

$$\dim \mathbb{S}_{k+\frac{1}{2}}(1) = \dim S_{2k}(1),$$

which implies that

$$\dim \mathbb{S}_{k+\frac{1}{2}}(1) = 1 \iff k = 6, 8, 9, 10, 11, 13 \iff 1 - k = -5, -7, -8, -9, -10, -12.$$

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Now we assume that $k \in \{6,8,9,10,11,13\}$. For $\kappa > 2$ there is an antilinear differential operator $\xi_{2-\kappa}$: $H_{2-\kappa}(\Gamma_0(4p)) \to S_\kappa(\Gamma_0(4p))$ defined by

$$\xi_{2-\kappa}(f)(\tau) := 2iy^{2-\kappa} \cdot \overline{\frac{\partial f}{\partial \bar{\tau}}}.$$

Fact 2. *For $k \in \{6,8,9,10,11,13\}$, it follows from [4, Theorem 1.1-(iii) and Lemma 4.2-(c)] that*

$$\xi_{\frac{3}{2}-k} : \mathbb{H}_{\frac{3}{2}-k}(1) \to \mathbb{S}_{k+\frac{1}{2}}(1) \ \textit{ is surjective.}$$

For any $\kappa \in \mathbb{Z}+\frac{1}{2}$, the Duke-Jenkins basis [5] for $\mathbb{M}^!_\kappa := \mathbb{M}^!_\kappa(1)$ is constructed as follows. Let $2\kappa - 1 = 12\ell_\kappa + k'$ with uniquely determined $\ell_\kappa \in \mathbb{Z}$ and $k' \in \{0,4,6,8,10,14\}$. If A_κ denotes the maximal order of a non-zero $f \in \mathbb{M}^!_\kappa$ at $i\infty$, then by the Shimura correspondence [3] one has

$$A_\kappa = \begin{cases} 2\ell_\kappa - (-1)^{\kappa-1/2} & \text{if } \ell_\kappa \text{ is odd,} \\ 2\ell_\kappa & \text{otherwise.} \end{cases} \tag{1}$$

A basis for $\mathbb{M}^!_\kappa$ then consists of functions of the form

$$f_{\kappa,m}(\tau) = q^{-m} + \sum_{n>A_\kappa} a_\kappa(m,n)q^n, \tag{2}$$

where $m \geq -A_\kappa$ satisfies $(-1)^{\kappa-3/2}m \equiv 0, 1 \pmod 4$. Using (1) and (2) we deduce the following facts.

Fact 3. *For $\kappa = \frac{3}{2} - k$ with $k \in \{6,8,9,10,11,13\}$, the maximal order A_κ of a non-zero $f \in M^!_\kappa$ at $i\infty$ is given by $A_\kappa = -4$. Thus for each $m \geq 4$ satisfying $(-1)^k m \equiv 0, 1 \pmod 4$, there exist unique modular forms $f_{\frac{3}{2}-k,m}(\tau) \in \mathbb{M}^!_{\frac{3}{2}-k}$ with Fourier development*

$$f_{\frac{3}{2}-k,m}(\tau) = q^{-m} + \sum_{n\geq -3} a_{\frac{3}{2}-k}(m,n)q^n,$$

which form a basis for the space $\mathbb{M}^!_{\frac{3}{2}-k}$.

Fact 4. *For $\kappa = k + \frac{1}{2}$ with $k \in \{6,8,9,10,11,13\}$, the space $\mathbb{S}_\kappa$ is spanned by*

$$f_k := f_{\kappa,-\alpha}$$

where α is given by

$$\alpha = \alpha_k = A_\kappa = A_{k+\frac{1}{2}} = \begin{cases} 1, & k \text{ even,} \\ 3, & k \text{ odd,} \end{cases}$$

and f_k has the form $q^\alpha + O(q^4)$.

Fact 5. *Let $\kappa = k+\frac{1}{2}$ and $f(z) \in H_\kappa(\Gamma_0(4p))$ with Fourier expansion $f(\tau) = \sum_{n\in\mathbb{Z}} c(y;n)e^{2\pi i n x}$ where $\tau = x+iy$. For each prime l with $\gcd(l,4p) = 1$, the l^2-th Hecke operator is defined by*

$$f|_\kappa T(l^2)(\tau) = \sum_{n\in\mathbb{Z}} \left(c(y/l^2; nl^2) + \left(\frac{(-1)^k n}{l}\right) l^{k-1} c(y;n) + l^{2k-1} c(l^2 y; n/l^2)\right) e^{2\pi i n x}.$$

Then for each $\mathcal{M} \in \mathbb{H}_{2-\kappa}(p)$, we obtain from [6, (2.6)] or [7, (7.2)] that for $\kappa > 2$,

$$\xi_{2-\kappa}(\mathcal{M})|_\kappa T(l^2) = l^{2\kappa-2}\xi_{2-\kappa}(\mathcal{M}|_{2-\kappa}T(l^2)).$$

As a corollary of Fact 5, one has that if $f(z) = \sum_{\substack{n\geq 1 \\ (-1)^k n \equiv 0,1 \ (4)}} a_f(n)q^n \in \mathbb{S}_{k+\frac{1}{2}}$, then

$$f|_{k+\frac{1}{2}} T(l^2) = \sum_{\substack{n\geq 1 \\ (-1)^k n \equiv 0,1 \ (4)}} \left(a_f(l^2 n) + \left(\frac{(-1)^k n}{l}\right) l^{k-1} a_f(n) + l^{2k-1} a_f(n/l^2)\right) q^n \in \mathbb{S}_{k+\frac{1}{2}}.$$

(or see [3, Theorem 1-(i)].)

Let (V, Q) be a non-degenerate rational quadratic space of signature (b^+, b^-) and L an even lattice with dual L'. Denote the standard basis elements of the group algebra $\mathbb{C}[L'/L]$ by $\mathfrak{e}_\gamma$ for $\gamma \in L'/L$. Let $\mathrm{Mp}_2(\mathbb{Z})$ denote the integral metaplectic group, which consists of pairs (γ, ϕ), where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi : \mathfrak{H} \to \mathbb{C}$ is a holomorphic function with $\phi(\tau)^2 = c\tau + d$. It is well known that $\mathrm{Mp}_2(\mathbb{Z})$ is generated by $S = \left(\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau}\right)$ and $T = \left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1\right)$. Then there is a unitary representation ρ_L of the group $Mp_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$, the so-called Weil representation, which is defined by

$$\rho_L(T)(\mathfrak{e}_\gamma) := e(Q(\gamma))\mathfrak{e}_\gamma,$$
$$\rho_L(S)(\mathfrak{e}_\gamma) := \frac{e((b^- - b^+)/8)}{\sqrt{|L'/L|}} \sum_{\delta \in L'/L} e(-(\gamma, \delta))\mathfrak{e}_\delta,$$

where $e(z) := e^{2\pi i z}$ and $(X, Y) := Q(X+Y) - Q(X) - Q(Y)$ is the associated bilinear form. One has the relations

$$S^2 = (ST)^3 = Z \qquad \left(Z = \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), i\right)\right)$$

from which we note that

$$\rho_L(Z)\mathfrak{e}_\gamma = i^{b^- - b^+}\mathfrak{e}_{-\gamma}. \tag{3}$$

We write $< \cdot, \cdot >$ for the standard scalar product on $\mathbb{C}[L'/L]$, i.e.

$$< \sum_{\gamma \in L'/L} \lambda_\gamma \mathfrak{e}_\gamma, \sum_{\gamma \in L'/L} \mu_\gamma \mathfrak{e}_\gamma >= \sum_{\gamma \in L'/L} \lambda_\gamma \overline{\mu_\gamma}.$$

For $\gamma, \delta \in L'/L$ and $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$ the coefficient $\rho_{\gamma\delta}(M, \phi)$ of the representation ρ_L is defined by

$$\rho_{\gamma\delta}(M, \phi) =< \rho_L(M, \phi)\mathfrak{e}_\delta, \mathfrak{e}_\gamma > .$$

Following [9], for an integer r we denote by $H_{r+1/2,\rho_L}$ (resp. $M^!_{r+1/2,\rho_L}$), the space of $\mathbb{C}[L'/L]$-valued harmonic weak Maass forms (resp. weakly holomorphic modular forms) of weight $r + 1/2$ and type ρ_L.

Let L_r be the lattice $2p\mathbb{Z}$ of signature $(1, 0)$ (resp. $(0, 1)$) when r is even (resp. odd) equipped with the quadratic form $Q_r(x) = (-1)^r x^2/4p$. Then its dual lattice L_r' is equal to $\mathbb{Z}$. For a vector valued modular form $F = \sum_\gamma F_\gamma \mathfrak{e}_\gamma$, we define a map Φ by

$$\Phi(F)(\tau) := \sum_\gamma F_\gamma(4p\tau). \tag{4}$$

It then follows from [8, Theorem 1] that the map Φ defines an isomorphism from $H_{r+1/2,\rho_{L_r}}$ to $\mathbb{H}_{r+1/2}(p)$ since $\bar{\rho}_{L_r} = \rho_{L_{r+1}}$.

For a $\mathbb{C}[L'/L]$-valued function f and $(M, \phi) \in Mp_2(\mathbb{Z})$ we define the Petersson slash operator by

$$(f|^L_{r+1/2}(M, \phi))(\tau) = \phi(\tau)^{-2r-1}\rho_L(M, \phi)^{-1} f(M\tau).$$

Let $L := L_k$ and $Q := Q_k$. Following [9] we define the vector valued cuspidal Poincaré series $P^L_{\beta,n}(\tau)$ as follows: for each $\beta \in \mathbb{Z}/2p\mathbb{Z}$ and $n \in \mathbb{Z} + Q(\beta)$ with $n > 0$,

$$P^L_{\beta,n}(\tau) := \frac{1}{2} \sum_{(M,\phi) \in \tilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \mathfrak{e}_\beta(n\tau)|_\kappa(M, \phi).$$

Then we know from [9] that $P^L_{\beta,n}(\tau)$ belongs to the space S_{κ,ρ_L}. Let $\mathfrak{D}_k$ denote the set of all integers D such that $(-1)^k D > 0$ and D is congruent to a square modulo $4p$.

Theorem 1.1 ([4, Theorem 1.1]). *For an integer $k > 2$ we let $\kappa = k + \frac{1}{2}$ and $L := L_k$. For each $D \in \mathfrak{D}_k$, we define*

$$P^+_D := \Phi(P^L_{\beta,\frac{|D|}{4p}}),$$

where β is an integer such that $D \equiv \beta^2 \pmod{4p}$. Then the following assertions are true.
(i) $P^+_D \in \mathbb{S}_\kappa(p)$ and the defintion of P^+_D does not depend on the choice of β.

(ii) For each $f = \sum_{n\geq 1} a_f(n)q^n \in \mathbb{S}_\kappa(p)$, *we have*

$$(f, c_{k,D}P_D^+) = a_f(|D|)$$

where $(\cdot,\cdot)$ *denotes the Petersson inner product and*

$$c_{k,D} = \begin{cases} \frac{(4\pi|D|)^{\kappa-1}}{\Gamma(\kappa-1)}\cdot\frac{s(D)}{3}, & \text{if } p = 1 \\ \frac{(4\pi|D|)^{\kappa-1}}{\Gamma(\kappa-1)}\cdot\frac{s(D)}{4}, & \text{if } p = 2 \\ \frac{(4\pi|D|)^{\kappa-1}}{\Gamma(\kappa-1)}\cdot\frac{s(D)}{6}, & \text{if } p > 2 \end{cases}$$

with $s(D) = \begin{cases} 1, & \text{if } p|D, \\ 2, & \text{otherwise.}\end{cases}$

(iii) The set $\{P_D^+ \mid D \in \mathfrak{D}_k\}$ *spans the space* $\mathbb{S}_\kappa(p)$. *Moreover, if we let* $t := \dim \mathbb{S}_\kappa(p)$ *and* $\{f_1, f_2, \ldots, f_t\}$ *be a basis for* $\mathbb{S}_\kappa(p)$ *satisfying* $f_i = q^{|D_i|} + O(q^{|D_i|+1})$ *for some* $D_i \in \mathfrak{D}_k$ $(i = 1,\ldots,t)$ *and* $0 < |D_1| < |D_2| < \cdots < |D_t|$, *then the set*

$$\{P_{D_1}^+, P_{D_2}^+, \ldots, P_{D_t}^+\}$$

forms a basis for $\mathbb{S}_\kappa(p)$.

(iv) Let I be a nonempty finite subset of $\mathbb{N}$. *Then the following two conditions are equivalent.*

(a) $\sum_{i\in I} \alpha_i P_{D_i}^+(\tau) \equiv 0$ *for some* $\alpha_i \in \mathbb{C}$ *and* $D_i \in \mathfrak{D}_k$.

(b) There exists $g \in \mathbb{M}^!_{2-\kappa}(p)$ *with the principal part* $\sum_{i\in I} \bar{\alpha}_i |D_i|^{1-\kappa} q^{-|D_i|}$.

Remark 1.2. *Let* $p = 1$ *and take*

$$D_k = \begin{cases} 1, & \text{if } k \text{ is even} \\ -3, & \text{if } k \text{ is odd.}\end{cases}$$

Then in Theorem 1.1 one can choose $\beta = 1$ *and*

$$P_{D_k}^+ := \Phi(P^{L_k}_{1,\frac{|D_k|}{4p}}).$$

We let $\tilde{\Gamma}_\infty :=< T >$. We define for $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$:

$$\mathcal{M}_s(y) = y^{-(2-\kappa)/2} M_{-(2-\kappa)/2, s-1/2}(y) \quad (y > 0),$$
$$\mathcal{W}_s(y) = |y|^{-(2-\kappa)/2} W_{\frac{2-\kappa}{2}\operatorname{sgn}(y), s-1/2}(|y|)$$

where $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ denote the usual Whittaker functions. Now we take $\kappa = k + 1/2 > 2$, $L := L_{1-k}$, and $Q := Q_{1-k}$. For each $\beta \in \mathbb{Z}/2p\mathbb{Z}$ and $m \in \mathbb{Z} + Q(\beta)$ with $m < 0$, modifying the Poincaré series in [9, (1.35)] we define the vector valued Maass Poincaré series $F^L_{\beta,m}$ of index (β, m) by

$$F^L_{\beta,m}(\tau, s) := \frac{1}{2\Gamma(2s)} \sum_{(M,\phi)\in\tilde{\Gamma}_\infty\backslash \mathrm{Mp}_2(\mathbb{Z})} [\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)]|^L_{2-\kappa}(M,\phi)$$

where $\tau = x + iy \in \mathfrak{H}$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$. Indeed, since $\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)$ is invariant under slash operator $|_{2-\kappa}T$, the Maass Poincaré series is well defined. This series has desirable properties as follows. As in Section 1.3 in [9] it converges normally for $\tau \in \mathbb{H}$ and $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1$ and hence defines a $\mathrm{Mp}_2(\mathbb{Z})$-invariant function on $\mathbb{H}$ under the slash operator $|_{2-\kappa}$. Moreover, $F^L_{\beta,m}(\tau, s)$ is an eigenfunction of $\Delta_{2-\kappa}$ with an eigenvalue $s(1-s) + \kappa(\kappa-2)/4$. Since $\mathfrak{e}_\beta(\tau)|_{2-\kappa}Z = \mathfrak{e}_{-\beta}$ by (3), the invariance of $F^L_{\beta,m}$ under the action of Z implies $F^L_{\beta,m} = F^L_{-\beta,m}$.

Let $\kappa = k + \frac{1}{2}$ and $L = L_{1-k}$ with k an integer > 2. For each $\beta \in \mathbb{Z}/2p\mathbb{Z}$ and $m \in \mathbb{Z} + Q(\beta)$ with $m < 0$, we obtain from [4, Corollary 1.5] that $F^L_{\beta,m}\left(\tau, \frac{\kappa}{2}\right)$ belongs to the space $H_{2-\kappa,\rho_L}$.

Let

$$Q = Q(k; z) := \Phi\left(F^{L_{1-k}}_{1,-\frac{\alpha}{4}}\left(\tau, \frac{\kappa}{2}\right)\right) = Q^+ + Q^-$$

where $Q^+ = Q^+(k;z)$ is the holomorphic part of $Q(k;z)$ and $Q^- = Q^-(k;z)$ is the nonholomorphic part of $Q(k;z)$. Let $Q(k;z)$ have the Fourier development as follows:

$$Q(k;z) = 2q^{-\alpha} + c_Q^+(0) + \sum_{\substack{n\geq 1\\ (-1)^{k-1}n\equiv 0,1\ (4)}} c_Q^+(n)q^n + \sum_{\substack{n\geq 1\\ (-1)^{k}n\equiv 0,1\ (4)}} c_Q^-(n)\Gamma(\kappa-1,4\pi ny)q^{-n}.$$

Now we are ready to state our main results.

Theorem 1.3. *With the same notations as above the following assertions are true.*
(1) Let

$$f_k|_{k+\frac{1}{2}}T(l^2) = \lambda_k(l^2)f_k$$

for some $\lambda_k(l^2)\in\mathbb{C}$. Then one has

$$\lambda_k(l^2) = a_{f_k}(l^2\alpha) + \left(\frac{(-1)^k\alpha}{l}\right)l^{k-1}.$$

(2) We have

$$Q|_{\frac{3}{2}-k}T(l^2) - l^{1-2k}\lambda_k(l^2)Q = Q^+|_{\frac{3}{2}-k}T(l^2) - l^{1-2k}\lambda_k(l^2)Q^+ \in \mathbb{M}^!_{\frac{3}{2}-k}.$$

Theorem 1.4. *For an odd prime l, the following assertions are true.*
(1) We have

$$c_Q^+(l^2\beta_k) \in \frac{\mathbb{Z}[c_Q^+(\beta_k)]}{l^{2k-1}} \subseteq \mathbb{Q}(c_Q^+(\beta_k)).$$

(2) Assume that $c_Q^+(\beta_k)$ is irrational. Then

$$a_{f_k}(l^2\alpha_k) = l^{k-1}\left(\left(\frac{(-1)^{k-1}\beta_k}{l}\right) - \left(\frac{(-1)^k\alpha_k}{l}\right)\right) \text{ if and only if } c_Q^+(l^2\beta_k)\in\mathbb{Q}.$$

(3) Assume that $\left(\frac{-\beta_k}{l}\right) = \left(\frac{\alpha_k}{l}\right)$ and $c_Q^+(\beta_k)$ is irrational. Then

$$a_{f_k}(l^2\alpha_k) = 0 \text{ if and only if } c_Q^+(l^2\beta_k)\in\mathbb{Q}.$$

Remark 1.5. *For simplicity, we dealt with the case $p = 1$ in our main results. But we remark that they can be extended to higher level cases whenever* $\dim \mathbb{S}_{k+\frac{1}{2}}(p) = 1$.

2 Proof of Theorem 1.3

First we are in need of two lemmas and one more fact.

Lemma 2.1 ([4, Lemma 4.1]). *Let $\kappa = k + \frac{1}{2}$ for an integer $k > 2$ and let $D \in \mathfrak{D}_k$. Then the following assertions are true.*
(a) For each $G \in H_{2-\kappa,\rho_{L_{1-k}}}$, we have

$$(4p)^{\kappa-1}\Phi\circ\xi_{2-\kappa}(G) = \xi_{2-\kappa}\circ\Phi(G).$$

(b) For each $f = \sum_{n\geq1} c_f(n)q^n \in \mathbb{S}_\kappa(p)$,

$$(f,(4p)^{\kappa-1}\Phi\xi_{2-\kappa}(F^{L_{1-k}}_{\beta,-\frac{|D|}{4p}}(\tau,\frac{\kappa}{2}))) = \begin{cases} \frac{3}{s(D)}\cdot c_f(|D|), & \text{if } p = 1\\ \frac{4}{s(D)}\cdot c_f(|D|), & \text{if } p = 2\\ \frac{6}{s(D)}\cdot c_f(|D|), & \text{if } p > 2.\end{cases}$$

Lemma 2.2 ([4, Lemma 4.2]). *With the same notations as in Lemma 2.1, we have the following assertions.*
(a) For a vector valued function $h = h_\beta(\tau)\mathfrak{e}_\beta$, one has

$$\xi_{2-\kappa}(h|_{2-\kappa}^{L_{1-k}}(M,\phi)) = (\xi_{2-\kappa}(h))|_\kappa^{L_k}(M,\phi).$$

(b) $\xi_{2-\kappa}(\Gamma(\kappa-1, 4\pi ny)) = -(4\pi n)^{\kappa-1}e^{-4\pi ny}$.
(c) Let $m = -\frac{|D|}{4p}$. Then one has

$$\xi_{2-\kappa}(F_{\beta,m}^{L_{1-k}}(\tau,\frac{\kappa}{2})) = \frac{(4\pi|m|)^{\kappa-1}}{\Gamma(\kappa-1)}P_{\beta,|m|}^{L_k}(\tau).$$

Fact 6. *Let $p = 1$ and $\kappa = k + \frac{1}{2}$.*
(1) It follows from Lemmas 2.1 and 2.2 that

$$\Phi\left(\xi_{2-\kappa}\left(F_{1,-\frac{|D_k|}{4}}^{L_{1-k}}\left(\tau,\frac{\kappa}{2}\right)\right)\right) = \Phi\left(\frac{(\pi|D_k|)^{\kappa-1}}{\Gamma(\kappa-1)}P_{1,\frac{|D_k|}{4}}^{L_k}(\tau)\right) = \frac{(\pi|D_k|)^{\kappa-1}}{\Gamma(\kappa-1)}P_{D_k}^{+}(\tau) \in \mathbb{S}_{k+\frac{1}{2}}.$$

(2) We obtain from Theorem 1.1-(ii) that

$$(f_k, c_{k,D_k}P_{D_k}^{+}) = a_{f_k}(|D_k|) = 1$$

where $c_{k,D_k} = \frac{(4\pi|D_k|)^{\kappa-1}}{3\Gamma(\kappa-1)} \in \mathbb{R}$.

For $k \in \{6, 8, 9, 10, 11, 13\}$, it follows from Fact 3, Fact 4, and Theorem 1.1-(iii), (iv) that $P_{D_k}^{+}$ does not vanish and

$$P_{D_k}^{+} = c_k f_k$$

for some $c_k \in \mathbb{C}^\times$. Thus one has from Fact 6 (2) that

$$1 = (f_k, c_{k,D_k}c_k f_k) = \overline{c_k}c_{k,D_k}||f_k||^2,$$

which implies

$$c_k = c_{k,D_k}^{-1}||f_k||^{-2}.$$

We compute that

$$\begin{aligned}
\xi_{2-\kappa}(Q(k;z)) &= \xi_{2-\kappa}\Phi\left(F_{1,-\frac{\alpha}{4}}^{L_{1-k}}\left(\tau,\frac{\kappa}{2}\right)\right) \\
&= 4^{\kappa-1}\Phi\xi_{2-\kappa}\left(F_{1,-\frac{\alpha}{4}}^{L_{1-k}}\left(\tau,\frac{\kappa}{2}\right)\right) \quad \text{by Lemma 2.1-(a)} \\
&= \frac{(4\pi|D_k|)^{\kappa-1}}{\Gamma(\kappa-1)}P_{D_k}^{+}(\tau) \quad \text{by Fact 6 (1)} \\
&= \frac{(4\pi|D_k|)^{\kappa-1}}{\Gamma(\kappa-1)}c_{k,D_k}^{-1}||f_k||^{-2}f_k \\
&= 3||f_k||^{-2}f_k.
\end{aligned} \tag{5}$$

Since

$$\begin{aligned}
f_k|_{k+\frac{1}{2}}T(l^2) &= \sum_{\substack{n\geq 1 \\ (-1)^k n\equiv 0,1\ (4)}} \left(a_{f_k}(l^2n) + \left(\frac{(-1)^k n}{l}\right)l^{k-1}a_{f_k}(n) + l^{2k-1}a_{f_k}(n/l^2)\right)q^n \\
&= \lambda_k(l^2)f_k,
\end{aligned}$$

one has

$$\lambda_k(l^2) = a_{f_k}(l^2\alpha) + \left(\frac{(-1)^k\alpha}{l}\right)l^{k-1},$$

which proves the first assertion. Hence for all $n \geq 1$ with $(-1)^k n \equiv 0, 1 \pmod 4$

$$\left(a_{f_k}(l^2\alpha) + \left(\frac{(-1)^k\alpha}{l}\right)l^{k-1}\right)a_{f_k}(n) = a_{f_k}(l^2 n) + \left(\frac{(-1)^k n}{l}\right)l^{k-1}a_{f_k}(n) + l^{2k-1}a_{f_k}(n/l^2). \tag{6}$$

It follows from (5) that

$$3\|f_k\|^{-2}f_k(z) = \xi_{2-\kappa}(Q(k;z)) = -\sum_{\substack{n\geq\alpha \\ (-1)^k n\equiv 0,1 \ (4)}} (4\pi n)^{\kappa-1}\overline{c_Q^-(n)}q^n,$$

which implies that

$$c_Q^-(n) = -3\|f_k\|^{-2}\cdot(4\pi n)^{1-\kappa}\cdot a_{f_k}(n). \tag{7}$$

Now we put $d_k := -3\|f_k\|^{-2}\cdot(4\pi)^{1-\kappa}$. We obtain that for all positive integers n with $(-1)^k n \equiv 0, 1 \pmod 4$,

$$\begin{aligned}
&n^{1-\kappa}\left(c_Q^-(nl^2)(nl^2)^{\kappa-1} + l^{2k-1}c_Q^-(n/l^2)(n/l^2)^{\kappa-1} + \left(\frac{(-1)^k n}{l}\right)l^{k-1}c_Q^-(n)n^{\kappa-1}\right)\\
=&n^{1-\kappa}d_k\left(a_{f_k}(nl^2) + a_{f_k}(n/l^2)l^{2k-1} + \left(\frac{(-1)^k n}{l}\right)l^{k-1}a_{f_k}(n)\right) \quad \text{by (7)}\\
=&n^{1-\kappa}d_k\left(a_{f_k}(l^2\alpha) + \left(\frac{(-1)^k\alpha}{l}\right)l^{k-1}\right)a_{f_k}(n) \quad \text{by (6)}\\
=&\lambda_k(l^2)c_Q^-(n).
\end{aligned}$$

Thus we have

$$\begin{aligned}
Q^-|T_{\frac{3}{2}-k}(l^2) &= \sum_{n\in\mathbb{Z}}\left(c_Q^-(nl^2) + \left(\frac{(-1)^k n}{l}\right)l^{-k}c_Q^-(n) + l^{1-2k}c_Q^-(n/l^2)\right)\Gamma(\kappa-1, 4\pi ny)q^{-n}\\
&= l^{1-2k}\sum_{n\in\mathbb{Z}}\left(c_Q^-(nl^2)l^{2k-1} + \left(\frac{(-1)^k n}{l}\right)l^{k-1}c_Q^-(n) + c_Q^-(n/l^2)\right)\Gamma(\kappa-1, 4\pi ny)q^{-n}\\
&= l^{1-2k}\sum_{n\in\mathbb{Z}}d_k n^{1-\kappa}\left(c_Q^-(nl^2)(nl^2)^{\kappa-1}d_k^{-1} + \left(\frac{(-1)^k n}{l}\right)l^{k-1}n^{\kappa-1}c_Q^-(n)d_k^{-1} + n^{\kappa-1}c_Q^-(n/l^2)d_k^{-1}\right)\\
&\qquad\Gamma(\kappa-1, 4\pi ny)q^{-n}\\
&= l^{1-2k}\sum_{n\in\mathbb{Z}}d_k n^{1-\kappa}\left(a_{f_k}(nl^2) + \left(\frac{(-1)^k n}{l}\right)l^{k-1}a_{f_k}(n) + l^{2k-1}a_{f_k}(n/l^2)\right)\Gamma(\kappa-1, 4\pi ny)q^{-n}\\
&= l^{1-2k}\sum_{n\in\mathbb{Z}}\lambda_k(l^2)c_Q^-(n)\Gamma(\kappa-1, 4\pi ny)q^{-n} \quad \text{since } f_k|T_{k+\frac{1}{2}}(l^2) = \lambda_k(l^2)f_k\\
&= l^{1-2k}\lambda_k(l^2)Q^-.
\end{aligned} \tag{8}$$

We obtain that

$$\begin{aligned}
l^{2\kappa-2}\xi_{2-\kappa}(Q|_{2-\kappa}T(l^2)) &= (\xi_{2-\kappa}(Q))|_\kappa T(l^2) \quad \text{by Fact 5}\\
&= 3\|f_k\|^{-2}f_k|_\kappa T(l^2) \quad \text{by (5)}\\
&= 3\|f_k\|^{-2}\lambda_k(l^2)f_k\\
&= \xi_{2-\kappa}(\lambda_k(l^2)Q) \quad \text{since } \lambda_k(l^2)\in\mathbb{R}.
\end{aligned}$$

Indeed, we observe that

$$\lambda_k(l^2) = a_{f_k}(l^2\alpha) + \left(\frac{(-1)^k\alpha}{l}\right)l^{k-1} \in \mathbb{Z}.$$

Thus we have

$$l^{2\kappa-2}Q|_{2-\kappa}T(l^2) - \lambda_k(l^2)Q \in \mathbb{M}^!_{2-\kappa},$$

which combined with (8) yields the second assertion.

3 Proof of Theorem 1.4

We observe that

$$\begin{aligned}
&Q|_{\frac{3}{2}-k}T(l^2) - l^{1-2k}\lambda_k(l^2)Q = Q^+|_{\frac{3}{2}-k}T(l^2) - l^{1-2k}\lambda_k(l^2)Q^+\\
&= \left(2q^{-\alpha} + c_Q^+(0) + \sum_{\substack{n\geq 1\\ (-1)^{k-1}n\equiv 0,1\ (4)}} c_Q^+(n)q^n\right)|_{\frac{3}{2}-k}T(l^2)\\
&\quad - l^{1-2k}\lambda_k(l^2)\left(2q^{-\alpha} + c_Q^+(0) + \sum_{\substack{n\geq 1\\ (-1)^{k-1}n\equiv 0,1\ (4)}} c_Q^+(n)q^n\right)\\
&= 2l^{1-2k}q^{-\alpha l^2} + 2\left(\left(\frac{(-1)^k\alpha}{l}\right)l^{-k} - l^{1-2k}\lambda_k(l^2)\right)q^{-\alpha} + c_Q^+(0)(1 + l^{1-2k} - l^{1-2k}\lambda_k(l^2))\\
&\quad + \sum_{\substack{n\geq 1\\ (-1)^{k-1}n\equiv 0,1\ (4)}}\left(c_Q^+(l^2n) + \left(\frac{(-1)^{k-1}n}{l}\right)l^{-k}c_Q^+(n) + l^{1-2k}c_Q^+(n/l^2) - l^{1-2k}\lambda_k(l^2)c_Q^+(n)\right)q^n\\
&= 2l^{1-2k}f_{\frac{3}{2}-k,\alpha l^2} \quad \text{since } -\alpha \geq -3.
\end{aligned}$$

So we find that

$$2f_{\frac{3}{2}-k,\alpha l^2} = l^{2k-1}Q^+|_{\frac{3}{2}-k}T(l^2) - \lambda_k(l^2)Q^+$$

has integral coefficients and for all positive integers n with $(-1)^{k-1}n \equiv 0, 1 \pmod 4$,

$$\begin{aligned}
&l^{2k-1}c_Q^+(l^2n) + \left(\frac{(-1)^{k-1}n}{l}\right)l^{k-1}c_Q^+(n) + c_Q^+(n/l^2) - \lambda_k(l^2)c_Q^+(n)\\
&= c_Q^+(n)\left(\left(\frac{(-1)^{k-1}n}{l}\right)l^{k-1} - \left(\frac{(-1)^k\alpha}{l}\right)l^{k-1} - a_{f_k}(l^2\alpha)\right) + l^{2k-1}c_Q^+(l^2n) + c_Q^+(n/l^2) \in 2\mathbb{Z}.
\end{aligned}$$

Then for $n = \beta_k$ with

$$\beta_k = \begin{cases} 3, & k \text{ even},\\ 1, & k \text{ odd},\end{cases}$$

we obtain that

$$c_Q^+(\beta_k)\left(\left(\left(\frac{(-1)^{k-1}\beta_k}{l}\right) - \left(\frac{(-1)^k\alpha}{l}\right)\right)l^{k-1} - a_{f_k}(l^2\alpha)\right) + l^{2k-1}c_Q^+(l^2\beta_k) \in 2\mathbb{Z}.$$

As a consequence of the above identity we get the assertions.

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