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Chaos and bifurcation in the controlled chaotic system

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Abstract: In this paper, chaos and bifurcation are explored for the controlled chaotic system, which is put forward based on the hybrid strategy in an unusual chaotic system. Behavior of the controlled system with variable parameter is researched in detain. Moreover, the normal form theory is used to analyze the direction and stability of bifurcating periodic solution.

Keywords: Controlled chaotic system, Chaos, Bifurcation, Normal form theory

MSC: 34C23,C34C28,C34C60,C34H20

1 Introduction

In 1963, the first three dimensional autonomous chaotic system (Lorenz system) was proposed in the literature [1]. After that, the construction and theoretical research of the chaotic system have become a hot issue in the nonlinear science. What is more, many three-dimensional chaotic systems have been put forward one after another, including Chen system [2] and Lü system [3], which have to be mentioned. In the sense of literature [4], Chen system is the dual system of Lorenz system, whereas Lü system acts as a bridge between the both. In 1994, with the aid of a computer, Sprott has found 19 three-dimensional chaotic systems [5]. Rössler system [6] and Chua system [7] have also received widespread attention from scholars. More about the contents of chaotic systems can be seen in the literature [8–11]. For most 3D systems, there may be only one equilibrium point, two symmetric equilibrium points, three equilibrium points or even more, their common features are that all of the equilibria are unstable [12]. In 2008, Yang and Chen [13] introduced a chaotic system [14] with a saddle point and two stable node-foci. Wei and Yang introduced a class of chaotic systems with only two stable equilibria [15]. In particular, the above mentioned chaotic systems have a common characteristic: the total stability of the two symmetric equilibria is always the same. Bifurcation analysis and control is one of the most popular research theme in the domain of bifurcation control [16–21].

Liu et al. [22] gave a new 3D chaotic system as follows:

$$\dot{x} = a(y - x), \quad \dot{y} = -c + xz, \quad \dot{z} = b - y^2.$$

There are only two nonlinear terms on the right side of the system, which is simple, but the local dynamic behavior is complex and interesting. There is a pair of equilibrium points in the system, whose positions are symmetric but the stability is opposite. The nature of this new chaotic system has attracted extensive interest [23–26]. In this article, we design a hybrid strategy which consists of a parameter and a state feedback , and

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it is added to the last system, and we can get the controlled chaotic system as follows:

$$\dot{x} = a(y - x), \quad \dot{y} = -c + xz + m(x - y), \quad \dot{z} = b - y^2.$$

Apparently, this strategy not only ensures that the equilibrium structure and the dimension of the original system are unchanged. To analyze complex dynamical behaviors of the controlled system and evaluate differences of chaos between the original system and controlled system, we study the chaos and bifurcation of the controlled system through analyzing the variable parameter. First, phase trajectories, Lyapunove exponents and bifurcation path are numerically investigated. Chaotic area, period area and several periodic windows within the range of control parameter in controlled system are found. Then a sufficient condition for the existence of Hopf bifurcation of controlled system are given. The next theorem describes following details: the existence and expression of the bifurcation periodic solutions are explored, and the stability, direction and the period size also are studied.

The remainder of the article is arranged as follows: In section 2, hybrid control strategy of Hopf bifurcation is examined. In section 3, bifurcation diagrams and Lyapunov exponents of the controlled system with varying parameter m are analyzed in detail. The Hopf bifurcation and its periodic solution of the controlled system are explored in section 4. Finally, section 5 is summing up the full text.

2 Hybrid control strategy

For the convenience of description, the original system proposed in [22] as follows:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -c + xz \\ \dot{z} = b - y^2, \end{cases} \tag{2.1}$$

when parameters a ($a > 0$), b and c are all real constants, and x, y, z are three state variables in this case. Apparently, when $c = 0$, system (2.1) is invariant while it is under the transformation $\mathrm{T}(x, y, z) \to \mathrm{T}(-x, -y, z)$. In the sense of this transformation, the system not itself invariant under the T transformation, and there is another orbit that corresponds to it. We calculate the next formulas:

$$a(y - x) = 0, \qquad -c + xz = 0, \qquad b - y^2 = 0.$$

It is easily obtained that if $b^2 + c^2 = 0$, there are non-isolated equilibria $O_z(0, 0, z)$; if either $b < 0$ or $b = 0$ and $c \neq 0$, there is not an equilibrium; if $b > 0$, there are two equilibria $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ and $E^-(-\sqrt{b}, -\sqrt{b}, -\frac{c}{\sqrt{b}})$.

In this part, based on the model (2.1), we design a controller to get the controlled system as below

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = -c + xz + m(x - y) \\ \dot{z} = b - y^2. \end{cases} \tag{2.2}$$

As needed, we only have to control the second equation, and the other two equations remain unchanged. Obviously, the characteristic of the control law is to ensure that the structure of the equilibrium and the dimension of the original system (2.1) are unchanged. Especially, if $m = 0$, the controlled system will restore to the original system.

The linearized system (2.2) acts at $E^{\pm}$, and we can get the Jacobian matrix

$$J^{\pm} = \begin{bmatrix} -a & a & 0 \\ m + z^{\pm} & -m & x^{\pm} \\ 0 & -2y^{\pm} & 0 \end{bmatrix},$$

its characteristic equation being

$$\Delta(\lambda) = \lambda^3 + (m + a)\lambda^2 + (2x^{\pm}y^{\pm} - az^{\pm})\lambda + 2ax^{\pm}y^{\pm} = 0. \tag{2.3}$$

Regarding the controlled system, we conclude that if $c > \frac{2bm\sqrt{b}}{a(a+m)}$, E^+ is unstable while E^- is stable, and if $c < \frac{2bm\sqrt{b}}{a(a+m)}$, E^+ is stable while E^- is unstable. The stability of the both is always opposite.

We just analyse the stability of equilibrium E^+. By using the Routh-Hurwitz criterion, we can easily prove the equilibrium E^+ is stable while $c < \frac{2bm\sqrt{b}}{a(a+m)}$.

The formula (2.3) at equilibrium E^+ is just as follows:

$$\Delta_+(\lambda) = \lambda^3 + (m+a)\lambda^2 + (2b - \frac{ac}{\sqrt{b}})\lambda + 2ab = 0. \tag{2.4}$$

Let $A = m + a$, $B = 2b - \frac{ac}{\sqrt{b}}$, $C = 2ab$. On the basis of the Routh-Hurwitz criterions, with the equation (2.4), while $A > 0$, $C > 0$, $AB - C > 0$, i.e. $a > 0$, $b > 0$, and $c < \frac{2bm\sqrt{b}}{a(a+m)}$, all of the real parts of the three roots are negative. We take into account the characteristic equation (2.3) at equilibrium E^- in the same way.

3 Complex dynamic behavior

To research and investigate the dynamic behaviors of the controlled system (2.2), a host of numerical simulations are carried out. The obtained results reveal that the controlled system (2.2) exhibits very significant and complex dynamical behaviors.

In this part, we fix $a = 1.5$, $b = 1.7$, $c = 0.05$, while varying $m \in [-1.6, 0.27]$. By analyzing Lyapunov exponents spectrum and corresponding bifurcation diagram for (2.2) just as manifested in Fig.1 and Fig.2, one can get the following conclusions.

(1) When the parameters $m \in [-1.6, -1.52]$ and $m \in [-1.5, -1.49]$, the controlled system of maximum Lyapunov exponent is often equal to zero here. System (2.2) keeps periodic motion.

(2) When the parameters $m = -1.51$, there are two Lyapunov exponents equal to zero. Combining with the phase portrait shown in Fig.3, it easily seen that the controlled system performs a quasi periodic motion.

(3) When the parameters $m \in [-1.48, -1.44]$, $m \in [-1.42, -1.37]$ and $m \in [-1.35, 0.01]$, their maximum Lyapunov exponents are positive exponents; the fact indicates that the controlled system is chaotic.

(4) When the parameters $m = -1.43$, $m = -1.36$ and $m = 0.02$, their corresponding maximum Lyapunov exponents all are equal to zero. The controlled system performs a periodic motion. The phase portraits are shown in Figs.4, 5 and 6.

(5) When the parameter $m \in [0.03, 0.27]$, the controlled system keeps a cycle motion. The cycle gets lower as the parameter m increases, and the maximum Lyapunov exponents are always kept negative and get smaller, which implies that the controlled system tends to converge to a sink.

From the above results, we can find that, within $m \in [-1.6, 0.27]$, controlled system (2.2) has chaotic attractor, period montion and periodic windows. The behaviors of the system (2.2) become complex with varying m [22].

4 Hopf bifurcation analysis

In this part we discuss the Hopf bifurcation for the controlled system (2.2), applying normal form theory and Mathemaitca software.

Theorem 4.1. *Suppose that $a > 0$, $b > 0$, $2b > \frac{ac}{\sqrt{b}}$. While c goes through the critical value $c = c_0 = \frac{2bm\sqrt{b}}{a(a+m)}$, system (2.2) plays a Hopf bifurcation at the equilibrium $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$.*

Proof. Suppose that there is a pair of pure imaginary roots $\lambda = \pm i\omega (\omega \in \mathbb{R}^+)$ in characteristic equation.

We obtained

$$\Delta(\lambda) = (i\omega)^3 + (m+a)(i\omega)^2 + (2b - \frac{ac}{\sqrt{b}})i\omega + 2ab = 0.$$

Fig. 1. The Lyapunov exponents spectrum of system (2.2) with $a = 1.5$, $b = 1.7$, $c = 0.05$ and $m \in [-1.6, 0.27]$.

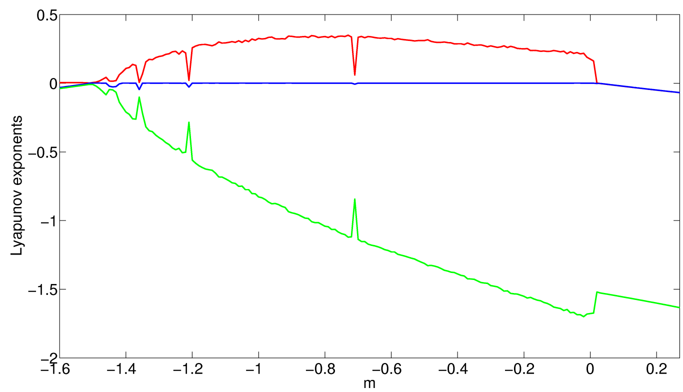


Fig. 2. Bifurcation diagram of system (2.2) with $a = 1.5$, $b = 1.7$, $c = 0.05$ and $m \in [-1.6,\ 0.27]$.

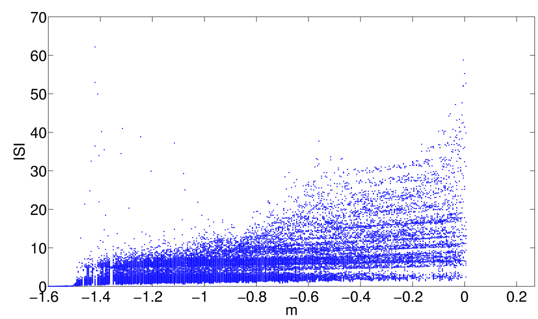
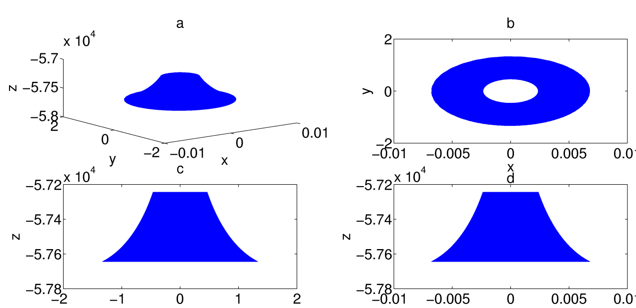


Fig. 3. The phase diagram of system (2.2) with $a = 1.5$, $b = 1.7$, $c = 0.05$ and $m = -1.51$.



Hence one can see that

$$\mathrm{Re}\Delta(\lambda) = 2ab - (a+m)\omega^2 = 0,$$
$$\mathrm{Im}\Delta(\lambda) = i(-\omega^3 + (2b - \frac{ac}{\sqrt{b}})\omega) = 0.$$

Thus we have

$$\omega^2 = \frac{2ab}{a+m},\ \omega^2 = 2b - \frac{ac}{\sqrt{b}}.$$

Then

$$\frac{2ab}{a+m} = 2b - \frac{ac}{\sqrt{b}}.$$

Fig. 4. The phase diagram of system (2.2) with $a = 1.5$, $b = 1.7$, $c = 0.05$ and $m = -1.43$.

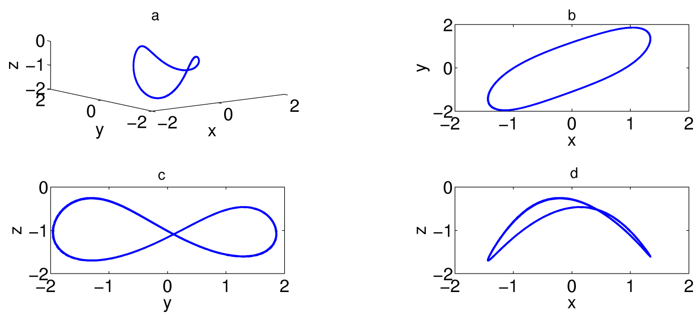


Fig. 5. The phase diagram of system (2.2) with $a = 1.5$, $b = 1.7$, $c = 0.05$ and $m = -1.36$.

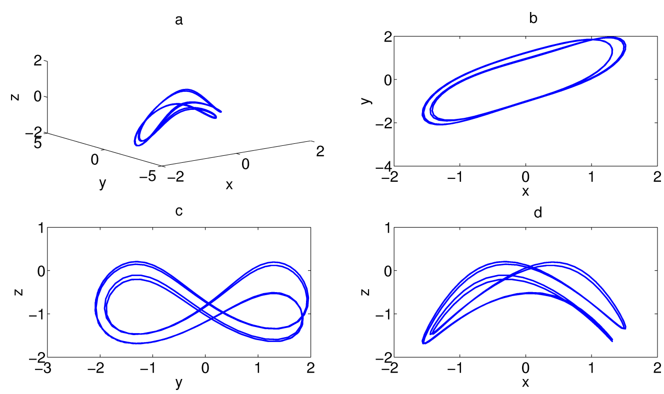
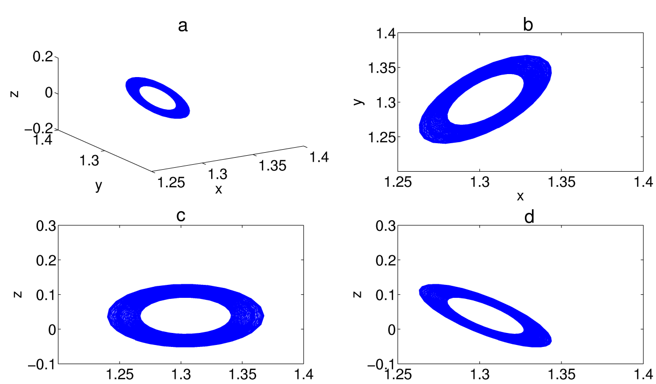


Fig. 6. The phase diagram of system (2.2) with $a = 1.5$, $b = 1.7$, $m = 0.02$ and $c = 0.05$.



By computing the equation, we get

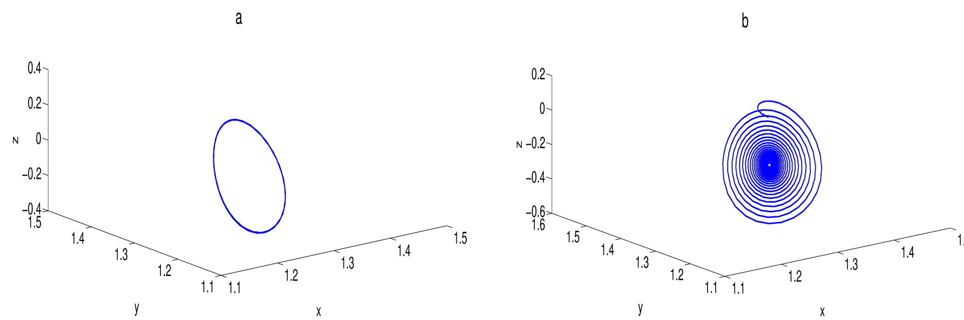
$$2b > \frac{ac}{\sqrt{b}}, \quad c_0 = \frac{2bm\sqrt{b}}{a(a+m)}.$$

When $c = c_0$, one obtains

$$\lambda_1 = i\omega, \quad \lambda_2 = -i\omega, \quad \lambda_3 = -a - m,$$

where $\omega = \sqrt{\frac{2ab}{a+m}}$. Thus, when $m > 0$, $c = c_0$, the first condition of the Hopf bifurcation is satisfied.

Fig. 7. (a) The phase diagram of system (2.2) with $a = 5$, $b = 1.7$, $m = -1$ and $c = -0.22$. (b) The phase diagram of system (2.2) with $a = 5$, $b = 1.7$, $m = -1$ and $c = -0.32$.



From $m > 0$, it follows that

$$0 = 3\lambda^2\lambda' + 2(a+m)\lambda\lambda' + (2b - \frac{ac}{\sqrt{b}})\lambda' - \frac{a}{\sqrt{b}}\lambda,$$

$$\lambda' = \frac{a\lambda}{[3\lambda^2+2(a+m)\lambda+2b]\sqrt{b}-ac},$$

which indicates that

$$\mathrm{Re}(\lambda'(c_0))|_{\lambda=i\omega} = -\frac{a(a+m)^2\sqrt{b}}{2b(a+m)^3+4ab^4} \neq 0.$$

Thus, the proof of Theorem 4.1 is completed [27]. □

Remark 4.2. *There is no Hopf bifurcation at the equilibrium* $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ *while* $2b < \frac{ac}{\sqrt{b}}$.

Next, the expression, stability and size of bifurcating period solution in system (2.2) are studied, by using the normal form theory [27] and many rigid symbolic computation.

Theorem 4.3. *If* $a > 0$, $b > 0$, *for the controlled system (2.2), bifurcating periodic solutions exist for sufficient small* $|c - c_0| > 0$ ($c_0 = \frac{2bm\sqrt{b}}{a(a+m)}$). *Furthermore, periodic solutions from Hopf bifurcation at* $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ *have the following properties:*
(i) if $M > 0$, *bifurcating periodic solutions of system (2.2) at* $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ *are non-degenerate, supercritical and unstable;*
(ii) if $M < 0$, *bifurcating periodic solutions of system (2.2) at* $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ *are non-degenerate, subcritical and stable;*
(iii) the characteristic exponent and period of the bifurcating periodic solution are

$$\beta = \beta_2\varepsilon^2 + O(\varepsilon^4), \quad T = \frac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2 + O(\varepsilon^4)),$$

where $\omega_0 = \sqrt{\frac{2ab}{a+m}}$, $\varepsilon^2 = \frac{c-c_0}{\mu_2} + O[(c-c_0)^2]$ *and*

$$\mu_2 = -\frac{ReC_1(0)}{\alpha'(0)} = \frac{M}{-\frac{a(a+m)^2\sqrt{b}}{4ab^4+2b(a+m)^3}},$$

$$\tau_2 = -\frac{\mu_2\omega'(0) + \mathrm{Im}\,C_1(0)}{\omega_0} = \frac{\mu_2 ab^2\sqrt{a}(a+m)}{2ab^2(a+m)^3 + 4a^2b^5} - \frac{N(a+m)}{\sqrt{2ab(a+m)}},$$

$$\beta_2 = 2ReC_1(0) = 2M,$$

where $M = \frac{Q_2Q_5+Q_1Q_6+\frac{1}{2}Im(g_{21})}{-2\sqrt{\frac{2ab}{a+m}}}$, $N = \frac{Q_1Q_5-Q_2Q_6-2(Q_1^2+Q_2^2)-\frac{1}{3}(Q_3^2+Q_4^2)+\frac{1}{2}Re(g_{21})}{2\sqrt{\frac{2ab}{a+m}}}$,
$Re(g_{21}) = (A_{13} + B_{23})w_{11} + \frac{1}{2}Re(w_{20})(A_{13} - B_{23}) - \frac{1}{2}Im(w_{20})(B_{13} + A_{23})$,

$Im(g_{21}) = (B_{13} - A_{23})w_{11} + \frac{1}{2}Im(w_{20})(A_{13} - B_{23}) - \frac{1}{2}Re(w_{20})(B_{13} + A_{23})$,
$Re(w_{20}) = \frac{(C11-C22)(a+m)-2C12\sqrt{\frac{2ab}{a+m}}}{(a+m)^2-\frac{8ab}{a+m}}$, $Im(w_{20}) = \frac{(C_{11}-C_{22})\sqrt{\frac{2ab}{a+m}}-(a+m)C_{12}}{(a+m)^2-\frac{8ab}{a+m}}$, $w_{11} = \frac{C_{11}+C_{22}}{-2(a+m)}$.
Q_i ($i = 1, 2, 3, \cdots, 6$), and $A_{13}, A_{23}, B_{13}, B_{23}, B_{13},$, are defined in Appendix.
(iv) the mathematical expression of Hopf bifurcation periodic solution of system (2.2) is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sqrt{b} \\ \sqrt{b} \\ \frac{c}{\sqrt{b}} \end{pmatrix} + \begin{pmatrix} \cos\frac{2\pi}{T}t \\ \sin\frac{2\pi}{T}t \\ 0 \end{pmatrix}\varepsilon + \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \frac{(C_{11}+C_{22})(1+Re(w_{20})\cos\frac{2\pi}{T}t-Im(w_{20})\sin\frac{2\pi}{T}t)}{-2(a+m)} \end{pmatrix}\varepsilon^2 + O(\varepsilon^3),$$

where $\gamma_1 = \frac{\sin\frac{4\pi}{T}t(Q_3+3Q_5)-\cos\frac{4\pi}{T}t(Q_4-3Q_6)-6Q_2}{6\sqrt{\frac{2ab}{a+m}}}$, $\gamma_2 = \frac{Q_3\cos\frac{4\pi}{T}t-3Q_5\cos\frac{4\pi}{T}t+6Q_1-Q_4\sin\frac{4\pi}{T}t-3Q_6\sin\frac{4\pi}{T}t}{6\sqrt{\frac{2ab}{a+m}}}$.
Q_i ($i = 1, 2, 3, \cdots, 6$) are defined in Appendix.

Proof. According to Theorem 4.1, while the parameter c changes and passes through the critical value c_0, Hopf bifurcation occurs at the equilibrium point E^+, and its direction and stability are discussed below.

Because equilibrium $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ is not in the origin, in order to shift it to the origin, let $x = \sqrt{b} + x_1$, $y = \sqrt{b} + y_1$ and $z = \frac{c}{\sqrt{b}} + z_1$. Then, system (2.2) becomes

$$\begin{cases} \dot{x_1} = a(y_1 - x_1) \\ \dot{y_1} = x_1 z_1 + \frac{c}{\sqrt{b}}x_1 + \sqrt{b}z_1 + m(x_1 - y_1) \\ \dot{z_1} = -y_1^2 - 2\sqrt{b}y_1. \end{cases}$$

Denoting c=c_0=$\frac{2bm\sqrt{b}}{a(a+m)}$, and linearizing system (2.2) at $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ now yields the Jacobian matrix

$$J = \begin{bmatrix} -a & a & 0 \\ m + \frac{c}{\sqrt{b}} & -m & \sqrt{b} \\ 0 & -2\sqrt{b} & 0 \end{bmatrix},$$

and according to the matrix calculated above, we can get its characteristic equation:

$$\Delta(\lambda) = \lambda^3 + (m + a)\lambda^2 + (2b - \frac{ac}{\sqrt{b}})\lambda + 2ab = 0.$$

By exact computations, we obtain λ_1=ωi, λ_2=$-\omega i$ and $\lambda_3 = -a - m$, when ω=$\sqrt{\frac{2ab}{a+m}}$, and the corresponding eigenvectors respectively are

$$v_1 = \begin{pmatrix} -a\sqrt{2b(a+m)} - \sqrt{a}(a^2 + am)i \\ -\sqrt{a}(a^2 + 2b + am)i \\ (a^2 + 2b + am)\sqrt{2(a+m)} \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2a\sqrt{b(a+m)} - \sqrt{2a}(a^2 + am)i \\ -\sqrt{a}(a^2 + 2b + am)i \\ 12(a^2 + 2b + am)\sqrt{a+m} \end{pmatrix}$$

$$\text{and } v_3 = \begin{pmatrix} -a(a+m) \\ m(a+m) \\ 2m\sqrt{b} \end{pmatrix}.$$

For system (2.2), define

$$\begin{aligned} \widetilde{P}_1 &= (\text{Re}v_1, -\text{Im}v_1, v_3) \\ &= \begin{pmatrix} -a\sqrt{2b(a+m)} & \sqrt{a}(a^2 + am) & -a(a+m) \\ 0 & \sqrt{a}(a^2 + 2b + am) & m(a+m) \\ (a^2 + 2b + am)\sqrt{2(a+m)} & 0 & 2m\sqrt{b} \end{pmatrix} \end{aligned} \tag{3.1}$$

and make a transformation as follows:

$$(x_1, y_1, z_1)^T = \widetilde{P}_1(x_2, y_2, z_2)^T.$$

Thus,

$$\begin{cases} \dot{x}_2 = -\sqrt{\frac{2ab}{a+m}}y_2 + F_1(x_2, y_2, z_2) \\ \dot{y}_2 = \sqrt{\frac{2ab}{a+m}}x_2 + F_2(x_2, y_2, z_2) \\ \dot{z}_2 = -(a+m)z_2 + F_3(x_2, y_2, z_2) \end{cases}$$

where

$$\begin{aligned} F_1(x_2, y_2, z_2) &= (A_{11}x_2^2 + A_{12}x_2y_2 + A_{22}y_2^2 + A_{13}x_2z_2 + A_{23}y_2z_2 + A_{33}z_2^2), \\ F_2(x_2, y_2, z_2) &= (B_{11}x_2^2 + B_{12}x_2y_2 + B_{22}y_2^2 + B_{13}x_2z_2 + B_{23}y_2z_2 + B_{33}z_2^2), \\ F_3(x_2, y_2, z_2) &= (C_{11}x_2^2 + C_{12}x_2y_2 + C_{22}y_2^2 + C_{13}x_2z_2 + C_{23}y_2z_2 + C_{33}z_2^2). \end{aligned}$$

Furthermore, the following marks are consistent with [28].

$$\begin{aligned} g_{11} &= \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial x_2^2} + \frac{\partial^2 F_1}{\partial y_2^2} + i\left(\frac{\partial^2 F_2}{\partial x_2^2} + \frac{\partial^2 F_2}{\partial y_2^2}\right)\right] = \frac{1}{2}[(A_{11} + A_{22}) + i(B_{11} + B_{22})], \\ g_{02} &= i\left(2\frac{\partial^2 F_1}{\partial x_2 \partial y_2} + \frac{\partial^2 F_2}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial y_2^2}\right) + \frac{1}{4}\left[\frac{\partial^2 F_1}{\partial x_2^2} - 2\frac{\partial^2 F_2}{\partial x_2 \partial y_2} - \frac{\partial^2 F_1}{\partial y_2^2}\right] \\ &= \frac{1}{2}[i(A_{12} + B_{11} - B_{22}) + (A_{11} - B_{12} - A_{22})], \\ g_{20} &= \frac{1}{4}\left[i\left(\frac{\partial^2 F_2}{\partial x_2^2} - \frac{\partial^2 F_2}{\partial y_2^2} - 2\frac{\partial^2 F_1}{\partial x_2 \partial y_2}\right) + \frac{\partial^2 F_1}{\partial x_2^2} + 2\frac{\partial^2 F_2}{\partial x_2 \partial y_2} - \frac{\partial^2 F_1}{\partial y_2^2}\right] \\ &= \frac{1}{2}[i(B_{11} - B_{22} - A_{12}) + (A_{11} + B_{12} - A_{22})], \\ G_{21} &= \frac{1}{8}\left[\frac{\partial^3 F_1}{\partial x_2^3} + \frac{\partial^3 F_1}{\partial x_2 \partial y_2^2} + \frac{\partial^3 F_2}{\partial x_2^2 \partial y_2} + \frac{\partial^3 F_2}{\partial y_2^3} + i\left(\frac{\partial^3 F_2}{\partial x_2 \partial y_2^2} - \frac{\partial^3 F_2}{\partial y_2^3} - \frac{\partial^3 F_1}{\partial x_2^2 \partial y_2} + \frac{\partial^3 F_2}{\partial x_2^3}\right)\right] = 0. \end{aligned}$$

Meanwhile, one has

$$\begin{aligned} h_{11}^1 &= \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_2^2} + \frac{\partial^2 F_3}{\partial y_2^2}\right) = \frac{1}{2}(C_{11} + C_{22}), \\ h_{20}^1 &= \frac{1}{4}\left(\frac{\partial^2 F_3}{\partial x_2^2} - \frac{\partial^2 F_3}{\partial y_2^2} - 2i\frac{\partial^2 F_3}{\partial x_2 \partial y_2}\right) = \frac{1}{2}(C_{11} - C_{22} - C_{12}i). \end{aligned}$$

Solving the following formula:

$$Dw_{11} = -h_{11}^1 \quad \text{and} \quad w_{20}(D - 2i\omega_0 I) = -h_{20}^1,$$

where $D = -(a+m)$, one obtains

$$w_{11} = \frac{C_{11} + C_{22}}{-2(a+m)}, \quad w_{20} = \frac{C_{11} - C_{22} - C_{12}i}{-2(a+m) - 4i\sqrt{\frac{2ab}{a+m}}} = \mathrm{Re}(w_{20}) + i\mathrm{Im}(w_{20}).$$

Where,

$$\mathrm{Re}(w_{20}) = \frac{(C_{11} - C_{22})(a+m) - 2C_{12}\sqrt{\frac{2ab}{a+m}}}{(a+m)^2 - \frac{8ab}{a+m}}, \mathrm{Im}(w_{20}) = \frac{(C_{11} - C_{22})\sqrt{\frac{2ab}{a+m}} - (a+m)C_{12}}{(a+m)^2 - \frac{8ab}{a+m}}.$$

Furthermore,

$$\begin{aligned} G_{110}^1 &= \frac{1}{2}\left[\left(i\left(\frac{\partial^2 F_2}{\partial x_2 \partial z_2} - \frac{\partial^2 F_1}{\partial y_2 \partial z_2}\right) + \frac{\partial^2 F_2}{\partial y_2 \partial z_2} + \frac{\partial^2 F_1}{\partial x_2 \partial z_2}\right)\right] \\ &= \frac{1}{2}[i(B_{13} - A_{23}) + B_{23} + A_{13}], \\ G_{101}^1 &= \frac{1}{2}\left[\left(i\left(\frac{\partial^2 F_2}{\partial x_2 \partial z_2} + \frac{\partial^2 F_2}{\partial y_2 \partial z_2}\right) + \frac{\partial^2 F_1}{\partial x_2 \partial z_2} - \frac{\partial^2 F_2}{\partial y_2 \partial z_2}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}[i(B_{13} + A_{23}) + A_{13} - B_{23}],\\
g_{21} &= G_{21} + (2G^1_{110}\omega_{11} + G^1_{101}\omega_{20})\\
&= \mathrm{Re}(g_{21}) + i\mathrm{Im}(g_{21}),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Re}(g_{21}) &= (A_{13} + B_{23})w_{11} + \frac{1}{2}\mathrm{Re}(w_{20})(A_{13} - B_{23}) - \frac{1}{2}\mathrm{Im}(w_{20})(B_{13} + A_{23}),\\
\mathrm{Im}(g_{21}) &= (B_{13} - A_{23})w_{11} + \frac{1}{2}\mathrm{Im}(w_{20})(A_{13} - B_{23}) - \frac{1}{2}\mathrm{Re}(w_{20})(B_{13} + A_{23}).
\end{aligned}
$$

Considering the above operation results, one can calculate the following formulas and get:

$$
\begin{aligned}
C_1(0) &= \frac{1}{2}g_{21} + \frac{i}{2\omega_0}\left(-2|g_{11}|^2 + g_{11}g_{20} - \frac{1}{3}|g_{02}|^2\right) = M + Ni,\\
\mu_2 &= -\frac{\mathrm{Re}\, C_1(0)}{\alpha'(0)} = -\frac{M}{\mathrm{Re}(\lambda'(c_0))},\\
\tau_2 &= -\frac{\mu_2\omega'(0) + \mathrm{Im}\, C_1(0)}{\omega_0} = -\frac{\mu_2\mathrm{Im}(\lambda'(c_0))) + N}{\omega_0},\ \beta_2 = 2\mathrm{Re}\, C_1(0) = 2M,
\end{aligned}
$$

where

$$
\omega_0 = \sqrt{\frac{2ab}{a+m}},\quad \alpha'(0) = \mathrm{Re}(\lambda'(c_0)) = -\frac{a(a+m)^2\sqrt{b}}{2b(a+m)^3 + 4ab^4},\quad \omega'(0) = -\frac{a\sqrt{2a(a+m)}}{2(a+m)^3 + 4ab^3}.
$$

$$
M = \frac{Q_2Q_5 + Q_1Q_6 + \frac{1}{2}\mathrm{Im}(g_{21})}{-2\sqrt{\frac{2ab}{a+m}}},\ N = \frac{Q_1Q_5 - Q_2Q_6 - 2(Q_1^2+Q_2^2) - \frac{1}{3}(Q_3^2+Q_4^2) + \frac{1}{2}\mathrm{Re}(g_{21})}{2\sqrt{\frac{2ab}{a+m}}}.
$$

From $a > 0$, $b > 0$, one gets that $\alpha'(0) < 0$, $\omega'(0)<0$ holds, so there are $\mu_2 > 0$ and $\beta_2 > 0$ with $M > 0$, which means that the Hopf bifurcation of system (2.2) at $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ is supercritical and non-degenerate, and the system's bifurcating periodic solution occurs and is unstable; if $M < 0$, then $\mu_2 < 0$ and $\beta_2 < 0$, which means that the Hopf bifurcation of system (2.2) at $E^+(\sqrt{b}, \sqrt{b}, \frac{c}{\sqrt{b}})$ is subcritical and non-degenerate, and the system's bifurcating periodic solution occurs and is stable.

Moreover, the characteristic exponent and period are

$$
\beta = \beta_2\varepsilon^2 + O(\varepsilon^4),\quad T = \frac{2\pi}{\omega_0}(1 + \tau_2\varepsilon^2 + O(\varepsilon^4)),
$$

where $\varepsilon^2 = \frac{c-c_0}{\mu_2} + O[(c-c_0)^2]$. The Hopf bifurcation periodic solution is

$$
X(x_1, y_1, z_1)^T = \widetilde{P}_1(x_2, y_2, z_2)^T = \widetilde{P}_1 Y,
$$

while $\widetilde{P}_1$ is a matrix and is described in (3.1),

$$
x_2 = \mathrm{Re}\, u,\ \ y_2 = \mathrm{Im}\, u,\ \ z_2 = \mathrm{Re}(\omega_{20}u^2) + |u|^2\omega_{11} + O(|u|^3),
$$

and

$$
u = e^{\frac{2it\pi}{T}}\varepsilon + \frac{\varepsilon^2 i}{6\omega_0}[6g_{11} + g_{02}e^{-\frac{4it\pi}{T}} - 3g_{20}e^{\frac{4it\pi}{T}}] + O(\varepsilon^3).
$$

By complicate computations, one gets

$$
\begin{pmatrix} x\\ y\\ z \end{pmatrix} = \begin{pmatrix} \sqrt{b}\\ \sqrt{b}\\ \frac{c}{\sqrt{b}} \end{pmatrix} + \begin{pmatrix} \cos\frac{2\pi}{T}t\\ \sin\frac{2\pi}{T}t\\ 0 \end{pmatrix}\varepsilon + \begin{pmatrix} \gamma_1\\ \gamma_2\\ \frac{(C_{11}+C_{22})(1+\mathrm{Re}(w_{20})\cos\frac{2\pi}{T}t - \mathrm{Im}(w_{20})\sin\frac{2\pi}{T}t)}{-2(a+m)} \end{pmatrix}\varepsilon^2 + O(\varepsilon^3),
$$

where γ_1, γ_2 are defined at Theorem 4.3. □

Remark 4.4. *The property of Hopf bifurcation at equilibrium E^- can be considered in the same way.*

Fix $a = 5$, $b = 1.7$, $m = -1$ and the critical value $c = c_0 = \frac{2bm\sqrt{b}}{a(a+m)} \approx -0.222$. From the above theorems, near the equilibrium point E^+, there are bifurcation periodic solutions for the controlled system (2.2) while $c > c_0$; E^+ is stable while $c < c_0$, just as demonstrated in Fig.7.

5 Conclusions

In this paper the chaos and bifurcation are explored for the controlled chaotic system, which is put forward based on the hybrid strategy in an unusual chaotic system. Behaviors of the controlled system are studied through analyzing its corresponding bifurcation diagrams and Lyapunov exponents, showing that there exist chaotic area, period area and several periodic windows within the range of variable parameter. The results show that the controlled system maintains the local structure and dimension of the original system, and the dynamics behavior of the controlled system is much richer than the original system. Furthermore, a sufficient condition for the existence of Hopf bifurcation of the controlled chaotic system is obtained. Using the normal form theory, the direction and stability of bifurcating periodic solution are verified. Numerical simulations demonstrate the theoretical analysis results.

Appendix

For convenience, some values of the symbols used in Theorem 4.3 are given in this section.

$$
\begin{aligned}
\tilde{D} &= -2bm + (a+m)(2b + (a+m)^2),\ \ D_1 = [2b + a(a+m)]\sqrt{2(a+m)\tilde{D}}:\\
D_2 &= (a^2 + 2b + am)\sqrt{a(a+m)\tilde{D}},\ \ D_3 = \sqrt{a+m\tilde{D}},\\
A_{11} &= \frac{4abm\sqrt{a+m}}{\sqrt{2\tilde{D}}},\ \ A_{12} = \frac{-2am\sqrt{ab}[2b(a+m) + a(a-m)^2]}{2b + a(a+m)\tilde{D}},\\
A_{22} &= \frac{-a[2b + a(a+m)][2b + (a+m)^2]}{\sqrt{2(a+m)\tilde{D}}},\ \ A_{13} = \frac{2am\sqrt{b}[a(a+m)^2 + 2b(a+m)]}{[2b + a(a+m)]\tilde{D}},\\
A_{23} &= \frac{-2m\sqrt{a(a+m)}[(a+m)(4ab + a(a+m)^2) + 2b(b+m^2)]}{\sqrt{2}[2b + a(a+m)]\tilde{D}},\\
A_{33} &= \frac{-m^2\sqrt{a+m}[(a+m)^3 - 2b(a-m)]}{\sqrt{2}[2b + a(a+m)]\tilde{D}},\ \ B_{11} = \frac{-2a^2\sqrt{b(a+m)}[2b + (a+m)^2]}{\sqrt{a\tilde{D}}}:\\
B_{12} &= \frac{\sqrt{2}a^2(s+m)(2b(a+m)^2)}{\tilde{D}},\ \ B_{22} = \frac{-am\sqrt{b(a+m)}(2b + a(a+m))}{\sqrt{a\tilde{D}}},\\
B_{13} &= \frac{-\sqrt{2}a^2[(a+m)^2 + 2b][a(a+m)^2 + am^2 + 2b(a+m)]}{\sqrt{a}(a^2 + 2b + am)\tilde{D}},\\
B_{23} &= \frac{2m\sqrt{ab(a+m)}[a^2(a^2 + 2b + am) - m(a+m)(2b - am)]}{\sqrt{a}(a^2 + 2b + am)\tilde{D}},\\
B_{33} &= \frac{-m\sqrt{b(a+m)}[(a+m)^2(2a^2 + m^2) + 4a^2b]}{\sqrt{a}(a^2 + 2b + am)\tilde{D}},\ \ C_{11} = \frac{-2a\sqrt{b}(2b + a(a+m))(a+m)^2}{\tilde{D}},\\
C_{12} &= \frac{a(a+m)^2\sqrt{2a(a+m)}[(2b + a(a+m)]}{\tilde{D}},\ \ C_{22} = \frac{a\sqrt{b}[2b + a(a+m)]^2}{\tilde{D}},\\
C_{13} &= \frac{-a\sqrt{2(a+m)}[(a+m)^2(a + 2b + am) + 2bm(a+m)]}{\tilde{D}},\ \ C_{23} = \frac{4m(a+m)\sqrt{ab}[b + a(a+m)]}{\tilde{D}},\\
C_{33} &= \frac{m\sqrt{b}(a+m)^2(m-2a)}{\tilde{D}},\ \ Q_1 = \frac{1}{2}(A_{11} + A_{22}),\ \ Q_2 = \frac{1}{2}(B_{11} + B_{22}),\\
Q_3 &= \frac{1}{2}(A_{11} - B_{12} - A_{22}),\ \ Q_4 = \frac{1}{2}(B_{11} + A_{12} - B_{22}),\\
Q_5 &= \frac{1}{2}(A_{11} + B_{12} - A_{22}),\ \ Q_6 = \frac{1}{2}(B_{11} - A_{12} - B_{22}).
\end{aligned}
$$

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