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Bounds of Strong EMT Strength for certain Subdivision of Star and Bistar

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Abstract: A *super edge-magic total (SEMT) labeling* of a graph $\wp(V, E)$ is a one-one map Υ from $V(\wp) \cup E(\wp)$ onto $\{1, 2, \ldots, |V(\wp) \cup E(\wp)|\}$ such that $\exists$ a constant *"a"* satisfying $\Upsilon(\upsilon) + \Upsilon(\upsilon\nu) + \Upsilon(\nu) = a$, for each edge $\upsilon\nu \in E(\wp)$, moreover all vertices must receive the smallest labels. The *super edge-magic total (SEMT) strength*, $sm(\wp)$, of a graph $\wp$ is the minimum of all magic constants $a(\Upsilon)$, where the minimum runs over all the SEMT labelings of $\wp$. This minimum is defined only if the graph has at least one such SEMT labeling. Furthermore, the *super edge-magic total (SEMT) deficiency* for a graph $\wp$, signified as $\mu_s(\wp)$, is the least non-negative integer n so that $\wp \cup nK_1$ has a SEMT labeling or $+\infty$ if such n does not exist. In this paper, we will formulate the results on SEMT labeling and deficiency of fork, H-tree and disjoint union of fork with star, bistar and path. Moreover, we will evaluate the SEMT strength for trees.

Keywords: SEMT labeling, SEMT deficiency, SEMT strength, Fork, H-tree

MSC: 05C78

1 Preliminaries

All graphs examined here are finite, simple, planar and undirected. The graph $\wp$ has vertex-set $V(\wp)$ and edge-set $E(\wp)$. Let $p = |V(\wp)|$ and $q = |E(\wp)|$. A bijection $\Upsilon : V(\wp) \cup E(\wp) \to \{1, 2, \ldots, p+q\}$ is called an *EMT labeling* of a graph $\wp$ if $\Upsilon(\upsilon) + \Upsilon(\upsilon\nu) + \Upsilon(\nu) = a$, where *"a"* is the constant called the *magic constant* of $\wp$. The graph that satisfies such a labeling is said to be an *EMT graph*. An EMT labeling Υ is called a *SEMT labeling* if $\Upsilon(V(\wp)) = \{1, 2, \ldots, p\}$. A graph that admits this type of labeling is called a *SEMT graph*. Kotzig and Rosa [1] and Enomoto *et al.* [2] were the first to introduce the concepts of EMT and SEMT graphs- Wallis [3] called this labeling a strong EMT labeling- respectively and conjectured that *every tree is EMT* [1], and *every tree is SEMT* [2]. These conjectures have become very prominent in the area of graph labeling. Many classes of trees have been verified to admit (super) EMT labelings, such as trees with upto 17 vertices by a computer search

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[4], stars [5, 6], paths, caterpillars [1] and subdivided stars [7–16] etc. However, in general, these conjectures are still open.

The *(super) EMT strength* of a graph $\wp$, denoted by $(sm(\wp))\ m(\wp)$, is defined as the minimum of all magic constants $a(\Upsilon)$, where the minimum is taken over all the (super) EMT labelings of $\wp$. This minimum is defined only if the graph has at least one such (super) EMT labeling. One can easily perceive that, since the labels of graph $\wp(V, E)$ are from the set $\{1, 2, \ldots, p+q\}$,

$$p + q + 3 \leq sm(\wp) \leq 3p.$$

Avadayappan *et al.* first introduced the notions of EMT strength [17] and SEMT strength [18] and found EMT strength for path, cycle etc., and also the exact values of SEMT strength for some graphs. In [19–21], the SEMT strengths of fire crackers, banana trees, unicyclic graphs, paths, stars, bistars, *y*-trees and the generalized Petersen graphs have been observed.

Kotzig and Rosa [1] verified that for any graph $\wp$, $\exists$ an EMT graph χ s.t. $\chi \cong \wp \cup nK_1$ for some non-negative integer n. This fact leads to the concept of *EMT deficiency* of a graph $\wp$, $\mu(\wp)$, which is the minimum non-negative integer n s.t. $\wp \cup nK_1$ is EMT. In particular,

$$\mu(\wp) = min\{n \geq 0 : \wp \cup nK_1 \ is\ EMT.\}$$

In the same paper [1], Kotzig and Rosa gave the upper bound for the EMT deficiency of a graph $\wp$ with n vertices i.e.,

$$\mu(\wp) \leq F_{n+2} - 2 - n - \frac{n(n-1)}{2}$$

where F_n is the n^{th} Fibonacci number. Figueroa-Centeno *et al.* [22] defined a similar concept for SEMT labeling i.e., the *SEMT deficiency* of a graph $\wp$, denoted by $\mu_s(\wp)$, is the minimum non-negative integer n s.t. $\wp \cup nK_1$ has a SEMT labeling, or $+\infty$ if there is no such n, more precisely,
If $M(\wp) = \{n \geq 0 : \wp \cup nK_1 \ is\ a\ SEMT\ graph\}$, then

$$\mu_s(\wp) = \begin{cases} min\, M(\wp) & if\ M(\wp) \neq \phi \\ +\infty & if\ M(\wp) = \phi \end{cases}$$

It can be seen easily that for every graph $\wp$, $\mu(\wp) \leq \mu_s(\wp)$. In [22, 23], Figueroa-Centeno *et al.* provided the exact values of SEMT deficiencies of several classes of graphs. They also proved that *all forests have finite deficiencies*. Ngurah *et al.* [24], Baig *et al.* [25] and Javed *et al.* [26] gave some upper bounds for the SEMT deficiency of various forests. In [27], Figueroa-Centeno *et al.* conjectured that *every forest with two components has SEMT deficiency at most 1*. The examination of deficiencies in this paper will put evidence on this conjecture. However, this conjecture is still open too.

In this paper, we established the results on SEMT labelings and deficiencies of fork, H-tree and disjoint union of fork with star, bistar and path. Also the SEMT strengths of fork and H-tree are discussed. A useful survey to know about the numerous graph labeling methods is the one by J. A Gallian [28] and for all graph-theoretic terminologies and notions we refer the reader to [29, 30].

2 The results

A *star* on n vertices is isomorphic to complete graph $K_{1,n-1}$. A *bistar* $BS(\upsilon, \nu)$ on n vertices is obtained from two stars $K_{1,\upsilon}$ and $K_{1,\nu}$ by joining their central vertices through an edge, where $\upsilon, \nu \geq 1$, $\upsilon + \nu = n - 2$. A *path* denoted by P_n is a graph consisting of n vertices and $n-1$ edges. The *subdivided star* $T(n_1, n_2, \ldots, n_\rho)$ is a tree obtained by inserting $n_\imath - 1$ vertices to each of the $\imath^{th}$ edge of the star $K_{1,\rho}$, where $1 \leq \imath \leq \rho$, $n_\imath \geq 1$ and $\rho \geq 3$. The vertex-set and edge-set are defined as

$$V(T(n_1, n_2, \ldots, n_\rho)) = \{k\} \cup \{x_\imath^{\ell_\imath} : 1 \leq \imath \leq \rho; 1 \leq \ell_\imath \leq n_\imath\}$$

and

$$E(T(n_1, n_2, \ldots, n_\rho)) = \{kx_\imath^1 : 1 \le \imath \le \rho\} \cup \{x_\imath^{\ell_\imath} x_\imath^{\ell_\imath+1} : 1 \le \imath \le \rho; 1 \le \ell_\imath \le n_\imath - 1\}$$

respectively. Moreover, $\forall n_\imath = 1$, $\underbrace{T(1, 1, \ldots, 1)}_{\rho-times} \cong K_{1,\rho}$.

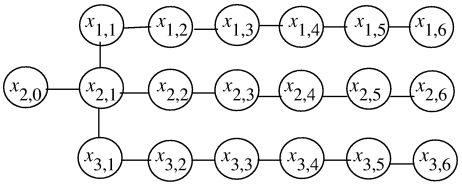
Definition 2.1. *A fork, denoted by Fr_ℓ, $\ell \in N\backslash\{1\}$, is a tree deduced from 3 equally sized paths of length ℓ that is $P_\ell : x_{1,\jmath}, x_{2,\jmath}, x_{3,\jmath}$, $1 \le \jmath \le \ell$, a single new vertex $x_{2,0}$ is added to the path $x_{2,\jmath}$; $1 \le \jmath \le \ell$ through an edge, these three paths are joined together by two edges that are $x_{\imath,1}x_{\imath+1,1}$, $1 \le \imath \le 2$. Precisely, the set of vertices and the set of edges of fork are as respectively:*

$$V(Fr_\ell) = \{x_{\imath,\jmath} : 1 \le \imath \le 3, 1 \le \jmath \le \ell\} \cup \{x_{2,0}\}$$

$$E(Fr_\ell) = \{x_{\imath,\jmath}x_{\imath,\jmath+1} : 1 \le \imath \le 3, 1 \le \jmath \le \ell - 1\} \cup \{x_{\imath,1}x_{\imath+1,1} : 1 \le \imath \le 2\} \cup \{x_{2,0}x_{2,1}\},$$

illustrated in Figure 1.

Fig. 1. Fork Fr_6

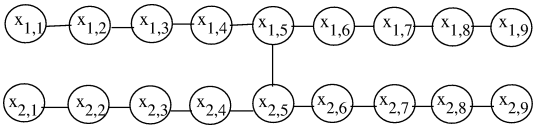


Definition 2.2. *H-tree is represented as H_ℓ, $\ell \in N$ consisting of four equally sized paths joined together by two new vertices forming alphabet H shape, illustrated in Figure 2. The vertex and edge sets of H-tree are as respectively:*

$$V(\wp) = \{x_{\imath,\jmath} : \imath = 1, 2, 1 \le \jmath \le 2\ell + 1\}$$

$$E(\wp) = \{x_{\imath,\jmath}x_{\imath,\jmath+1} : 1 \le \imath \le 2, 1 \le \jmath \le 2\ell\} \cup \{x_{1,\ell+1}x_{2,\ell+1}\}.$$

Fig. 2. H-tree H_4



Note 1. Fork Fr_ℓ can also be written as $T(1, \ell, \ell - 1, \ell)$ where $\ell \in N\backslash\{1\}$, as we can see that it is basically a subdivision of star $K_{1,4}$. Javed, Hussain, Ali and Shaker [8] have discussed the SEMT labelings on subdivisions of star $K_{1,4}$ but the advantage of SEMT labeling scheme presented in this paper over the previous ones mentioned in [8] is that it holds for all positive integers $\ell > 1$, not only for odd positive integers. H-tree can be taken as a subdivision of bistar $BS(2, 2)$ and this subdivision is carried out for all positive integers but the point to remember is that all the four legs of H should be equal in order.

The following lemma gives us a necessary and sufficient condition for a graph to be SEMT and in proving the main results, we will frequently use this. Conditions given in this Lemma are easier to work with than the original definition.

Lemma 2.3 ([6]). *A (p, q)-graph $\wp$ is SEMT if and only if there exists a bijective function $\Upsilon : V(\wp) \to \{1, 2, \ldots, p\}$ such that the set*

$$S = \{\Upsilon(\upsilon) + \Upsilon(\nu) : \upsilon\nu \in E(\wp)\}$$

consists of q consecutive integers. In such a case, $\wp$ extends to a SEMT labeling of $\wp$ with the magic constant $a = p + q + min(S)$, where

$$S = \{a - (p + q), a - (p + q) + 1, \ldots, a - (p + 1)\}.$$

Avadayappan *et al.* made a following remark about SEMT graphs i.e.,

Note 2. ([18]). Let Υ be a SEMT labeling of $\wp$ with the magic constant $a(\Upsilon)$. Then, adding all the magic constants obtained at each edge, we get

$$q\ a(\Upsilon) = \sum_{\nu \in V(\wp)} deg_{\wp}(\nu)\Upsilon(\nu) + \sum_{e \in E(\wp)} \Upsilon(e), \qquad q = |E(\wp)| \tag{1}$$

This condition holds also for EMT labelings. The term $deg_{\wp}(\nu)$ in above expression is the degree of vertex $\nu \in V(\wp)$, which can be defined as the number of vertices that are adjacent to ν, form a set denoted by $N_{\wp}(\nu)$, and $deg_{\wp}(\nu) = |N_{\wp}(\nu)|$ is the *degree* of ν in $\wp$.

There may exist a variety of SEMT labeling schemes for a single graph- if any graph admits a SEMT labeling then another distinct SEMT labeling will surely exist for the same graph because of the *dual super labeling* detailed in [31]- and of course there will be as many different magic constants as the distint labeling schemes. Many researchers have found the lower and upper bounds of magic constants for various graphs. In [7], Ngurah *et al.* obtained lower and upper bounds of the SEMT magic constants for subdivision of star $K_{1,3}$ i.e.,

Lemma 2.4 ([7]). *If $T(m, n, k)$ is a SEMT graph, then magic constant "a" is in the following interval: $\frac{1}{2t}(5t^2 + 3t + 6) \le a \le \frac{1}{2t}(5t^2 + 11t - 6)$, where $t = m + n + k$.*

Javaid [32] gave upper and lower bounds of SEMT magic constants for subdivided stars $T(n_1, n_2, \ldots, n_r)$ with any $n_\imath \ge 1$, $1 \le \imath \le r$, in the form of following lemma:

Lemma 2.5 ([32]). *If $T(n_1, n_2, \ldots, n_r)$ is a super $(a, 0)$-EAT graph, then $\frac{1}{2l}(5l^2 + r^2 - 2lr + 9l - r) \le a \le \frac{1}{2l}(5l^2 - r^2 + 2lr + 5l + r)$, where $l = \sum_{\imath=1}^{r} n_\imath$.*

Now we find the upper and lower bounds of magic constants for H-tree. Clearly, H-tree $H_\ell; \ell \ge 1$ has $4\ell + 2$ vertices and $4\ell + 1$ edges. Among these vertices, two vertices have degree 3, four vertices have degree 1, and the remaining vertices have degree 2, see fig 2. Suppose H_ℓ has an EMT labeling with magic constant *"a"*, then qa where $q = 4\ell + 1$, can not be smaller than the sum obtained by assigning the smallest two labels to the vertices of degree 3, the $q - 5$ next smallest labels to the vertices of degree 2, and four next smallest labels to the vertices of degree 1; in other words:

$$\begin{aligned} q\,a &\ge 3\sum_{\imath=1}^{2} \imath + 2\sum_{\imath=3}^{q-3} \imath + \sum_{\imath=q-2}^{q+1} \imath + \sum_{\imath=q+2}^{2q+1} \imath \\ &= \frac{18 + 2q(q-5) + 4(2q-1) + 3q(q+1)}{2} \\ &= \frac{5q^2 + q + 14}{2} \end{aligned}$$

An upper bound for qa can be achieved by giving the largest labels to the vertices of degree 3, and the $q - 5$ next largest labels to the vertices of degree 2, and four next largest labels to the vertices of degree 1, in other

words:

$$\begin{aligned} q\,a &\le 3\sum_{\imath=2q}^{2q+1}\imath + 2\sum_{\imath=q+5}^{2q-1}\imath + \sum_{\imath=q+1}^{q+4}\imath + \sum_{\imath=1}^{q}\imath \\ &= \frac{6(4q+1)+2(3q+4)(q-5)+4(2q+5)+q(q+1)}{2} \\ &= \frac{7q^2+11q-14}{2} \end{aligned}$$

Thus, we have the following result,

Lemma 2.6. *If H_ℓ is an EMT graph, then magic constant "a" is in the following interval:*

$$\frac{1}{2q}(5q^2+q+14) \le a \le \frac{1}{2q}(7q^2+11q-14)$$

By a similar argument, it is easy to verify that the following lemma holds.

Lemma 2.7. *If H_ℓ is a SEMT graph, then magic constant "a" is in the following interval:*

$$\frac{1}{2q}(5q^2+q+14) \le a \le \frac{1}{2q}(5q^2+13q-14)$$

In the next results of this section, we will construct the SEMT labeling and strength for Fork and H-tree.

Theorem 2.8. *For $\ell \ge 2$, the graph $\wp \cong Fr_\ell$ is SEMT with magic constant $a = 6\ell + \lceil \frac{3\ell}{2} \rceil + 4$.*

Proof. Let $\wp \cong Fr_\ell$, $\ell \ge 2$, where

$$V(\wp) = \{x_{\imath,\jmath} : 1 \le \imath \le 3, 1 \le \jmath \le \ell\} \cup \{x_{2,0}\}$$

$$E(\wp) = \{x_{\imath,\jmath}x_{\imath,\jmath+1} : 1 \le \imath \le 3, 1 \le \jmath \le \ell-1\} \cup \{x_{\imath,1}x_{\imath+1,1} : 1 \le \imath \le 2\} \cup \{x_{2,0}x_{2,1}\}$$

Let $p = |V(\wp)|$ and $q = |(E(\wp)|$, then $p = 3\ell+1$ and $q = 3\ell$

Consider the vertex labeling $\Upsilon : V(\wp) \to \{1, 2, \ldots, p\}$ as follows:

$$\Upsilon(x_{2,0}) = \ell + 1$$

$$\Upsilon(x_{\imath,\jmath}) = \begin{cases} 1 + (\ell+1)(\frac{\imath-1}{2}) + (\frac{\jmath-1}{2}) & ;\imath \equiv 1(mod2), \imath = 1,3 \\ & \jmath \equiv 1(mod2), \jmath \ge 1 \\ \ell - \frac{\jmath-2}{2} & ;\imath \equiv 0(mod2), \imath = 2 \\ & \jmath \equiv 0(mod2), \jmath \ge 2 \\ \lceil \frac{3\ell}{2} \rceil + \ell(\frac{\imath-1}{2}) + \frac{\jmath-2}{2} + 2 & ;\imath \equiv 1(mod2), \imath = 1,3 \\ & \jmath \equiv 0(mod2), \jmath \ge 2 \\ \lceil \frac{3\ell}{2} \rceil + \ell - \frac{\jmath-1}{2} + 1 & ;\imath \equiv 0(mod2), \imath = 2 \\ & \jmath \equiv 1(mod2), \jmath \ge 1 \end{cases}$$

The edge-sums generated by the above labeling "Υ" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar+q\}$, where $\hbar = \lceil \frac{3\ell}{2} \rceil + 2$. Thus by Lemma 2.3, "$\Upsilon$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = min(S)$. □

From this theorem, we obtain the magic constant $a(\Upsilon) = 6\ell + \lceil \frac{3\ell}{2} \rceil + 4; \ell \ge 2$ for Fork tree and by given lower bound of magic constants in Lemma 2.5, we have $a(\Upsilon) \ge \frac{5q^2+q+12}{2q}$, where $q = 3\ell$, thus we can conclude:

Theorem 2.9. *The SEMT strength for Fork $Fr_\ell; \ell \ge 2$ (subdivision of star $K_{1,4}$) is in the following interval:*

$$\frac{15\ell^2+\ell+4}{2\ell} \le sm(Fr_\ell) \le 6\ell + \lceil \frac{3\ell}{2} \rceil + 4, \quad \ell \ge 2.$$

Theorem 2.10. *For $\ell \geq 1$, the graph $\wp \cong H_\ell$ is SEMT with magic constant $a = 2(5\ell + 3)$.*

Proof. Let $\wp \cong H_\ell$, $\ell \geq 1$, where

$$V(\wp) = \{x_{\imath,\jmath} : \imath = 1, 2, 1 \leq \jmath \leq 2\ell + 1\}$$

$$E(\wp) = \{x_{\imath,\jmath}x_{\imath,\jmath+1} : 1 \leq \imath \leq 2, 1 \leq \jmath \leq 2\ell\} \cup \{x_{1,\ell+1}x_{2,\ell+1}\}$$

Let $v = |V(\wp)|$ and $e = |E(\wp)|$, then $p = 4\ell + 2$ and $q = 4\ell + 1$.

Consider the vertex labeling $\Upsilon : V(\wp) \to \{1, 2, \ldots, p\}$ as follows:

$$\Upsilon(x_{\imath,\jmath}) = \begin{cases} 1 + \frac{\jmath-1}{2} & ; \imath \equiv 1(mod2), \imath = 1 \\ & \jmath \equiv 1(mod2), \jmath \geq 1 \\ 2\ell - \frac{\jmath-2}{2} + 1 & ; \imath \equiv 0(mod2), \imath = 2 \\ & \jmath \equiv 0(mod2), \jmath \geq 2 \\ 2(\ell+1) + \frac{\jmath-2}{2} & ; \imath \equiv 1(mod2), \imath = 1 \\ & \jmath \equiv 0(mod2), \jmath \geq 2 \\ 4\ell - \frac{\jmath-1}{2} + 2 & ; \imath \equiv 0(mod2), \imath = 2 \\ & \jmath \equiv 1(mod2), \jmath \geq 1 \end{cases}$$

The edge-sums generated by the above labeling "Υ" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q\}$, where $\hbar = 2(\ell + 1)$. Thus by Lemma 2.3, "Υ" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = min(S)$. □

This theorem gives us the magic constant $a(\Upsilon) = 2(5\ell + 3), \ell \geq 1$ for H-tree and by given lower bound of magic constants in Lemma 2.7, we have $\frac{5q^2+q+14}{2q}$, where $q = 4\ell + 1$. Thus we can conclude:

Theorem 2.11. *The SEMT strength for H-tree H_ℓ, $\ell \geq 1$ (subdivision of bistar $BS(2, 2)$) is in the following interval:*

$$\frac{40\ell^2 + 22\ell + 10}{4\ell + 1} \leq sm(H_\ell) \leq 10\ell + 6, \quad \ell \geq 1.$$

In the next section, we will study the SEMT labelings and deficiencies of forests consisting of fork, star, bistar and path.

2.1 Semt labeling and deficiency of forests formed by fork, star, bistar and path

Theorem 2.12. *For $\ell \geq 2$,*
(a): $Fr_\ell \cup K_{1,\varpi}$ is SEMT.
(b): $\mu_s(Fr_\ell \cup K_{1,\varpi-1}) \leq 1$.
where $\varpi = \ell - 1$.

Proof. **(a):** Consider the graph $\wp \cong Fr_\ell \cup K_{1,\varpi}$.

Let $p = |V(\wp)|$ and $q = |(E(\wp)|$, then

$$p = 3\ell + \varpi + 2$$

$$q = 3\ell + \varpi$$

We define a labeling $\Upsilon : V(Fr_\ell) \to \{1, 2, \ldots, 3\ell + 1\}$, as

$$\Upsilon(x_{\imath,\jmath}) = \begin{cases} \lfloor \frac{\ell}{2} \rfloor + \ell(\frac{\imath-1}{2}) - \frac{\jmath-2}{2} & ; \imath \equiv 1(mod2), \imath = 1, 3 \\ & \jmath \equiv 0(mod2), \jmath \geq 2 \\ \lfloor \frac{\ell}{2} \rfloor + \frac{\jmath-1}{2} + 1 & ; \imath \equiv 0(mod2), \imath = 2 \\ & \jmath \equiv 1(mod2), \jmath \geq 1 \end{cases}$$

Now consider the labeling $\Psi : V(\wp) \to \{1, 2, \ldots, p\}$.

For $1 \leq k \leq \varpi + 1$,

$$\Psi(y_k) = \begin{cases} \lfloor \frac{3\ell}{2} \rfloor + 1 \; ; k = 1 \\ \\ 3\ell + k \quad ; k \neq 1 \end{cases}$$

Let $A = \lfloor \frac{3\ell}{2} \rfloor + 1$

$$\Upsilon(x_{\imath,\jmath}) = \begin{cases} A + \lceil \frac{\ell}{2} \rceil + \ell(\frac{\imath-1}{2}) - \frac{\jmath-1}{2} \; ; \imath \equiv 1(mod2), \imath = 1,3 \\ \qquad\qquad\qquad\qquad\qquad\quad \jmath \equiv 1(mod2), \jmath \geq 1 \\ A + \lceil \frac{\ell}{2} \rceil + \frac{\jmath-2}{2} + 1 \qquad ; \imath \equiv 0(mod2), \imath = 2 \\ \qquad\qquad\qquad\qquad\qquad\quad \jmath \equiv 0(mod2), \jmath \geq 2 \end{cases}$$

$$\Upsilon(x_{2,0}) = \Psi(x_{2,0}) = 3\ell + \varpi + 2$$

$$\Psi(x_{\imath,\jmath}) = \Upsilon(x_{\imath,\jmath}), \;\; 1 \leq \imath \leq 3, 1 \leq \jmath \leq \ell$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + 2$. Thus by Lemma 2.3, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = min(S)$.

(b): Let $\mho \cong Fr_\ell \cup K_{1,\varpi-1} \cup K_1$. Here

$$V(\mho) = V(Fr_\ell) \cup V(K_{1,\varpi-1}) \cup \{z\}$$

$V(K_{1,\varpi-1}) = \{y_k : \quad 1 \leq k \leq \varpi\}$ and $E(K_{1,\varpi-1}) = \{y_1 y_k : \quad 2 \leq k \leq \varpi\}$.

Let $p' = |(V(\mho)|$ and $q' = |E(\mho)|$, then

$$p' = 3\ell + \varpi + 2$$

and

$$q' = 3\ell + \varpi - 1$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \ldots, p'\}$, keep in view the labeling Υ defined in (a). We define the labeling Ψ' as follows:

$$\Upsilon(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \;\; 1 \leq \imath \leq 3, 1 \leq \jmath \leq \ell$$

with $A = \Psi(y_1) = \Psi'(y_1)$

$$\Psi'(y_k) = \Psi(y_k); \quad 1 \leq k \leq \varpi$$

$$\Psi'(z) = 3\ell + \varpi + 1$$

$$\Psi'(x_{2,0}) = \Psi(x_{2,0}) = \Upsilon(x_{2,0}) = 3\ell + \varpi + 2$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q'\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + 2$. Thus by Lemma 2.3, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + \hbar + 1$, where $\hbar + 1 = min(S)$. □

Theorem 2.13. *For $\ell \geq 2$,*
(a): $Fr_\ell \cup BS(\zeta, \xi)$ is SEMT.
(b): $\mu_s(Fr_\ell \cup BS(\zeta, \xi - 1)) \leq 1$.
where $\xi = \ell - 2, \zeta \geq 0$.

Proof. **(a):** Consider the graph $\wp \cong Fr_\ell \cup BS(\zeta, \xi)$.

Let $p = |V(\wp)|$ and $q = |E(\wp)|$, then

$$p = 3\ell + \zeta + \xi + 3$$

$$q = 3\ell + \zeta + \xi + 1$$

Before formulating the labeling $\Psi : V(\wp) \to \{1, 2, \ldots, p\}$, keep in view the labeling Υ defined in theorem 2.12. We define the labeling Ψ as follows:

$$\Upsilon(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}); \;\; 1 \leq \imath \leq 3, 1 \leq \jmath \leq \ell$$

with $A = \lfloor \frac{3\ell}{2} \rfloor + \zeta + 1$

$$\Psi(z_{\flat t}) = \begin{cases} \lfloor \frac{3\ell}{2} \rfloor + t & ; \flat = 1, 1 \le t \le \zeta \\ \lfloor \frac{3\ell}{2} \rfloor + \zeta + 1 & ; \flat = 2, t = 0 \\ 3\ell + \zeta + 2 & ; \flat = 1, t = 0 \\ 3\ell + \zeta + t + 2 & ; \flat = 2, 1 \le t \le \xi \end{cases}$$

$$\Upsilon(x_{2,0}) = \Psi(x_{2,0}) = 3\ell + \zeta + \xi + 3$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + \zeta + 2$. Thus by Lemma 2.3, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = min(S)$.

(b): Let $\mho \cong Fr_\ell \cup BS(\zeta, \xi - 1) \cup K_1$. Here

$$V(\mho) = V(Fr_\ell) \cup V(BS(\zeta, \xi - 1)) \cup \{z\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, then

$$p' = 3\ell + \zeta + \xi + 3$$

and

$$q' = 3\ell + \zeta + \xi$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \ldots, p'\}$, keep in view the labeling Υ defined in theorem 2.12. We define the labeling Ψ' as follows:

$$\Upsilon(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \ \ 1 \le \imath \le 3, 1 \le \jmath \le \ell$$

with $A = \Psi(z_{20}) = \Psi'(z_{20}) = \lfloor \frac{3\ell}{2} \rfloor + \zeta + 1$

$$\Psi'(z_{1t}) = \Psi(z_{1t}); \quad 0 \le t \le \zeta$$

$$\Psi'(z_{2t}) = \Psi(z_{2t}); \ \ 0 \le t \le \xi - 1$$

$$\Psi'(z) = 3\ell + \zeta + \xi + 2$$

$$\Psi'(x_{2,0}) = \Psi(x_{2,0}) = \Upsilon(x_{2,0}) = 3\ell + \zeta + \xi + 3$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q'\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + \zeta + 2$. Thus by Lemma 2.3, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + \hbar + 1$, where $\hbar + 1 = min(S)$. □

In the next two theorems, we will present two distinct SEMT labelings- which are non-dual of each other- for disjoint union of path P_m and fork.

Theorem 2.14. *For $\ell \ge 2$*
(a)(i): $Fr_\ell \cup P_r$ is SEMT.
(a)(ii): $Fr_\ell \cup P_{r-1}$ is SEMT.
(b)(i): $\mu_s(Fr_\ell \cup P_{r-2}) \le 1$.
(b)(ii): $\mu_s(Fr_\ell \cup P_{r-3}) \le 1$.
where $r = 2\ell - 1$.

Proof. **(a):** Consider the graph $\wp \cong Fr_\ell \cup P_\varrho$, where

$$V(P_\varrho) = \{x_t : \quad 1 \le t \le \varrho\}$$

and

$$E(P_\varrho) = \{x_t x_{t+1} : \quad 1 \le t \le \varrho - 1\}$$

Let $p = |V(\wp)|$ and $q = |E(\wp)|$, so we get

$$p = 3\ell + \varrho + 1$$

$$q = 3\ell + \varrho - 1$$

where

$$\varrho = \begin{cases} r & ;for\ \ a(i) \\ r-1 & ;for\ \ a(ii) \end{cases}$$

Before formulating the labeling $\Psi : V(\wp) \to \{1, 2, \ldots, p\}$, keep in view the labeling Υ defined in Theorem 2.12.

$$\Psi(x_{\imath,\jmath}) = \Upsilon(x_{\imath,\jmath}) \quad ; 1 \le \imath \le 3, 1 \le \jmath \le \ell$$

with $A = \lfloor \frac{3\ell}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor$. We define the labeling Ψ as follows:

$$\Psi(x_t) = \begin{cases} \lfloor \frac{3\ell}{2} \rfloor + k & ; t = 2k-1, 1 \le k \le \lfloor \frac{\varrho+1}{2} \rfloor \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor & ; t = 2k, 1 \le k \le \lfloor \frac{\varrho}{2} \rfloor, for\ \ a(i) \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 1 & ; t = 2k, 1 \le k \le \frac{\varrho}{2}, for\ \ a(ii) \end{cases}$$

$$\Psi(x_{2,0}) = \Upsilon(x_{2,0}) = \begin{cases} 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \lfloor \frac{\varrho}{2} \rfloor - \lfloor \frac{\ell}{2} \rfloor + 1 & ;for\ \ a(i) \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \frac{\varrho}{2} - \lfloor \frac{\ell}{2} \rfloor & ;for\ \ a(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor + 1$. Thus by Lemma 2.3, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + min(S)$, where $min(S) = \hbar + 1$.

(b): Let $\mho \cong Fr_\ell \cup P_\varrho \cup K_1$, where

$$V(P_\varrho) = \{x_t : \quad 1 \le t \le \varrho\},$$

$$V(K_1) = \{z\}$$

and

$$E(P_\varrho) = \{x_t x_{t+1} : \quad 1 \le t \le \varrho - 1\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, so we get

$$p' = 3\ell + \varrho + 2$$

$$q' = 3\ell + \varrho - 1$$

where

$$\varrho = \begin{cases} r-2 & ;for\ \ b(i) \\ r-3 & ;for\ \ b(ii) \end{cases}$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \ldots, p'\}$, keep in view the labeling Υ defined in theorem 2.12.

$\Upsilon(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le 3, 1 \le \jmath \le \ell, for\ \ b(i)\ \ and\ \ b(ii) both$ with $A = \lfloor \frac{3\ell}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor$

$$\Psi'(x_t) = \Psi(x_t), \quad t \equiv 1 (mod 2)$$

$$\Psi'(x_t) = \begin{cases} 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 1 & ; t = 2k, 1 \le k \le \frac{\varrho-1}{2}, for\ \ b(i) \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 2 & ; t = 2k, 1 \le k \le \lceil \frac{\varrho-1}{2} \rceil, for\ \ b(ii) \end{cases}$$

Let $B = 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \frac{\varrho-1}{2} - \lfloor \frac{\ell}{2} \rfloor - 1$ and $C = 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil - \lfloor \frac{\ell}{2} \rfloor - 2$, then

$$\Psi'(z) = \begin{cases} B+1 \ ; for \ \ b(i) \\ \\ C+1 \ ; for \ \ b(ii) \end{cases}$$

$$\Psi'(x_{2,0}) = \begin{cases} B+2 \ ; for \ \ b(i) \\ \\ C+2 \ ; for \ \ b(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar+q'\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor + 1$. Thus by Lemma 2.3, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + min(S)$, where $min(S) = \hbar + 1$. □

Theorem 2.15. *For* $\ell \geq 2$
(a)(i): $Fr_\ell \cup P_r$ *is SEMT.*
(a)(ii): $Fr_\ell \cup P_{r-1}$ *is SEMT,* $\ell \neq 2$
(b)(i): $\mu_s(Fr_\ell \cup P_{r-2}) \leq 1$,
(b)(ii): $\mu_s(Fr_\ell \cup P_{r-3}) \leq 1$; $\ell \neq 2, 3$
where $r = 2\ell - 2$.

Proof. **(a):** Consider the graph $\wp \cong Fr_\ell \cup P_\varrho$, where

$$V(P_\varrho) = \{x_t : \quad 1 \leq t \leq \varrho\}$$

and

$$E(P_\varrho) = \{x_t x_{t+1} : \quad 1 \leq t \leq \varrho - 1\}$$

Let $p = |V(\wp)|$ and $q = |E(\wp)|$, so we get

$$p = 3\ell + \varrho + 1$$

$$q = 3\ell + \varrho - 1$$

where

$$\varrho = \begin{cases} r \quad ; for \ \ a(i) \\ \\ r-1 \ ; for \ \ a(ii) \end{cases}$$

Before formulating the labeling $\Psi : V(\wp) \to \{1, 2, \ldots, p\}$, keep in view the labeling Υ defined in theorem 2.12.

$$\Psi(x_{\imath,\jmath}) = \Upsilon(x_{\imath,\jmath}) \quad ; 1 \leq \imath \leq 3, 1 \leq \jmath \leq \ell$$

with $A = \lfloor \frac{3\ell}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil$. We define the labeling Ψ as follows:

$$\Psi(x_t) = \begin{cases} \lfloor \frac{3\ell}{2} \rfloor + k & ; t = 2k, 1 \leq k \leq \lceil \frac{\varrho-1}{2} \rceil \\ \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 1 & ; t = 2k-1, 1 \leq k \leq \frac{\varrho}{2}, for \ \ a(i) \\ \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 2 & ; t = 2k-1, 1 \leq k \leq \lceil \frac{\varrho}{2} \rceil, for \ \ a(ii) \end{cases}$$

$$\Psi(x_{2,0}) = \Upsilon(x_{2,0}) = \begin{cases} 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \frac{\varrho}{2} - \lfloor \frac{\ell}{2} \rfloor & ; for \ \ a(i) \\ \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \lceil \frac{\varrho}{2} \rceil - \lfloor \frac{\ell}{2} \rfloor - 1 & ; for \ \ a(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar+q\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil + 1$. Thus by Lemma 2.3, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + min(S)$, where $min(S) = \hbar + 1$.

(b): Let $\mho \cong Fr_\ell \cup P_\varrho \cup K_1$, where

$$V(P_\varrho) = \{x_t : \quad 1 \le t \le \varrho\},$$

$$V(K_1) = \{z\}$$

and

$$E(P_\varrho) = \{x_t x_{t+1} : \quad 1 \le t \le \varrho - 1\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, so we get

$$p' = 3\ell + \varrho + 2$$

$$q' = 3\ell + \varrho - 1$$

where

$$\varrho = \begin{cases} r-2 \ ; for \ \ b(i) \\ \\ r-3 \ ; for \ \ b(ii) \end{cases}$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \ldots, p'\}$, keep in view the labeling Υ defined in theorem 2.12.

$\Upsilon(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le 3, 1 \le \jmath \le \ell, for \ \ b(i) \ \ and \ \ b(ii) both$ with $A = \lfloor \frac{3\ell}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil$

$$\Psi'(x_t) = \Psi(x_t), \quad t \equiv 0(mod 2)$$

$$\Psi'(x_t) = \begin{cases} 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 2 \ ; t = 2k-1, 1 \le k \le \lfloor \frac{\varrho+1}{2} \rfloor, for \ \ b(i) \\ \\ 3\ell + \lfloor \frac{3\ell}{2} \rfloor + k - \lfloor \frac{\ell}{2} \rfloor - 3 \ ; t = 2k-1, 1 \le k \le \frac{\varrho+1}{2}, for \ \ b(ii) \end{cases}$$

Let $B = 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor - \lfloor \frac{\ell}{2} \rfloor - 2$ and $C = 3\ell + \lfloor \frac{3\ell}{2} \rfloor + \frac{\varrho+1}{2} - \lfloor \frac{\ell}{2} \rfloor - 3$, then

$$\Psi'(z) = \begin{cases} B+1 \ ; for \ \ b(i) \\ \\ C+1 \ ; for \ \ b(ii) \end{cases}$$

$$\Psi'(x_{2,0}) = \begin{cases} B+2 \ ; for \ \ b(i) \\ \\ C+2 \ ; for \ \ b(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar+q'\}$, where $\hbar = \lfloor \frac{3\ell}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil + 1$. Thus by Lemma 2.3, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + min(S)$, where $min(S) = \hbar + 1$. □

Concluding remarks

In this paper, we defined new terminologies for a particular class of subdivided stars and subdivided bistars named as fork Fr_ℓ and H-tree H_ℓ respectively. Furthermore, we established the results on SEMT labelings and deficiencies of fork, H-tree and disjoint union of fork with star, bistar and path.

Javaid [32] gave upper and lower bounds of SEMT magic constants for subdivided stars $T(n_1, n_2, \ldots, n_r)$ with any $n_\imath \ge 1$, $1 \le \imath \le r$. This paper extended the key concept for evaluating the bounds for H-tree. Consequently, we ended up on the SEMT strengths of fork and H-tree. We conclude the paper with the subsequent open problems:

Open problem 1. *Make SEMT forests of existing trees with newly defined trees in this manuscript.*

Open problem 2. *Find the SEMT labeling for disjoint union of any number of isomorphic or non-isomorphic copies of Fork tree and H-tree and determine the bounds for their deficiencies.*

Open problem 3. *Find the exact values of the SEMT strength for Fr_ℓ and H_ℓ.*

Open problem 4. *Find the SEMT strength of forests with more than one component, mentioned in this paper.*

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