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An effective algorithm for globally solving quadratic programs using parametric linearization technique

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Abstract: In this paper, we present an effective algorithm for globally solving quadratic programs with quadratic constraints, which has wide application in engineering design, engineering optimization, route optimization, etc. By utilizing new parametric linearization technique, we can derive the parametric linear programming relaxation problem of the quadratic programs with quadratic constraints. To improve the computational speed of the proposed algorithm, some interval reduction operations are used to compress the investigated interval. By subsequently partitioning the initial box and solving a sequence of parametric linear programming relaxation problems the proposed algorithm is convergent to the global optimal solution of the initial problem. Finally, compared with some known algorithms, numerical experimental results demonstrate that the proposed algorithm has higher computational efficiency.

Keywords: Quadratic programs with quadratic constraints, Global optimization, Parametric linearization technique, Reduction operation

MSC: 90C20, 90C26, 65K05

1 Introduction

This paper considers the following quadratic programs with quadratic constraints:

$$(\mathrm{QP}): \begin{cases} \min F_0(y) = \sum_{k=1}^n d_k^0 y_k + \sum_{j=1}^n \sum_{k=1}^n p_{jk}^0 y_j y_k \\ \text{s.t.} \ F_i(y) = \sum_{k=1}^n d_k^i y_k + \sum_{j=1}^n \sum_{k=1}^n p_{jk}^i y_j y_k \le \beta_i, \ i = 1, \dots, m, \\ y \in Y^0 = \{ y \in R^n : l^0 \le y \le u^0 \}, \end{cases}$$

where $l^0 = (l_1^0, \dots, l_n^0)^T$, $u^0 = (u_1^0, \dots, u_n^0)^T$; p_{jk}^i , d_k^i and β_i are all arbitrary real numbers. QP has wide application in route optimization, engineering design, investment portfolio, management decision, production programs, etc. In addition, QP usually owns multiple local optimal solutions which are not global optimal solutions, i.e., in these classes of problems there exist important theoretical difficulties and computational complexities. Thus, it is necessary to present an efficient algorithm for globally solving QP.

In last several decades, many algorithms have been developed for solving QP and its special cases, such as duality-bounds algorithm [1], branch-and-reduce methods [2-6], approximation approach [7], branch-and-

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bound approaches [8-10], and so on. Except for the above ones, some algorithms for polynomial programming [11-15] and quadratic fractional programming [16-17] also can be used to solve OP. Although these algorithms can be used to solve OP and its special cases, less work has been still done for globally solving the investigated quadratic programs with quadratic constraints.

This paper will present a new global optimization branch-and-bound algorithm for solving QP. First of all, we derive a new parametric linearization technique. By utilizing this linearization technique, the initial QP can be converted into a parametric linear programming relaxation problem, which can be used to determine the lower bounds of the global optimal values of the initial QP and its subproblems. Based on the branch-andbound framework, a new global optimization branch-and-bound algorithm is designed for solving QP, the proposed algorithm is convergent to the global optimal solution of the initial QP by successively subdividing the initial box and by solving the converted parametric linear programming relaxation problems. To improve the computational speed of the proposed branch-and-bound algorithm, some interval reduction operations are used to compress the investigated interval. Finally, compared with some known algorithms, numerical experimental results show higher computational efficiency of the proposed branch-and-bound algorithm.

The remaining sections of this paper are listed as follows. Firstly, in order to derive the parametric linear programming relaxation problem of QP, Section 2 presents a new parametric linearization technique. Secondly, based on the branch-and-bound framework in Section 3, by combing the derived parametric linear programming relaxation problem with the interval reduction operations, an effective branch-and-bound algorithm is constructed for globally solving QP. Thirdly, compared with some known methods, some existent test problems are used to verify the computational feasibility of the proposed algorithm in Section 4. Finally, some conclusions are obtained.

2 New parametric linearization approach

In this section, we will present a new parametric linearization approach for constructing the parametric linear programming relaxation problem of QP. The detailed deriving process of the parametric linearization approach is given as follows. Without loss of generality, we assume that $Y = \{(y_1, y_2, \dots, y_n)^T \in R^n : l_i \leq y_i \leq 1\}$ $u_j, j = 1, ..., n$ } $\subseteq Y^0, \gamma = (\gamma_{jk})_{n \times n} \in R^{n \times n}$ is a symmetric matrix, and $\gamma_{jk} \in \{0, 1\}$.

For convenience in expression, for any $y \in Y$, for any $j \in \{1, 2, ..., n\}$, $k \in \{1, 2, ..., n\}$, $j \neq k$, we define

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y_k(\gamma_{kk}) = l_k + \gamma_{kk}(u_k - l_k),
y_k(1-\gamma_{kk}) = l_k + (1-\gamma_{kk})(u_k - l_k),
f_{kk}(y) = y_k^2,
f_{kk}(y, Y, \gamma_{kk}) = [y_k(\gamma_{kk})]^2 + 2y_k(\gamma_{kk})[y_k - y_k(\gamma_{kk})],
\overline{f}_{kk}(y, Y, \gamma_{kk}) = [y_k(\gamma_{kk})]^2 + 2y_k(1 - \gamma_{kk})[y_k - y_k(\gamma_{kk})],
y_i(\gamma_{jk}) = l_j + \gamma_{jk}(u_j - l_j),
y_k(\gamma_{jk}) = l_k + \gamma_{jk}(u_k - l_k),
y_j(1-\gamma_{jk})=l_j+(1-\gamma_{jk})(u_j-l_j),
y_k(1-\gamma_{jk}) = l_k + (1-\gamma_{jk})(u_k - l_k),
 (y_j + y_k)(\gamma_{jk}) = (l_j + l_k) + \gamma_{jk}(u_j + u_k - l_j - l_k),
(y_j + y_k)(1 - \gamma_{jk}) = (l_j + l_k) + (1 - \gamma_{jk})(u_j + u_k - l_j - l_k),
f_{jk}(y) = y_j y_k = \frac{(y_j + y_k)^2 - y_j^2 - y_k^2}{2},
f_{jk}(y, Y, \gamma_{jk}) = \frac{1}{2} \{ \{ [(y_j + y_k)(\gamma_{jk})]^2 + 2(y_j + y_k)(\gamma_{jk})[y_j + y_k - (y_j + y_k)(\gamma_{jk})] \}
                     -\{[y_i(\gamma_{jk})]^2 + 2y_i(1-\gamma_{jk})[y_i-y_j(\gamma_{jk})]\}
                     -\{[y_k(\gamma_{ik})]^2 + 2y_k(1-\gamma_{ik})[y_k-y_k(\gamma_{ik})]\}\},
\overline{f}_{jk}(y, Y, \gamma_{jk}) = \frac{1}{2} \left\{ \left\{ \left[ (y_j + y_k)(\gamma_{jk}) \right]^2 + 2(y_j + y_k)(1 - \gamma_{jk}) \left[ y_j + y_k - (y_j + y_k)(\gamma_{jk}) \right] \right\} \right\}
                     -\{[y_{j}(\gamma_{jk})]^{2}+2y_{j}(\gamma_{jk})[y_{j}-y_{j}(\gamma_{jk})]\}
                     -\{[y_k(\gamma_{ik})]^2+2y_k(\gamma_{ik})[y_k-y_k(\gamma_{ik})]\}\}.
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It is obvious that

$$y_k(0) = l_k$$
, $y_k(1) = u_k$, $(y_i + y_k)(0) = l_i + l_k$, $(y_i + y_k)(1) = u_i + u_k$.

Theorem 2.1. For any $k \in \{1, 2, ..., n\}$, for any $y \in Y$, then we have:

(i) The following inequalities hold:

$$\underline{f}_{kk}(y, Y, \gamma_{kk}) \leq f_{kk}(y) \leq \overline{f}_{kk}(y, Y, \gamma_{kk}), \tag{1}$$

$$[y_{j}(\gamma_{jk})]^{2} + 2y_{j}(\gamma_{jk})[y_{j} - y_{j}(\gamma_{jk})] \leq y_{j}^{2} \leq [y_{j}(\gamma_{jk})]^{2} + 2y_{j}(1 - \gamma_{jk})[y_{j} - y_{j}(\gamma_{jk})], \tag{1}$$

$$y_{k}(\gamma_{jk})]^{2} + 2y_{k}(\gamma_{jk})[y_{k} - y_{k}(\gamma_{jk})] \leq y_{k}^{2} \leq [y_{k}(\gamma_{jk})]^{2} + 2y_{k}(1 - \gamma_{jk})[y_{k} - y_{k}(\gamma_{jk})], \tag{2}$$

$$(y_{j} + y_{k})^{2} \leq [(y_{j} + y_{k})(\gamma_{jk})]^{2} + 2(y_{j} + y_{k})(\gamma_{jk})[y_{j} + y_{k} - (y_{j} + y_{k})(\gamma_{jk})], \tag{2}$$

$$(y_{i} + y_{k})^{2} \geq [(y_{i} + y_{k})(\gamma_{ik})]^{2} + 2(y_{i} + y_{k})(\gamma_{ik})[y_{j} + y_{k} - (y_{j} + y_{k})(\gamma_{jk})], \tag{3}$$

and

$$\underline{f}_{jk}(y, Y, \gamma_{jk}) \le f_{jk}(y) \le \overline{f}_{jk}(y, Y, \gamma_{jk}). \tag{2}$$

(ii) The following limitations hold:

$$\lim_{\|y-l\|\to 0} [f_{kk}(y) - f_{-kk}(y, Y, \gamma_{kk})] = 0,$$
(3)

$$\lim_{\|u-l\|\to 0} [\bar{f}_{kk}(y, Y, \gamma_{kk}) - f_{kk}(y)] = 0, \tag{4}$$

$$\lim_{\|u-l\|\to 0} [y_j^2 - \{[y_j(\gamma_{jk})]^2 + 2y_j(\gamma_{jk})[y_j - y_j(\gamma_{jk})]\}] = 0,$$

$$\lim_{\|y_i-I\|\to 0} [[y_j(\gamma_{jk})]^2 + 2y_j(1-\gamma_{jk})[y_j-y_j(\gamma_{jk})] - y_j^2] = 0,$$

$$\lim_{\|u-l\|\to 0} [y_k^2 - \{[y_k(\gamma_{jk})]^2 + 2y_k(\gamma_{jk})[y_k - y_k(\gamma_{jk})]\}] = 0,$$

$$\lim_{\|u-l\|\to 0} [[y_k(\gamma_{jk})]^2 + 2y_k(1-\gamma_{jk})[y_k-y_k(\gamma_{jk})] - y_k^2] = 0,$$

$$\lim_{\|y_{i-1}\| \to 0} \left[\left[(y_j + y_k)(\gamma_{jk}) \right]^2 + 2(y_j + y_k)(1 - \gamma_{jk}) \left[y_j + y_k - (y_j + y_k)(\gamma_{jk}) \right] - (y_j + y_k)^2 \right] = 0,$$

$$\lim_{\|y_k-l\|\to 0} [(y_j+y_k)^2 - \{[(y_j+y_k)(\gamma_{jk})]^2 + 2(y_j+y_k)(\gamma_{jk})[(y_j+y_k) - (y_j+y_k)(\gamma_{jk})]\}] = 0,$$

$$\lim_{\|u-l\|\to 0} [f_{jk}(y) - f_{-jk}(y, Y, \gamma_{jk})] = 0$$
 (5)

and

$$\lim_{\|u-I\|\to 0} [\bar{f}_{jk}(y, Y, \gamma_{jk}) - f_{jk}(y)] = 0.$$
 (6)

Proof. (i) By the mean value theorem, for any $y \in Y$, there exists a point $\xi_k = \alpha y_k + (1 - \alpha) y_k(\gamma_{kk})$, where $\alpha \in [0, 1]$, such that

$$y_k^2 = [y_k(\gamma_{kk})]^2 + 2\xi_k[y_k - y_k(\gamma_{kk})].$$

If $\gamma_{kk} = 0$, then we have

$$\xi_k \ge l_k = \gamma_k(\gamma_{kk})$$
 and $\gamma_k - \gamma_k(\gamma_{kk}) = \gamma_k - l_k \ge 0$.

If $\gamma_{kk} = 1$, then it follows that

$$\xi_k \le u_k = y_k(\gamma_{kk})$$
 and $y_k - y_k(\gamma_{kk}) = y_k - u_k \le 0$.

Thus, we can get that

$$f_{kk}(y) = y_k^2$$

$$= [y_k(\gamma_{kk})]^2 + 2\xi_k[y_k - y_k(\gamma_{kk})]$$

$$\geq [y_k(\gamma_{kk})]^2 + 2y_k(\gamma_{kk})[y_k - y_k(\gamma_{kk})]$$

$$= f_{kk}(y, Y, \gamma_{kk}).$$

Similarly, if γ_{kk} = 0, then we have

$$\xi_k \le u_k = y_k (1 - \gamma_{kk})$$
 and $y_k - y_k (\gamma_{kk}) = y_k - l_k \ge 0$.

If $\gamma_{kk} = 1$, then it follows that

$$\xi_k \ge l_k = y_k(1 - \gamma_{kk})$$
 and $y_k - y_k(\gamma_{kk}) = y_k - u_k \le 0$.

Thus, we can get that

$$f_{kk}(y) = y_k^2$$

$$= [y_k(\gamma_{kk})]^2 + 2\xi_k[y_k - y_k(\gamma_{kk})]$$

$$\leq [y_k(\gamma_{kk})]^2 + 2y_k(1 - \gamma_{kk})[y_k - y_k(\gamma_{kk})]$$

$$= \overline{f}_{kk}(y, Y, \gamma_{kk}).$$

Therefore, for any $y \in Y$, we have that

$$f_{kk}(y, Y, \gamma_{kk}) \leq f_{kk}(y) \leq \overline{f}_{kk}(y, Y, \gamma_{kk}).$$

From the inequality (1), replacing γ_{kk} by γ_{jk} , and replacing y_k by y_j , we can get that

$$[y_i(\gamma_{ik})]^2 + 2y_i(\gamma_{ik})[y_i - y_i(\gamma_{ik})] \le y_i^2 \le [y_i(\gamma_{ik})]^2 + 2y_i(1 - \gamma_{ik})[y_i - y_i(\gamma_{ik})].$$

From the inequality (1), replacing γ_{kk} by γ_{jk} , we can get that

$$[y_k(\gamma_{jk})]^2 + 2y_k(\gamma_{jk})[y_k - y_k(\gamma_{jk})] \le y_k^2 \le [y_k(\gamma_{jk})]^2 + 2y_k(1 - \gamma_{jk})[y_k - y_k(\gamma_{jk})].$$

From (1), replacing γ_{kk} and y_k by γ_{ik} and $(y_i + y_k)$, respectively, we can get that

$$(y_j + y_k)^2 \le [(y_j + y_k)(\gamma_{jk})]^2 + 2(y_j + y_k)(1 - \gamma_{jk})[(y_j + y_k) - (y_j + y_k)(\gamma_{jk})],$$

$$(y_i + y_k)^2 \ge [(y_i + y_k)(\gamma_{ik})]^2 + 2(y_i + y_k)(\gamma_{ik})[(y_i + y_k) - (y_i + y_k)(\gamma_{ik})].$$

From the former several inequalities, it is easy to follow that

$$f_{jk}(y) = y_{j}y_{k} = \frac{(y_{j}+y_{k})^{2}-y_{j}^{2}-y_{k}^{2}}{2}$$

$$\geq \frac{1}{2}\{\{[(y_{j}+y_{k})(\gamma_{jk})]^{2}+2(y_{j}+y_{k})(\gamma_{jk})[y_{j}+y_{k}-(y_{j}+y_{k})(\gamma_{jk})]\}\}$$

$$-\{[y_{j}(\gamma_{jk})]^{2}+2y_{j}(1-\gamma_{jk})[y_{j}-y_{j}(\gamma_{jk})]\}\}$$

$$-\{[y_{k}(\gamma_{jk})]^{2}+2y_{k}(1-\gamma_{jk})[y_{k}-y_{k}(\gamma_{jk})]\}\},$$

$$= f_{-jk}(y, Y, \gamma_{jk})$$

and

$$f_{jk}(y) = y_{j}y_{k} = \frac{(y_{j}+y_{k})^{2}-y_{j}^{2}-y_{k}^{2}}{2}$$

$$\leq \frac{1}{2}\left\{\left[\left(y_{j}+y_{k}\right)\left(\gamma_{jk}\right)\right]^{2}+2\left(y_{j}+y_{k}\right)\left(1-\gamma_{jk}\right)\left[y_{j}+y_{k}-\left(y_{j}+y_{k}\right)\left(\gamma_{jk}\right)\right]\right\}$$

$$-\left\{\left[y_{j}\left(\gamma_{jk}\right)\right]^{2}+2y_{j}\left(\gamma_{jk}\right)\left[y_{j}-y_{j}\left(\gamma_{jk}\right)\right]\right\}$$

$$-\left\{\left[y_{k}\left(\gamma_{jk}\right)\right]^{2}+2y_{k}\left(\gamma_{jk}\right)\left[y_{k}-y_{k}\left(\gamma_{jk}\right)\right]\right\}\right\},$$

$$=\overline{f}_{jk}(y,Y,\gamma_{jk}).$$

Therefore, we have

$$\underline{f}_{jk}(y,Y,\gamma_{jk}) \leq f_{jk}(y) \leq \overline{f}_{jk}(y,Y,\gamma_{jk}).$$

(ii) Since

$$f_{kk}(y) - \underline{f}_{kk}(y, Y, \gamma_{kk}) = y_k^2 - \{ [y_k(\gamma_{kk})]^2 + 2y_k(\gamma_{kk})[y_k - y_k(\gamma_{kk})] \}$$

$$= (y_k - y_k(\gamma_{kk}))^2$$

$$\leq (u_k - l_k)^2,$$
(7)

we have

$$\lim_{\|u-l\|\to 0} [f_{kk}(y) - \underline{f}_{-kk}(y, Y, \gamma_{kk})] = 0.$$

Also since

$$\bar{f}_{kk}(y, Y, \gamma_{kk}) - f_{kk}(y) = [y_k(\gamma_{kk})]^2 + 2y_k(1 - \gamma_{kk})[y_k - y_k(\gamma_{kk})] - y_k^2
= (y_k(\gamma_{kk}) + y_k)(y_k(\gamma_{kk}) - y_k)
+ 2y_k(1 - \gamma_{kk})(y_k - y_k(\gamma_{kk}))
= [y_k - y_k(\gamma_{kk})][2y_k(1 - \gamma_{kk}) - y_k(\gamma_{kk}) - y_k]
= [y_k - y_k(\gamma_{kk})][y_k(1 - \gamma_{kk}) - y_k(\gamma_{kk})]
+ [y_k - y_k(\gamma_{kk})][y_k(1 - \gamma_{kk}) - y_k]
\le 2(u_k - l_k)^2.$$
(8)

Therefore, it follows that

$$\lim_{\|u-l\|\to 0} \left[\overline{f}_{kk}(y,Y,\gamma_{kk}) - f_{kk}(y)\right] = 0.$$

From the limitations (3) and (4), replacing γ_{kk} and y_k by γ_{ik} and y_i , respectively, we have

$$\lim_{\|y_j-1\|\to 0} [y_j^2 - \{[y_j(\gamma_{jk})]^2 + 2y_j(\gamma_{jk})[y_j - y_j(\gamma_{jk})]\}] = 0$$

and

$$\lim_{\|u-l\|\to 0} [[y_j(\gamma_{jk})]^2 + 2y_j(1-\gamma_{jk})[y_j-y_j(\gamma_{jk})] - y_j^2] = 0.$$

From the limitations (3) and (4), replacing γ_{kk} by γ_{jk} , it follows that

$$\lim_{\|u-l\|\to 0} [y_k^2 - \{[y_k(\gamma_{jk})]^2 + 2y_k(\gamma_{jk})[y_k - y_k(\gamma_{jk})]\}\}] = 0$$

and

$$\lim_{\|u-l\|\to 0} [[y_k(\gamma_{jk})]^2 + 2y_k(1-\gamma_{jk})[y_k-y_k(\gamma_{jk})] - y_k^2] = 0.$$

By the limitations (3) and (4), replacing γ_{kk} and y_k by γ_{jk} and $(y_j + y_k)$, respectively, we can get that

$$\lim_{\|y_{k}-1\|\to 0} \left[\left[(y_{j}+y_{k})(\gamma_{jk}) \right]^{2} + 2(y_{j}+y_{k})(1-\gamma_{jk}) \left[y_{j}+y_{k}-(y_{j}+y_{k})(\gamma_{jk}) \right] - (y_{j}+y_{k})^{2} \right] = 0$$

and

$$\lim_{\|u-l\|\to 0} \left[(y_j + y_k)^2 - \left\{ \left[(y_j + y_k)(\gamma_{jk}) \right]^2 + 2(y_j + y_k)(\gamma_{jk}) \left[y_j + y_k - (y_j + y_k)(\gamma_{jk}) \right] \right\} \right] = 0.$$

From the inequalities (7) and (8), we have

$$\begin{split} f_{jk}(y) - & \underline{f}_{jk}(y,Y,\gamma_{jk}) = y_j y_k - \underline{f}_{-jk}(y,Y,\gamma_{jk}) \\ &= \frac{(y_j + y_k)^2 - y_j^2 - y_k^2}{2} - \frac{1}{2} \big\{ \big\{ \big[\big(y_j + y_k \big) \big(\gamma_{jk} \big) \big]^2 \\ &+ 2 \big(y_j + y_k \big) \big(\gamma_{jk} \big) \big[y_j + y_k - \big(y_j + y_k \big) \big(\gamma_{jk} \big) \big] \big\} \\ &- \big\{ \big[y_j \big(\gamma_{jk} \big) \big]^2 + 2 y_j \big(1 - \gamma_{jk} \big) \big[y_j - y_j \big(\gamma_{jk} \big) \big] \big\} \big\} \\ &- \big\{ \big[y_k \big(\gamma_{jk} \big) \big]^2 + 2 y_k \big(1 - \gamma_{jk} \big) \big[y_k - y_k \big(\gamma_{jk} \big) \big] \big\} \big\}, \\ &= \frac{1}{2} \big[\big\{ \big[y_j \big(\gamma_{jk} \big) \big]^2 + 2 y_j \big(1 - \gamma_{jk} \big) \big[y_j - y_j \big(\gamma_{jk} \big) \big] \big\} - y_j^2 \big] \\ &+ \frac{1}{2} \big[\big\{ \big[y_k \big(\gamma_{jk} \big) \big]^2 + 2 y_k \big(1 - \gamma_{jk} \big) \big[y_k - y_k \big(\gamma_{jk} \big) \big] \big\} - y_k^2 \big] \\ &+ \frac{1}{2} \big\{ \big(y_j + y_k \big)^2 - \big\{ \big[\big(y_j + y_k \big) \big(\gamma_{jk} \big) \big]^2 \\ &+ 2 \big(y_j + y_k \big) \big(\gamma_{jk} \big) \big[y_j + y_k - \big(y_j + y_k \big) \big(\gamma_{jk} \big) \big] \big\} \\ &\leq \big(u_j - l_j \big)^2 + \big(u_k - l_k \big)^2 + \frac{1}{2} \big(u_k + u_j - l_j - l_k \big)^2 \,. \end{split}$$

Thus, we can get that

$$\lim_{\|y-l\|\to 0} [f_{jk}(y) - \underline{f}_{jk}(y, Y, \gamma_{jk})] = 0.$$

Also from the inequalities (7) and (8), we get that

$$\begin{split} \overline{f}_{jk}(y,Y,\gamma_{jk}) - f_{jk}(y) &= \overline{f}_{jk}(y,Y,\gamma_{jk}) - y_{j}y_{k} \\ &= \frac{1}{2} \{ \{ [(y_{j} + y_{k})(\gamma_{jk})]^{2} \\ &+ 2(y_{j} + y_{k})(1 - \gamma_{jk})[y_{j} + y_{k} - (y_{j} + y_{k})(\gamma_{jk})] \} \\ &- \{ [y_{j}(\gamma_{jk})]^{2} + 2y_{j}(\gamma_{jk})[y_{j} - y_{j}(\gamma_{jk})] \} \\ &- \{ [y_{k}(\gamma_{jk})]^{2} + 2y_{k}(\gamma_{jk})[y_{k} - y_{k}(\gamma_{jk})] \} \} \\ &- \frac{(y_{j} + y_{k})^{2} - y_{j}^{2} - y_{k}^{2}}{2} \\ &= \frac{1}{2} \{ \{ [(y_{j} + y_{k})(\gamma_{jk})]^{2} \\ &+ 2(y_{j} + y_{k})(1 - \gamma_{jk})[y_{j} + y_{k} - (y_{j} + y_{k})(\gamma_{jk})] - (y_{j} + y_{k})^{2} \} \\ &+ y_{j}^{2} - \{ [y_{j}(\gamma_{jk})]^{2} + 2y_{j}(\gamma_{jk})[y_{j} - y_{j}(\gamma_{jk})] \} \\ &+ y_{k}^{2} - \{ [y_{k}(\gamma_{jk})]^{2} + 2y_{k}(\gamma_{jk})[y_{k} - y_{k}(\gamma_{jk})] \} \} \\ &\leq \frac{1}{2} (u_{j} - l_{j})^{2} + \frac{1}{2} (u_{k} - l_{k})^{2} + (u_{k} + u_{j} - l_{j} - l_{k})^{2}. \end{split}$$

Thus, it follows that

$$\lim_{\|y-I\|\to 0} \left[\overline{f}_{jk}(y,Y,\gamma_{jk}) - f_{jk}(y)\right] = 0.$$

Without loss of generality, for any $Y = [l, u] \subseteq Y^0$, for any parameter matrix $\gamma = (\gamma_{ik})_{n \times n}$, for any $y \in Y$ and $i \in \{0, 1, ..., m\}$, we let

$$\begin{split} f_{kk}^{i}(y,Y,\gamma_{kk}) &= \begin{cases} p_{kk}^{i} f_{\underline{k}k}(y,Y,\gamma_{kk}), & \text{if } p_{kk}^{i} > 0, \\ p_{kk}^{i} f_{kk}(y,Y,\gamma_{kk}), & \text{if } p_{kk}^{i} < 0, \end{cases} \\ \bar{f}_{kk}^{i}(y,Y,\gamma_{kk}) &= \begin{cases} p_{kk}^{i} \bar{f}_{kk}(y,Y,\gamma_{kk}), & \text{if } p_{kk}^{i} > 0, \\ p_{kk}^{i} f_{kk}(y,Y,\gamma_{kk}), & \text{if } p_{kk}^{i} > 0, \end{cases} \\ f_{jk}^{i}(y,Y,\gamma_{jk}) &= \begin{cases} p_{jk}^{i} f_{jk}(y,Y,\gamma_{jk}), & \text{if } p_{jk}^{i} > 0, j \neq k, \\ p_{jk}^{i} f_{jk}(y,Y,\gamma_{jk}), & \text{if } p_{jk}^{i} > 0, j \neq k, \end{cases} \\ \bar{f}_{jk}^{i}(y,Y,\gamma_{jk}) &= \begin{cases} p_{jk}^{i} \bar{f}_{jk}(y,Y,\gamma_{jk}), & \text{if } p_{jk}^{i} > 0, j \neq k, \\ p_{jk}^{i} f_{jk}(y,Y,\gamma_{jk}), & \text{if } p_{jk}^{i} > 0, j \neq k, \end{cases} \\ f_{i}^{i}(y,Y,\gamma_{jk}) &= \begin{cases} p_{jk}^{i} \bar{f}_{jk}(y,Y,\gamma_{jk}), & \text{if } p_{jk}^{i} < 0, j \neq k, \end{cases} \\ F_{i}^{i}(y,Y,\gamma) &= \sum_{k=1}^{n} (d_{k}^{i} y_{k} + f_{kk}^{i}(y,Y,\gamma_{kk})) + \sum_{j=1}^{n} \sum_{k=1,k\neq j}^{n} f_{jk}^{i}(y,Y,\gamma_{jk}). \end{cases} \\ F_{i}^{U}(y,Y,\gamma) &= \sum_{k=1}^{n} (d_{k}^{i} y_{k} + f_{kk}^{i}(y,Y,\gamma_{kk})) + \sum_{j=1}^{n} \sum_{k=1,k\neq j}^{n} f_{jk}^{i}(y,Y,\gamma_{jk}). \end{cases} \end{split}$$

Theorem 2.2. For any $y \in Y = [l, u] \subseteq Y^0$, for any parameter matrix $\gamma = (\gamma_{jk})_{n \times n}$, for any i = 0, 1, ..., m, we can get the following conclusions:

$$F_i^L(y, Y, \gamma) \le F_i(y) \le F_i^U(y, Y, \gamma),$$

$$\lim_{\|u-I\| \to 0} [F_i(y) - F_i^L(y, Y, \gamma)] = 0$$

and

$$\lim_{\|u-l\|\to 0} \left[F_i^U(y,Y,\gamma) - F_i(y)\right] = 0.$$

Proof. (i) From (1) and (2), for any $j, k \in \{1, ..., n\}$, we can get that

$$\underline{f}_{kk}^{i}(y, Y, \gamma_{kk}) \le p_{kk}^{i} y_{k}^{2} \le \overline{f}_{kk}^{i}(y, Y, \gamma_{kk}), \tag{9}$$

and

$$\underline{f}_{jk}^{i}(y, Y, \gamma_{jk}) \leq p_{jk}^{i} y_{j} y_{k} \leq \overline{f}_{jk}^{i}(y, Y, \gamma_{jk}). \tag{10}$$

By (9) and (10), for any $y \in Y \subseteq Y^0$, we have that

$$\begin{split} F_{i}^{L}(y,Y,\gamma) &= \sum_{k=1}^{n} (d_{k}^{i} y_{k} + \underline{f}_{kk}^{i}(y,Y,\gamma_{kk})) + \sum_{j=1}^{n} \sum_{k=1,k\neq j}^{n} \underline{f}_{jk}^{i}(y,Y,\gamma_{jk}) \\ &\leq \sum_{k=1}^{n} d_{k}^{i} y_{k} + \sum_{k=1}^{n} p_{kk}^{i} y_{k}^{2} + \sum_{j=1}^{n} \sum_{k=1,k\neq j}^{n} p_{jk}^{i} y_{j} y_{k} = F_{i}(y) \\ &\leq \sum_{k=1}^{n} (d_{k}^{i} y_{k} + \overline{f}_{kk}^{i}(y,Y,\gamma_{kk})) + \sum_{j=1}^{n} \sum_{k=1,k\neq j}^{n} \overline{f}_{jk}^{i}(y,Y,\gamma_{jk}) \\ &= F_{i}^{U}(y,Y,\gamma). \end{split}$$

Therefore, we obtain that

$$F_i^L(y, Y, \gamma) \leq F_i(y) \leq F_i^U(y, Y, \gamma).$$

(ii)
$$F_{i}(y) - F_{i}^{L}(y, Y, \gamma) = \sum_{k=1}^{n} d_{k}^{i} y_{k} + \sum_{k=1}^{n} p_{kk}^{i} y_{k}^{2} + \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} p_{jk}^{i} y_{j} y_{k} - \left[\sum_{k=1}^{n} d_{k}^{i} y_{k} + \sum_{j=1}^{n} p_{kk}^{i} y_{k}^{2} + \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} p_{jk}^{i} (y, Y, \gamma_{jk})\right]$$

$$+ \sum_{k=1}^{n} \int_{-kk}^{n} \left[p_{ik}^{i} y_{j}^{2} - f_{-kk}^{i} (y, Y, \gamma_{kk}) \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1, k \neq j}^{n} \left[p_{jk}^{i} y_{j} y_{k} - f_{-jk}^{i} (y, Y, \gamma_{jk}) \right]$$

$$= \sum_{k=1, p_{kk}^{i} > 0}^{n} p_{kk}^{i} \left[f_{kk}(y) - f_{-kk}(y, Y, \gamma_{kk}) \right]$$

$$+ \sum_{k=1, p_{kk}^{i} < 0}^{n} p_{kk}^{i} \left[f_{kk}(y) - f_{-jk}(y, Y, \gamma_{jk}) \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1, k \neq j, p_{jk}^{i} > 0}^{n} p_{jk}^{i} \left[f_{jk}(y) - f_{-jk}(y, Y, \gamma_{jk}) \right]$$

$$+ \sum_{j=1}^{n} \sum_{k=1, k \neq j, p_{jk}^{i} > 0}^{n} p_{jk}^{i} \left[f_{jk}(y) - f_{-jk}(y, Y, \gamma_{jk}) \right].$$

From (3)-(6), we can obtain that $\lim_{\|u-l\|\to 0} [f_{kk}(y) - \underline{f}_{kk}(y, Y, \gamma_{kk})] = 0$, $\lim_{\|u-l\|\to 0} [\overline{f}_{kk}(y, Y, \gamma_{kk}) - f_{kk}(y)] = 0$, $\lim_{\|u-l\|\to 0} [f_{jk}(y) - \underline{f}_{jk}(y, Y, \gamma_{jk})] = 0$ and $\lim_{\|u-l\|\to 0} [\overline{f}_{jk}(y, Y, \gamma_{jk}) - f_{jk}(y)] = 0$.

Therefore, we obtain that

$$\lim_{\|u-l\|\to 0} \left[F_i(y) - F_i^L(y, Y, \gamma)\right] = 0.$$

Similarly to the proof above, we can get that

$$\lim_{\|u-l\|\to 0} \left[F_i^U(y,Y,\gamma) - F_i(y)\right] = 0.$$

The proof is completed.

By Theorem 2.2, we can establish the following parametric linear programming relaxation problem (PLPRP) of QP over *Y*:

$$(\mathbf{PLPRP}): \begin{cases} \min F_0^L(y, Y, \gamma), \\ \text{s.t.} & F_i^L(y, Y, \gamma) \leq \beta_i, \ i = 1, \dots, m, \\ & y \in Y = \{y : l \leq y \leq u\}. \end{cases}$$

where

$$F_i^L(y, Y, \gamma) = \sum_{k=1}^n (d_k^i y_k + \underline{f}_{kk}^i(y, Y, \gamma_{kk})) + \sum_{j=1}^n \sum_{k=1, k \neq j}^n \underline{f}_{jk}^i(y, Y, \gamma_{jk}).$$

Based on the above parametric linearization process, we know that the PLPRP can provide a reliable lower bound for the minimum value of QP in the region *Y*. In addition, Theorem 2.2 ensures that the PLPRP will sufficiently approximate the QP as $||u - l|| \rightarrow 0$, and this ensures the global convergence of the proposed branch-and-bound algorithm.

Branch-and-bound algorithm

In this section, a new global optimization branch-and-bound algorithm is presented for solving the OP. In this algorithm, there are several important operations, which are given as follows.

3.1 Basic operations

Branching Operation: The branching operation will produce a more precise subdivision. Here we select a rectangle bisection method, which is sufficient to guarantee the global convergence of the branch-and-bound algorithm. For any selected rectangle $Y^{'} = [l^{'}, u^{'}] \subseteq Y^{0}$, let $\eta \in \arg\max\{u^{'}_{j} - l^{'}_{j} : j = 1, 2, ..., n\}$, we can subdivide $Y^{'}$ into two new sub-rectangles $Y^{'1}$ and $Y^{'2}$ by partitioning interval $[\underline{y}_{n}^{'}, \overline{y}_{n}^{'}]$ into two sub-intervals $[\underline{y}'_{\eta}, (\underline{y}'_{\eta} + \overline{y}'_{\eta})/2]$ and $[(\underline{y}'_{\eta} + \overline{y}'_{\eta})/2, \overline{y}'_{\eta}]$.

Bounding Operation: For each sub-rectangle $Y \subseteq Y^0$, which has not been fathomed, determining the lower bound operation need to solve the parametric linear programming relaxation problem over the corresponding rectangle, and denote by $LB_s = \min\{LB(Y)|Y \in \Omega_s\}$, where Ω_s is the remaining set of sub-rectangle after s iterations. Determining the upper bound operation need to judge the feasibility of the midpoint of each inspected sub-rectangle Y and the optimal solution of the PLPRP over each inspected sub-rectangle Y, where $Y \in \Omega_s$. At the same time, we need to compute the objective function values of these known feasible points of QP, and we let $UB_s = \min\{F_0(Y) : Y \in \Theta\}$ be the best upper bound, where Θ is the set of the known feasible points.

Interval Reduction Operation: To enhance the running speed of the proposed algorithm, some interval reduction operations are given as follows.

For convenience in expression, for any $y \in Y$ and $i \in \{0, 1, ..., m\}$, we let *UB* be the current upper bound of the (QP), and let

$$F_i^L(y, Y, \gamma) = \sum_{j=1}^n c_{ij}(\gamma)y_j + e_i(\gamma),$$

$$LB_i(\gamma) = \sum_{i=1}^n \min\{c_{ij}(\gamma)l_j, c_{ij}(\gamma)u_j\} + e_i(\gamma).$$

Similarly to Theorem 3.1 of Ref. [17], for any investigated sub-rectangle $Y = (Y_i)_{1 \times n} \subseteq Y^0$, we have the following conclusions:

- (a) If $LB_0(\gamma) > UB$, then: the entire sub-rectangle Y can be abandoned.
- (b) If $LB_0(\gamma) \leq UB$ and $c_{0q}(\gamma) > 0$ for some $q \in \{1, 2, ..., n\}$, then the region Y_q can be replaced by
- (b) If $LB_0(\gamma) \leq UB$ and $C_{0q}(\gamma)$, $Q_0(\gamma) = \frac{1}{c_{0q}(\gamma)} \left[l_q, \frac{UB LB_0(\gamma) + \min\{c_{0q}(\gamma)l_q, c_{0q}(\gamma)u_q\}}{c_{0q}(\gamma)} \right] \cap Y_q$.

 (c) If $LB_0(\gamma) \leq UB$ and $c_{0q}(\gamma) < 0$ for some $q \in \{1, 2, \ldots, n\}$, then the region Y_q can be replaced by $\left[\frac{UB LB_0(\gamma) + \min\{c_{0q}(\gamma)l_q, c_{0q}(\gamma)u_q\}}{c_{0q}(\gamma)}, u_q \right] \cap Y_q.$
- (*d*) If $LB_i(\gamma) > \beta_i$ for some $i \in \{1, ..., m\}$, then the entire sub-rectangle Y can be abandoned.
- (e) If $LB_i(\gamma) \leq \beta_i$ for some $i \in \{1, \ldots, m\}$ and $c_{iq}(\gamma) > 0$ for some $q \in \{1, 2, \ldots, n\}$, then the region Y_q can be replaced by $[l_q, \frac{\beta_i LB_i(\gamma) + \min\{c_{iq}(\gamma)l_q, c_{iq}(\gamma)u_q\}}{c_{iq}(\gamma)}] \cap Y_q$.
- (f) If $LB_i(\gamma) \leq \beta_i$ for some $i \in \{1, \ldots, m\}$ and $c_{iq}(\gamma) < 0$ for some $q \in \{1, 2, \ldots, n\}$, then the region Y_q can be replaced by $\left[\frac{\beta_i LB_i(\gamma) + \min\{c_{iq}(\gamma)l_q, c_{iq}(\gamma)u_q\}}{c_{iq}(\gamma)}, u_q\right] \cap Y_q$.

From the above conclusions, to improve the convergent speed of the proposed algorithm, we can construct some interval reduction operations to compress the investigated rectangular area.

3.2 New branch-and-bound algorithm

For any sub-rectangle $Y^s \subseteq Y^0$, let $LB(Y^s)$ be the optimal value of the PLPRP over the sub-rectangle Y^s , and let $y^s = y(Y^s)$ be the optimal solution of the PLPRP over the sub-rectangle Y^s . Combining the former branching operation and bounding operation with interval reduction operations, a new global optimization branch-and-bound algorithm is described as follows.

Algorithm Steps:

Step 0. Given the termination error ϵ and the random parameter matrix γ . For the rectangle Y^0 , solve the PLPRP to obtain its optimal solution y^0 and optimal value $LB(Y^0)$, let $LB_0 = LB(Y^0)$ be the initial lower bound. If y^0 is feasible to the QP, let $UB_0 = F_0(y^0)$ be the initial upper bound, else let the initial upper bound $UB_0 = +\infty$. If $UB_0 - LB_0 \le \epsilon$, the algorithm stops, y^0 is an ϵ -global optimal solution of the QP. Else, let $\Omega_0 = \{Y^0\}$, $\Lambda = \emptyset$ and S = 1.

Step 1. Let the new upper bound be $UB_s = UB_{s-1}$. By using the branching operation, partition the selected rectangle Y^{s-1} into two sub-rectangles $Y^{s,1}$ and $Y^{s,2}$, and let $\Lambda = \Lambda \cup \{Y^{s-1}\}$ be the set of the deleted sub-rectangles.

Step 2. For each sub-rectangle $Y^{s,t}$, t=1,2, use the former interval reduction operations to compress its interval range, still let $Y^{s,t}$ be the remaining sub-rectangle.

Step 3. For each $t \in \{1,2\}$, solve the PLPRP over the sub-rectangle $Y^{s,t}$ to get its optimal solution $y^{s,t}$ and optimal value $LB(Y^{s,t})$, respectively. And denote by $\Omega_s = \{Y|Y \in \Omega_{s-1} \cup \{Y^{s,1},Y^{s,2}\}, Y \notin \Lambda\}$ and $LB_s = \min\{LB(Y)|Y \in \Omega_s\}$.

Step 4. If the midpoint y^{mid} of each sub-rectangle $Y^{s,t}$ is feasible to the QP, let $\Theta := \Theta \cup \{y^{mid}\}$ and $UB_s = \min\{UB_s, F_0(y^{mid})\}$. If the optimal solution $y^{s,t}$ of the PLPRP is feasible to the QP, let $UB_s = \min\{UB_s, F_0(y^{s,t})\}$, and let y^s be the best known feasible point which is satisfied with $UB_s = F_0(y^s)$.

Step 5. If $UB_s - LB_s \le \epsilon$, then the algorithm stops, and y^s is an ϵ -global optimal solution of the QP. Otherwise, let s = s + 1, and go to Step 1.

3.3 Global convergence

Without loss of generality, let *v* be the global optimal value of the QP, the global convergence of the proposed algorithm is proved as follows.

Theorem 3.1. If the presented algorithm terminates after finite s iterations, then y^s is an ϵ -global optimal solution of the QP; if the presented algorithm does not stop after finite iterations, then an infinite sub-sequence $\{Y^s\}$ of the rectangle Y^0 will be generated, and its accumulation point will be the global optimal solution of the QP.

Proof. If after s finite iterations, where s is a finite number such that $s \ge 0$, the presented algorithm stops, then it will follow that $UB_s \le LB_s + \epsilon$. From Step 4, we can obtain that there must exist a feasible point y^s , which is satisfied with $v \le UB_s = F_0(y^s)$. By the structure of the presented branch-and-bound algorithm, we have $LB_k \le v$. Combining the above inequalities together, we have

$$v \leq UB_s = F_0(\gamma^s) \leq LB_s + \epsilon \leq v + \epsilon$$
.

So that y^s is an ϵ -global optimal solution of the QP.

If the presented algorithm does not stop after finite iterations, since the selected branching operation is the bisection of rectangle, then the branching process is exhaustive, i.e., the branching operation will ensure that the intervals of all variables are convergent to 0, i.e., $\|u - l\| \to 0$. From Theorem 2.2, as $\|u - l\| \to 0$, the optimal solution of the PLPRP will sufficiently approximate the optimal solution of the QP, and this ensures that $\lim_{s\to\infty} (UB_s - LB_s) = 0$, therefore the bounding operation is consistent. Since the subdivided rectangle which obtains the actual lower bound is selected for further branching operation at the later immediate iteration, therefore, the proposed selecting operation is bound improving. By Theorem IV.3 in Ref.[18], the presented algorithm satisfies that the branching operation is exhaustive, the bounding method is consistent and the selecting operation is improvement, i.e., the presented algorithm satisfies the sufficient condition for global convergence, so that the presented algorithm is globally convergent to the optimal solution of the QP.

4 Numerical experiments

Let the parameter matrix $\gamma = (\gamma_{jk})_{n \times n} \in \mathbb{R}^{n \times n}$, where $\gamma_{jk} \in \{0, 1\}$, and the termination error $\epsilon = 10^{-6}$. Compared with the known algorithms, several test problems in literatures are run on microcomputer, and the program is coded in C++, all parametric linear programming relaxation problems are computed by simplex method. These test problems and their numerical results are given as follows. In Tables 1 and 2, we denote by "Iter." and "Time(s)" number of iteration and running time of the algorithm, respectively.

Problem 4.1 (Ref. [11]).

$$\begin{cases} \min F_0(y) = y_1 \\ \text{s.t. } F_1(y) = \frac{1}{4}y_1 + \frac{1}{2}y_2 - \frac{1}{16}y_1^2 - \frac{1}{16}y_2^2 \le 1, \\ F_2(y) = \frac{1}{14}y_1^2 + \frac{1}{14}y_2^2 - \frac{3}{7}y_1 - \frac{3}{7}y_2 \le -1, \\ 1 \le y_1 \le 5.5, \ 1 \le y_2 \le 5.5. \end{cases}$$

Problem 4.2 (Refs. [4,12]).

$$\begin{cases} \min F_0(y) = y_1^2 + y_2^2 \\ \text{s.t. } F_1(y) = 0.3y_1y_2 \ge 1, \\ 2 \le y_1 \le 5, \ 1 \le y_2 \le 3. \end{cases}$$

Problem 4.3 (Ref. [11]).

$$\begin{cases} \min F_0(y) = y_1y_2 - 2y_1 + y_2 + 1 \\ \text{s.t. } F_1(y) = 8y_2^2 - 6y_1 - 16y_2 \le -11, \\ F_2(y) = -y_2^2 + 3y_1 + 2y_2 \le 7, \\ 1 \le y_1 \le 2.5, \ 1 \le y_2 \le 2.225. \end{cases}$$

Problem 4.4 (Refs. [4,5,10]).

$$\begin{cases} \min \ F_0(y) = 6y_1^2 + 4y_2^2 + 5y_1y_2 \\ \text{s.t.} \ F_1(y) = -6y_1y_2 \le -48, \\ 0 \le y_1, y_2 \le 10. \end{cases}$$

Problem 4.5 (Refs. [12,13]).

$$\begin{cases} \min F_0(y) = y_1 \\ \text{s.t. } F_1(y) = 4y_2 - 4y_1^2 \le 1, \\ F_2(y) = -y_1 - y_2 \le -1, \\ 0.01 \le y_1, y_2 \le 15. \end{cases}$$

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Problem 4.6 (Refs. [10,15]).

$$\begin{cases} \min F_0(y) = -4y_2 + (y_1 - 1)^2 + y_2^2 - 10y_3^2 \\ \text{s.t. } F_1(y) = y_1^2 + y_2^2 + y_3^2 \le 2, \\ F_2(y) = (y_1 - 2)^2 + y_2^2 + y_3^2 \le 2, \\ 2 - \sqrt{2} \le y_1 \le \sqrt{2}, \ 0 \le y_2, y_3 \le \sqrt{2}. \end{cases}$$

Problem 4.7 (Ref. [14]).

$$\begin{cases} \min F_0(y) = -y_1 + y_1 y_2^{0.5} - y_2 \\ \text{s.t. } F_1(y) = -6y_1 + 8y_2 \le 3, \\ F_2(y) = 3y_1 - y_2 \le 3, \\ 1 \le y_1, y_2 \le 1.5. \end{cases}$$

Table 1. Numerical comparisons for Problems 4.1-4.7

Problem	Refs.	Global optimal solution	Optimal value	Iter.	Time(s)
1	[11]	(1.177124327, 2.177124353)	1.177124327	434	1.0000
	ours	(1.177124344, 2.177124344)	1.177124344	20	0.0091
2	[4]	(2.000000000, 1.666666667)	6.777778340	30	0.0068
	[12]	(2.00003, 1.66665)	6.7780	44	0.1800
	ours	(2.000000000, 1.666666667)	6.777778340	10	0.0018
3	[11]	(2.000000, 1.000000)	-1.0	24	0.0129
	ours	(2.000000, 1.000000)	-0.999999	22	0.0080
4	[4]	(2.555409888, 3.130613160)	118.383672050	49	0.0744
	[5]	(2.5557793695, 3.1301646393)	118.383756475	210	0.7800
	[10]	(2.555745855, 3.130201688)	118.383671904	59	0.0385
	ours	(2.560178568, 3.125000000)	118.392375898	46	0.0189
5	[11]	(0.5, 0.5)	0.5	91	0.8500
	[13]	(0.5, 0.5)	0.5	96	1.0000
	ours	(0.5, 0.5)	0.5	26	0.0065
6	[10]	(1.0, 0.181818470, 0.983332113)	-11.363636364	420	0.2845
	[15]	(0.998712,0.196213,0.979216)	-10.35	1648	0.3438
	ours	(1.0, 0.181815072, 0.983332745)	-11.363636364	97	0.1568
7	[14]	(1.5, 1.5)	-1.16288	84	0.1257
	our	(1.5, 1.5)	-1.16288	38	0.0658

Problem 4.8 (Ref. [9]).

$$\begin{cases} \min F_0(y) = \frac{1}{2} \langle y, Q^0 y \rangle + \langle y, d^0 \rangle \\ \text{s.t.} \quad F_i(y) = \frac{1}{2} \langle y, Q^i y \rangle + \langle y, d^i \rangle \leq \beta_i, \ i = 1, \dots, m, \\ 0 \leq y_j \leq 10, \ j = 1, \dots, n. \end{cases}$$

Each element of Q^0 is randomly generated in [0,1], each element of Q^i ($i=1,\cdots,m$) is randomly generated in [-1,0], each element of d^0 is randomly generated in [0,1], each element of d^i ($i=1,\cdots,m$) is randomly generated in [-1,0], each element of β_i ($i=1,\cdots,m$) is randomly generated in [-300,-90], and each element of $\gamma=(\gamma_{jk})_{n\times n}\in R^{n\times n}$ is randomly generated from 0 or 1.

We denote by n the dimension of our problem, our problem and by m the constraint number of our problem. Numerical results for Problem 4.8 are given in Table 2.

Compared with the known algorithms, numerical experimental results of Problems 4.1-4.8 demonstrate that the proposed algorithm can globally solve the QP with the higher computational efficiency.

Table 2. Computational	comparisons with	Ref [9] for	Problem 4.8
Table 2. Computational	CUIIIDAIISUIIS WILII	1101	FIUDICIII 4.0

(Dimension, Constraints)	Algorithm of Ref. [9]	This paper	
(n,m)	Time(s)	Time(s)	
(4,6)	2.37678	1.989758	
(5, 11)	6.39897	5.234890	
(14, 6)	9.22732	7.246780	
(18,7)	15.8410	12.84189	
(20, 5)	11.9538	9.527432	
(35, 10)	74.8853	68.44086	
(37,9)	77.1476	69.40288	
(45,8)	86.7174	80.51039	
(46, 5)	44.2502	36.85856	
(60, 11)	315.659	280.25859	

5 Concluding remarks

In this article, based on branch-and-bound framework, we present a new global optimization algorithm for solving the quadratic programs with quadratic constraints. In this algorithm, a new parametric linearization technique is derived. By utilizing the parametric linearization technique, we can derive the parametric linear programming relaxation problem of the QP. In addition, some interval reduction operations are proposed for improving the computational speed of the proposed branch-and-bound algorithm. The presented algorithm is convergent to the global optimal solution of the QP by subsequently partitioning the initial rectangle and by solving a sequence of parametric linear programming relaxation problems. Finally, numerical results show that the proposed algorithm has higher computational efficiency than those existent algorithms.

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