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Uniqueness theorems for L-functions in the extended Selberg class

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Abstract: In this paper, we obtain uniqueness theorems of L-functions from the extended Selberg class, which generalize and complement some recent results due to Li, Wu-Hu, and Yuan-Li-Yi.

Keywords: Meromorphic function, L-function, Selberg class, Value distribution

MSC: 11M36, 30D35, 30D30

1 Introduction

The Riemann hypothesis as one of the millennium problems has been given a lot of attention by many scholars for a long time. Selberg guessed that the Riemann hypothesis also holds for the L-function in the Selberg class. Such an L-function based on the Riemann zeta function as a prototype is defined to be a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \tag{1}$$

of a complex variable $s = \sigma + it$ satisfying the following axioms [1]:
(i) Ramanujan hypothesis: $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.
(ii) Analytic continuation: There exists a nonnegative integer m such that $(s-1)^m L(s)$ is an entire function of finite order.
(iii) Functional equation: L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j, and complex numbers ν_j, ω with $\mathrm{Re}\nu_j \geq 0$ and $|\omega| = 1$.
(iv) Euler product: $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

It is mentioned that there are many Dirichlet series but only those satisfying the axioms (i)-(iii) are regarded as the extended Selberg class [1, 2]. All the L-functions which are studied in this article are from the extended

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Selberg class. Therefore, the conclusions proved in this article are also true for L-functions in the Selberg class. Theorems in this paper will be proved by means of Nevanlinna's Value distribution theory. Suppose that F and G are two nonconstant meromorphic functions in the complex plane $\mathbb{C}$, c denotes a value in the extended complex plane $\mathbb{C} \cup \{\infty\}$. If $F - c$ and $G - c$ have the same zeros counting multiplicities, we say that F and G share c CM. If $F - c$ and $G - c$ have the same zeros ignoring multiplicities, then we say that F and G share c IM. It is well known that two nonconstant meromorphic functions in $\mathbb{C}$ are identically equal when they share five distinct values IM [3, 4]. The following uniqueness theorem of two L-functions was proved by Steuding [1].

Theorem 1.1 (see [1]). *If two L-functions with $a(1) = 1$ share a complex value $c \neq \infty$ CM, then they are identically equal.*

Remark 1.2. *In [5], the authors gave an example that $L_1 = 1 + \frac{2}{4^s}$ and $L_2 = 1 + \frac{3}{9^s}$, which showed that Theorem 1.1 is actually false when $c = 1$.*

In 2011, Li [6] considered values which are shared IM and got

Theorem 1.3 (see [6]). *Let L_1 and L_2 be two L-functions satisfying the same functional equation with $a(1) = 1$ and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. If $L_1^{-1}(a_j) = L_2^{-1}(a_j)$, $j = 1, 2$, then $L_1 \equiv L_2$.*

In 2001, Lahiri [7] put forward the concept of weighted sharing as follows.

Let k be a nonnegative integer or ∞, $c \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(c, f)$ the set of all zeros of $f - c$, where a zero of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(c, f) = E_k(c, g)$, we say that f and g share the value c with weight k (see [7]).

In 2015, Wu and Hu [8] removed the assumption that both L-functions satisfy the same functional equation in Theorem 1.3. By including weights, they had shown the following result.

Theorem 1.4 (see [8]). *Let L_1 and L_2 be two L-functions, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. Take two positive integers k_1, k_2 with $k_1k_2 > 1$. If $E_{k_j}(a_j, L_1) = E_{k_j}(a_j, L_2)$, $j = 1, 2$, then $L_1 \equiv L_2$.*

In 2003, the following question was posed by C.C. Yang [9].

Question 1.5 (see [9]). *Let f be a meromorphic function in the complex plane and a, b, c are three distinct values, where $c \neq 0, \infty$. If f and the Riemann zeta function ζ share a, b CM and c IM, will then $f \equiv \zeta$?*

The L-function is based on the Riemann zeta function as the model. It is then valuable that we study the relationship between an L-function and an arbitrary meromorphic function [10–14]. This paper concerns the problem of how meromorphic functions and L-functions are uniquely determined by their c-values. Firstly, we introduced the following theorem.

Theorem 1.6 (see [10]). *Let a and b be two distinct finite values and f be a meromorphic function in the complex plane with finitely many poles. If f and a nonconstant L-function L share a CM and b IM, then $L \equiv f$.*

Then, using the idea of weighted sharing, we will prove the following theorem.

Theorem 1.7. *Let f be a meromorphic function in the complex plane with finitely many poles, let L be a nonconstant L-function, and let $a_1, a_2 \in \mathbb{C}$ be two distinct values. Take two positive integers k_1, k_2 with $k_1k_2 > 1$. If $E_{k_j}(a_j, f) = E_{k_j}(a_j, L)$, $j = 1, 2$, then $L \equiv f$.*

Remark 1.8. *Note that an L-function itself can be analytically continued as a meromorphic function in the complex plane. Therefore, an L-function will be taken as a special meromorphic function. We can also see that Theorem 1.4 is included in Theorem 1.7.*

In 1976, the following question was mentioned by Gross in [15].

Question 1.9 (see [15]). *Must two nonconstant entire functions f_1 and f_2 be identically equal if f_1 and f_2 share a finite set S?*

Recently, Yuan, Li and Yi [16] considered this question leading to the theorem below.

Theorem 1.10 (see [16]). *Let $S = \{\omega_1, \omega_2, \cdots, \omega_l\}$, where $\omega_1, \omega_2, \cdots, \omega_l$ are all distinct roots of the algebraic equation $\omega^n + a\omega^m + b = 0$. Here l is a positive integer satisfying $1 \le l \le n$, n and m are relatively prime positive integers with $n \ge 5$ and $n > m$, and a, b, c are nonzero finite constants, where $c \ne \omega_j$ for $1 \le j \le l$. Let f be a nonconstant meromorphic function such that f has finitely many poles in $\mathbb{C}$, and let L be a nonconstant L-function. If f and L share S CM and c IM, then $f \equiv L$.*

Concerning shared set, we prove the following theorem.

Theorem 1.11. *Let f be an entire function with $\lim_{\Re(s)\to+\infty} f(s) = k$ $(k \ne \infty)$ and let $R(a) = 0$ be a algebraic equation with $n \ge 2$ distinct roots, and $R(k), R(b), R(1) \ne 0$. Suppose that $f(s_0) = L(s_0) = b$ for some $s_0 \in \mathbb{C}$. If f and a nonconstant L-function L share S CM, where $S = \{a : R(a) = 0\}$, then $R(L) \equiv R(f)$.*

Furthermore, we obtain a result which is similar to Theorem 1.10 by different means.

Theorem 1.12. *Let f be an entire function with $\lim_{\Re(s)\to+\infty} f(s) = k$ $(k \ne \infty)$. Let $S = \{\omega_1, \omega_2, \cdots, \omega_i\} \subset \mathbb{C}\backslash\{1, k, b\}$, where $\omega_1, \omega_2, \cdots, \omega_i$ are all distinct roots of the algebraic equation $\omega^{n+m} + \alpha\omega^n + \beta = 0$, $1 \le i \le n+m$, n, m are two positive integers with $n > m+2$, α, β are finite nonzero constants. If f and a nonconstant L-function L share S CM and $f(s_0) = L(s_0) = b$ for some $s_0 \in \mathbb{C}$, then $f \equiv tL$, where t is a constant such that $t^d = 1$, $d = GCD(n, m)$.*

2 Some lemmas

In this section, we present some important lemmas which will be needed in the sequel. Firstly, let f be a meromorphic function in $\mathbb{C}$. The order $\rho(f)$ is defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

Lemma 2.1 (see [4], Lemma 1.22). *Let f be a nonconstant meromorphic function and let $k \ge 1$ be an integer. Then $m\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. Further if $\rho(f) < +\infty$, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r).$$

Lemma 2.2 (see [4], Corollary of Theorem 1.5). *Let f be a nonconstant meromorphic function. Then f is a rational function if and only if $\liminf\limits_{r\to\infty} \frac{T(r,f)}{\log r} < \infty$.*

Lemma 2.3 (see [4], Theorem 1.19). *Let $T_1(r)$ and $T_2(r)$ be two nonnegative, nondecreasing real functions defined in $r > r_0 > 0$. If $T_1(r) = O(T_2(r))$ $(r \to \infty, r \notin E)$, where E is a set with finite linear measure, then*

$$\limsup_{r\to\infty} \frac{\log^+ T_1(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log^+ T_2(r)}{\log r}$$

and

$$\liminf_{r\to\infty} \frac{\log^+ T_1(r)}{\log r} \le \liminf_{r\to\infty} \frac{\log^+ T_2(r)}{\log r},$$

which imply that the order and the lower order of $T_1(r)$ are not greater than the order and the lower order of $T_2(r)$ respectively.

Lemma 2.4 (see [4], Theorem 1.14). *Let f and g be two nonconstant meromorphic functions. If the order of f and g is $\rho(f)$ and $\rho(g)$ respectively, then*

$$\rho(f \cdot g) \leq \max\{\rho(f), \rho(g)\},$$

$$\rho(f + g) \leq \max\{\rho(f), \rho(g)\}.$$

Lemma 2.5 (see [17], Lemma 2.7). *Let $R(\omega) = \omega^n + a\omega^m + b$, where n, m are positive integers satisfying $n > m$, a, b are finite nonzero complex numbers. Then the algebraic equation $R(\omega) = 0$ has at least $n - 1$ distinct roots.*

Lemma 2.6 (see [18], Lemma 8). *Let $s > 0$ and t be relatively prime integers, and let c be a finite complex number such that $c^s = 1$. Then there exists one and only one common zero of $\omega^s - 1$ and $\omega^t - c$.*

3 Proofs of the theorems

3.1 Proof of Theorem 1.7

First of all, we denote by d the degree of L. Then $d = 2\sum_{j=1}^{k} \lambda_j > 0$, where k and λ_j are respectively the positive integer and the positive real number in the functional equation of the axiom (iii) of the definition of L-functions. According to a result due to Steuding [1], p.150, we have

$$T(r, L) = \frac{d}{\pi} r \log r + O(r). \tag{2}$$

Therefore $\rho(L) = 1$ and $S(r, L) = O(\log r)$.

Noting that f has finitely many poles and L at most has one pole at $s = 1$ in the complex plane, it follows that

$$N(r, f) = O(\log r), \qquad N(r, L) = O(\log r). \tag{3}$$

Because f and L share a_1, a_2 weighted k_1, k_2 respectively, by (3), from the first and second fundamental theorems we have

$$\begin{aligned} T(r,f) &\leq \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + \overline{N}(r, f) + S(r, f) \\ &= \overline{N}\left(r, \frac{1}{L - a_1}\right) + \overline{N}\left(r, \frac{1}{L - a_2}\right) + O(\log r) + S(r, f) \\ &\leq T\left(r, \frac{1}{L - a_1}\right) + T\left(r, \frac{1}{L - a_2}\right) + O(\log r) + S(r, f) \\ &= 2T(r, L) + O(\log r) + S(r, f). \end{aligned} \tag{4}$$

Then from (4) and Lemma 2.3 we obtain

$$\rho(f) \leq \rho(L). \tag{5}$$

Similarly,

$$\rho(L) \leq \rho(f). \tag{6}$$

Combining (5) with (6) yields

$$\rho(f) = \rho(L). \tag{7}$$

Thus

$$S(r, f) = O(\log r). \tag{8}$$

We introduce two auxiliary functions below.

$$F_1 = \frac{L'}{L-a_1} - \frac{f'}{f-a_1}, \tag{9}$$

$$F_2 = \frac{L'}{L-a_2} - \frac{f'}{f-a_2}. \tag{10}$$

Next, we assume that $F_1 \not\equiv 0$ and $F_2 \not\equiv 0$. By (8) and Lemma 2.1 we get

$$m(r, F_1) = O(\log r). \tag{11}$$

By the assumption L and f share (a_1, k_1), (a_2, k_2), from (3), (9) and (11) we have

$$\begin{aligned} k_2\overline{N}_{(k_2+1}\left(r, \frac{1}{L-a_2}\right) &\le N\left(r, \frac{1}{F_1}\right) \le T(r, F_1) + O(1) \le N(r, F_1) + m(r, F_1) + O(1) \\ &\le \overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) + \overline{N}(r, L) + \overline{N}(r, f) + O(\log r) \\ &\le \overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) + O(\log r). \end{aligned} \tag{12}$$

Similarly, from (3), (10) and (11) we have

$$\begin{aligned} k_1\overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) &\le N\left(r, \frac{1}{F_2}\right) \le T(r, F_2) + O(1) \le N(r, F_2) + m(r, F_2) + O(1) \\ &\le \overline{N}_{(k_2+1}\left(r, \frac{1}{L-a_2}\right) + \overline{N}(r, L) + \overline{N}(r, f) + O(\log r) \\ &\le \overline{N}_{(k_2+1}\left(r, \frac{1}{L-a_2}\right) + O(\log r). \end{aligned} \tag{13}$$

Combining (12) with (13) yields

$$\begin{aligned} \overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) &\le \frac{1}{k_1}\overline{N}_{(k_2+1}\left(r, \frac{1}{L-a_2}\right) + O(\log r) \\ &\le \frac{1}{k_1k_2}\overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) + O(\log r). \end{aligned} \tag{14}$$

Since $k_1k_2 > 1$, from (14) we obtain

$$\overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) = O(\log r). \tag{15}$$

Substituting (15) into (12) implies

$$\overline{N}_{(k_2+1}\left(r, \frac{1}{L-a_2}\right) = O(\log r). \tag{16}$$

Set

$$G = \frac{L-a_1}{f-a_1}.$$

Noting L and f share (a_1, k_1), (a_2, k_2), combining (15) with (16) yields

$$\overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) = \overline{N}_{(k_1+1}\left(r, \frac{1}{f-a_1}\right) = O(\log r),$$

$$\overline{N}_{(k_2+1}\left(r, \frac{1}{L-a_2}\right) = \overline{N}_{(k_2+1}\left(r, \frac{1}{f-a_2}\right) = O(\log r).$$

Clearly,

$$\overline{N}(r, G) \le N(r, L) + \overline{N}_{(k_1+1}\left(r, \frac{1}{f-a_1}\right) = O(\log r), \tag{17}$$

$$\overline{N}\left(r, \frac{1}{G}\right) \le N(r, f) + \overline{N}_{(k_1+1}\left(r, \frac{1}{L-a_1}\right) = O(\log r). \tag{18}$$

Set

$$G_1 = \frac{Q(L - a_1)}{f - a_1}, \tag{19}$$

where Q is a rational function satisfying that G_1 is a zero-free entire function. From (17) and (18), it is easy to see that such a Q does exist. By Lemma 2.2 and Lemma 2.4 we get

$$\rho(G_1) \le \max\{\rho(Q), \rho(L), \rho(f)\} = 1.$$

By the Hadamard factorization theorem [19], p.384, we know

$$G_1 = \frac{Q(L - a_1)}{f - a_1} = e^{\varphi}, \tag{20}$$

where φ is a polynomial of degree at most $deg(\varphi) \le 1$. We may write $\varphi = a_0 s + b_0$ for some complex numbers a_0, b_0. In view of (20) and Hayman [3], p.7, we have

$$T(r, G_1) = T(r, e^{a_0 s + b_0}) = O(r). \tag{21}$$

By (19), the assumption that L and f share a_2, we get that every a_2-point of L has to be 1-point of $\frac{G_1}{Q} - 1$. Now (20), (21) and the first fundamental theorem yield

$$\begin{aligned} \overline{N}\left(r, \frac{1}{L - a_2}\right) &\le N\left(r, \frac{1}{\frac{G_1}{Q} - 1}\right) \le T\left(r, \frac{1}{\frac{G_1}{Q} - 1}\right) \\ &= T\left(r, \frac{G_1}{Q} - 1\right) + O(1) \\ &\le T(r, G_1) + T(r, Q) + O(1) = O(r). \end{aligned} \tag{22}$$

Similarly, set

$$G_2 = \frac{L - a_2}{f - a_2}.$$

We also get

$$\overline{N}\left(r, \frac{1}{L - a_1}\right) = O(r). \tag{23}$$

By (22), (23) and the second fundamental theorem it follows that

$$T(r, L) \le \overline{N}\left(r, \frac{1}{L - a_1}\right) + \overline{N}\left(r, \frac{1}{L - a_2}\right) + \overline{N}(r, L) + O(\log r) = O(r). \tag{24}$$

This contradicts (2). Thus, $F_1 \equiv 0$ or $F_2 \equiv 0$. By integration, we have from (9) that

$$L - a_1 \equiv A(f - a_1),$$

where $A(\neq 0)$ is a constant. This implies that L and f share a_1 CM. Hence by Theorem 1.6 we deduce Theorem 1.7 holds. If $F_2 \equiv 0$, using the same manner, we also have the conclusion.

This completes the proof of Theorem 1.7.

3.2 Proof of Theorem 1.11

First we consider the following function

$$G = \frac{QR(L)}{R(f)}, \tag{25}$$

where

$$Q(s) = A(s - 1)^{nm} \tag{26}$$

is a rational function satisfying that G has no zeros and no poles in $\mathbb{C}$; A is a nonzero finite value; m is the nonnegative integer in the axiom (ii) of the definition of L-functions.

We claim that such a Q does exist. By the condition that f and L share S CM, set

$$F = \frac{R(L)}{R(f)}. \tag{27}$$

We can see that there can be only a pole of f or L such that $F = 0$ or $F = \infty$. Since f has no pole and L has only one possible pole at $s = 1$, it follows that F has no zero and only one possible pole at $s = 1$. Hence such a Q does exist.

Next, assume that $a_1, a_2, \cdots, a_n$ are all distinct roots of $R(a)$. Using the first fundamental theorem we get

$$T(r, L - a_i) = T(r, L) + O(1), \quad i = 1, 2, \cdots, n.$$

Noting $n \geq 2$, by the second fundamental theorem we have

$$\begin{aligned}(n-1)T(r,f) &\leq \sum_{i=1}^{n} \overline{N}\left(r, \frac{1}{f - a_i}\right) + \overline{N}(r,f) + S(r,f)\\ &= \sum_{i=1}^{n} \overline{N}\left(r, \frac{1}{L - a_i}\right) + \overline{N}(r,f) + S(r,f)\\ &\leq \sum_{i=1}^{n} T\left(r, \frac{1}{L - a_i}\right) + S(r,f)\\ &= nT(r,L) + S(r,f),\end{aligned} \tag{28}$$

which gives

$$T(r,f) \leq \frac{n}{n-1} T(r,L) + S(r,f).$$

This together with Lemma 2.3 yields

$$\rho(f) \leq \rho(L). \tag{29}$$

Similarly,

$$\rho(L) \leq \rho(f). \tag{30}$$

By (29), (30) and (2) we obtain

$$\rho(f) = \rho(L) = 1. \tag{31}$$

Also, from the first fundamental theorem we get

$$\rho\left(\frac{1}{f - a_i}\right) = \rho(f) = 1,$$

and then by Lemma 2.2 and Lemma 2.4 we deduce

$$\rho(G) \leq \max\{\rho(Q), \rho(L), \rho(f)\} = 1.$$

From the Hadamard factorization theorem [19], p.384 we see

$$G = e^{h(s)}, \tag{32}$$

where $h(s)$ is a polynomial of degree $\deg(h(s)) \leq 1$. One can write

$$\Re h(\sigma + it) = \alpha(t)\sigma + \beta(t), \tag{33}$$

a polynomial in σ with $\alpha(t)$, $\beta(t)$ being polynomials in t. Now the claim is $\alpha(t) \equiv 0$. From (25), (27) and (32) we get

$$F = \frac{R(L)}{R(f)} = e^{h(s)} Q^{-1}. \tag{34}$$

Since $\lim_{\sigma \to +\infty} L(s) = 1$, $\lim_{\sigma \to +\infty} f(s) = k (k \neq \infty)$, $R(k) \neq 0$ and $R(1) \neq 0$, it follows that

$$\lim_{\sigma \to +\infty} \frac{R(L)}{R(f)} = C, \tag{35}$$

where $C \neq 0$ is a finite value. If $\alpha(t) \not\equiv 0$, we obtain $\alpha(t_0) \neq 0$ for some value t_0. If $\alpha(t_0) > 0$, from (34) we know that

$$\left|\frac{R(L)}{R(f)}\right| = \left|Q^{-1}\right| e^{\Re h(\sigma+it)}. \tag{36}$$

Thus from (26), (33), (35) and (36) we can deduce that, $|C| = \infty$ when $\sigma \to +\infty$ with $t = t_0$, which is a contradiction. Similarly, if $\alpha(t_0) < 0$, we have that, $|C| = 0$ when $\sigma \to +\infty$ with $t = t_0$, which is also a contradiction. Therefore $\alpha(t) \equiv 0$. Now by (33) and (36) we get

$$\left|\frac{R(L)}{R(f)}\right| = \left|Q^{-1}\right| e^{\beta(t)}. \tag{37}$$

Combining (35) with (37) yields

$$\lim_{\sigma\to+\infty} |Q| = \frac{e^{\beta(t)}}{|C|} \tag{38}$$

for a fixed t. Considering that the limit of $|Q|$ as $\sigma \to +\infty$ is a nonzero finite constant for some value t and $n \geq 2$, in view of (26) we see that $m = 0$, and then $Q(s) \equiv A$. From (38) we have $e^{\beta(t)} = |A||C|$. Thus it follows by (37) that

$$\left|\frac{R(L)}{R(f)}\right| = |C|. \tag{39}$$

Since $C \neq 0$ is a finite complex number, from (39) we deduce that $\frac{R(L)}{R(f)}$ is a constant. Then by (35) we know that

$$\frac{R(L)}{R(f)} \equiv C. \tag{40}$$

From the assumption in the theorem we have $f(s_0) = L(s_0) = b$ for some $s_0 \in \mathbb{C}$. It now follows from (40) that $C = 1$. Thus

$$\frac{R(L)}{R(f)} \equiv 1. \tag{41}$$

That is $R(L) \equiv R(f)$.

This completes the proof of Theorem 1.11.

3.3 Proof of Theorem 1.12

First, we have that the algebraic equation $\omega^{n+m} + \alpha\omega^n + \beta = 0$ has at least $n + m - 1 > 3m + 1 \geq 4$ distinct roots in view of Lemma 2.5. By Theorem 1.11, we get

$$L^{n+m} + \alpha L^n \equiv f^{n+m} + \alpha f^n. \tag{42}$$

Set $H = \frac{f}{L}$. Then by (42) we deduce

$$\frac{1}{\alpha} L^m = \frac{H^n - 1}{H^{n+m} - 1}. \tag{43}$$

We discuss two cases:

Case 1. H is a constant. If $H^{n+m} \neq 1$, by (43), we get that L is a constant, which contradicts the assumption that L is a nonconstant L-function. Therefore, $H^{n+m} = 1$, and so it follows by (43) that $H^m = H^n = 1$, that is $f^n = L^n$ and $f^m = L^m$. We get $f^d = L^d$.

Case 2. H is a nonconstant meromorphic function. Note that L has at most one pole. Now we discuss the following two subcases again.

Subcase 2.1. L has no poles. Then, from (43) we get that every 1-point of H^{n+m} has to be 1-point of H^n. Since $H^{n+m} = H^n H^m$, we have any 1-point of H^{n+m} to be a 1-point of H^m. Because $n > m + 2$, it follows that H is a constant, contradicting the assumption.

Subcase 2.2. L has one and only one pole. Then by (43) we know every zero of $H^{n+m} - 1$ has to be zero of $H^n - 1$ with one exception. Put

$$H^n - 1 = (H - 1)(H - \zeta_1)\cdots(H - \zeta_{n-1}),$$

$$H^{n+m} - 1 = (H-1)(H-\tau_1)\cdots(H-\tau_{n+m-1}),$$

where $\zeta_1, \zeta_2, \cdots, \zeta_{n-1}$ are $n-1$ distinct finite complex numbers satisfying $\zeta_i^n = 1$, $\zeta_i \neq 1$, $1 \le i \le n-1$; $\tau_1, \tau_2, \cdots, \tau_{n+m-1}$ are $n+m-1$ distinct finite complex numbers satisfying $\tau_j^{n+m} = 1$, $\tau_j \neq 1$, $1 \le j \le n+m-1$.

Let $m = 1$. By Lemma 2.6 we see $H^n - 1$ and $H^{n+1} - 1$ have only one common zero, so H cannot be equal to any $n+m-2$ values of $\{\tau_1, \tau_2, \cdots, \tau_{n+m-1}\}$. From $n > m+2$ it follows that H is a constant, contradicting the assumption.

Let $m \ge 2$. If any 1-point of H^n is a 1-point of H^{n+m}, then any 1-point of H^n is a 1-point of H^m. Note that $n > m+2$. This contradicts the assumption that H is nonconstant. If there is at least one $\zeta_i \neq \tau_j$, $1 \le i \le n-1$, $1 \le j \le n+m-1$, then H cannot be equal to any $m+1$ values of $\{\tau_1, \tau_2, \cdots, \tau_{n+m-1}\}$. From $m \ge 2$, we know H is a constant, contradicting the assumption.

This completes the proof of Theorem 1.12.

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