Open Mathematics

Research Article

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Semi-Hurewicz-Type properties in ditopological texture spaces

https://doi.org/10.1515/math-2018-0104 Received January 11, 2018; accepted September 14, 2018.

Abstract: In this paper we will define and discuss semi-Hurewicz type covering properties in ditopological texture spaces. We consider the behaviour of semi-Hurewicz and co-semi-Hurewicz selection properties under direlation and difunction between ditopological texture spaces.

Keywords: Semi-open set, Ditopological texture space, Semi-compact space, Semi-cocompact space, s-Hurewicz space, co-s-Hurewicz space

MSC: 54A05, 54A10, 54C08, 54H12, 54D20

1 Introduction

The theory of selection principles can be traced back to the first half of 19th century. The general form of classical selection principles in topological spaces have been defined as follows:

Let S be an infinite set and let A and B be collections of subsets of S. Then the symbol $S_1(A, B)$ defines the statement:

For each sequence $(A_n)_{n<\infty}$ of elements of \mathcal{A} there is a sequence $(b_n)_{n<\infty}$ such that for each n, we have $b_n \in A_n$, and $\{b_n\}_{n<\infty} \in \mathcal{B}$.

If \mathbb{O} denotes the collection of open covers of a topological space (X, T) then the property $S_1(\mathbb{O}, \mathbb{O})$ is called the *Rothberger covering property* and was introduced by Rothberger in [1].

Similarly, the selection hypothesis $S_{fin}(A, B)$ is defined as:

For each sequence $(A_n)_{n<\infty}$ of elements of $\mathcal A$ there is a sequence $(B_n)_{n<\infty}$ such that for each n, we have B_n is finite subset of A_n , and $\bigcup B_n \in \mathcal B$.

The property introduced in [2] by K. Menger in 1924 is equivalent to $S_{fin}(\mathcal{O}, \mathcal{O})$ and was proved by W. Hurewicz in [3] in 1925. The property $S_{fin}(\mathcal{O}, \mathcal{O})$ is known as the *Menger covering property*.

In 2016, Sabah et al. in [4] proved that X has the s-Menger (resp. s-Rothberger covering property [5]) in topological space X, if X satisfies $S_{fin}(s \circ 0, s \circ 0)$ (resp. $S_1(s \circ 0, s \circ 0)$) where $s \circ 0$ denotes the family of all semi-open covers of X.

For readers more interested in theory of selection principles and its relations with various branches of mathematics, we refer to see [6-10].

In this paper we study the properties of ditopological texture spaces related to the following classical *Hurewicz property* [3] $U_{fin}(A, B)$: For each sequence $\{A_n\}_{n\in\mathbb{N}}$ of elements of A, which do not contain a finite

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sub-cover, there exist finite (possibly empty) subsets $B_n \subseteq A_n$, $n \in \mathbb{N}$ and $\{\bigcup B_n\}_{n \in \mathbb{N}} \in \mathcal{B}$. It was shown in [11] that the Hurewicz property is of the S_{fin} -type for appropriate classes of \mathcal{A} and \mathcal{B} .

2 Preliminaries

L. M. Brown in 1992 at a conference on Fuzzy systems and artificial intelligence held in Trabzon introduced the notion of a texture space under the name fuzzy structure. Textures first arose in representation of connection of lattices of L – fuzzy sets and Hutton algebras under a point based settings. This representation provided a fruitful atmosphere to study complement free concepts in mathematics. We now recall the definition of the texture space as follows.

Texture space: [12] If *S* is a set, a texturing $\Im \subseteq P(S)$ is complete, point separating, completely distributive lattice containing *S* and \emptyset , and, for which finite join \bigvee coincides with union \bigcup and arbitrary meet \bigwedge coincides with intersection \bigcap . Then the pair (S, \Im) is called the texture space.

A mapping $\sigma: \Im \to \Im$ satisfying $\sigma^2(A) = A$, for each $A \in \Im$ and $A \subseteq B$ implies $\sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \Im$ is called a complementation on (S, \Im) and (S, \Im, σ) is then said to be a complemented texture [12]. The sets $P_S = \bigcap \{A \in \Im \mid s \in A\}$ and $Q_S = \bigvee \{P_t \mid t \in S, s \notin P_t\}$ defines conveniently most of the properties of the texture space and are known as *p-sets* and *q-sets* respectively.

For $A \in \Im$ the core A^b of A is defined by $A^b = \{s \in S \mid A \nsubseteq Q_s\}$. The set A^b does not necessarily belong to \Im .

If (S, P(s)), (\mathcal{L}, \Im_2) are textures, then the product texture of (S, P(S)) and (\mathcal{L}, \Im_2) is $P(S) \otimes \Im_2$ for which $\overline{P}_{(s,t)}$ and $\overline{Q}_{(s,t)}$ denotes the *p-sets* and *q-sets* respectively. For $s \in S$, $t \in \mathcal{L}$ we have p-sets and q-sets in the product space as following:

$$\overline{P}_{(s,t)} = \{s\} \times P_t$$

$$\overline{Q}_{(s,t)} = (S \setminus \{s\} \times T) \cup (S \times Q_t).$$

Direlation: [13] Let (S, \Im_1) , (\mathcal{L}, \Im_2) be textures. Then for $r \in \mathcal{P}(S) \otimes \Im_2$ satisfying:

 (R_1) $r \nsubseteq \overline{Q}_{(s,t)}$ and $P_{\hat{s}} \nsubseteq Q_s$ implies $r \nsubseteq \overline{Q}_{(\hat{s},t)}$,

 (R_2) $r \nsubseteq \overline{Q}_{(s,t)}$ then there is $\hat{s} \in S$ such that $P_s \nsubseteq Q_{\hat{s}}$ and $r \nsubseteq \overline{Q}_{(\hat{s},t)}$,

is called *relation* and for $R \in \mathcal{P}(S) \otimes \mathcal{F}_2$ such that:

 $(CR_1) \overline{P}_{(s,t)} \nsubseteq R$ and $P_s \nsubseteq Q_{\hat{s}}$ implies $\overline{P}_{(\hat{s},t)} \nsubseteq R$,

(CR₂) If $\overline{P}_{(s,t)} \nsubseteq R$ then there exists $\hat{s} \in S$ such that $P_{\hat{s}} \nsubseteq Q_s$ and $\overline{P}_{(\hat{s},t)} \nsubseteq R$,

is called a *corelation* from (S, P(S)) to (\mathcal{L}, \Im_2) . The pair (r, R) together is a direlation from (S, \Im_1) to (\mathcal{L}, \Im_2) .

Lemma 2.1 ([13]). *Let* (r, R) *be a direlation from* (S, \Im_1) *to* (T, \Im_2) , J *be an index set,* $A_j \in \Im_1$, $\forall j \in J$ *and* $B_j \in \Im_2$, $\forall j \in J$. *Then:*

(1)
$$r \leftarrow (\bigcap_{j \in J} B_j) = \bigcap_{j \in J} r \leftarrow B_j \text{ and } R \rightarrow (\bigcap_{j \in J} A_j) = \bigcap_{j \in J} R \rightarrow A_j,$$

(2) $r \rightarrow (\bigvee_{j \in J} A_j) = \bigvee_{j \in J} r \rightarrow A_j \text{ and } R \leftarrow (\bigvee_{j \in J} B_j) = \bigvee_{j \in J} R \leftarrow B_j.$

Difunction: Let (f, F) be a direlation from (S, \Im_1) to (\mathcal{L}, \Im_2) . Then $(f, F) : (S, \Im_1) \to (\mathcal{L}, \Im_2)$ is a difunction if it satisfies the following two conditions:

(DF1) For
$$s, \hat{s} \in S$$
, $P_s \nsubseteq Q_{\hat{s}} \Longrightarrow t \in \mathcal{L}$ with $f \nsubseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(\hat{s},t)} \nsubseteq F$.

(DF2) For
$$t, t' \in \mathcal{L}$$
 and $s \in S, f \nsubseteq Q_{(s,t)}$ and $\overline{P}_{(s,t')} \nsubseteq F \Longrightarrow P_{t'} \nsubseteq Q_{t}$.

Definition 2.2 ([13]). Let $(f, F) : (S, \Im_1) \to (\mathcal{L}, \Im_2)$ be a diffunction. For $A \in \Im_1$, the image $f \to (A)$ and coimage $F \to (A)$ are defined as:

$$f \xrightarrow{\longrightarrow} (A) = \bigcap \{Q_t : \forall s, f \nsubseteq \overline{Q}_{(s,t)} \Longrightarrow A \subseteq Q_s\},$$

$$F \rightarrow (A) = \bigvee \{P_t : \forall s, \ \overline{P}_{(s,t)} \not\subseteq F \Longrightarrow P_s \subseteq A\},$$

and for $B \in \Im_2$, the inverse image $f \leftarrow (B)$ and inverse coimage $F \leftarrow (B)$ are defined as:

$$f \leftarrow (B) = \bigvee \{P_s : \forall t, f \nsubseteq \overline{Q}_{(s,t)} \Longrightarrow P_t \subseteq B\},$$

$$F \leftarrow (B) = \bigcap \{Q_s : \forall t, \ \overline{P}_{(s,t)} \nsubseteq F \Longrightarrow B \subseteq Q_t\}.$$

For a difunction, the inverse image and the inverse coimage are equal, but the image and coimage are usually not.

Lemma 2.3 ([13]). For a direlation (f, F) from (S, \Im_1) to (T, \Im_2) the following are equivalent:

- (1) (f, F) is a direlation.
- (2) The following inclusion holds:
- (a) $f \leftarrow (F \rightarrow (A)) \subseteq A \subseteq F \leftarrow (f \rightarrow (A)); \forall A \in \mathcal{F}_1$, and
- (b) $f \rightarrow (F \leftarrow (B)) \subseteq B \subseteq F \rightarrow (f \leftarrow (B)); \forall B \in \Im_2$
- (3) $f \leftarrow (B) = F \leftarrow (B); \forall B \in \mathfrak{F}_2.$

Definition 2.4 ([13]). Let $(f, F) : (S, \Im_1) \to (\mathcal{L}, \Im_2)$ be a diffunction. Then (f, F) is called surjective if it satisfies the condition:

(SUR) For $t, t' \in \mathcal{L}$, $P_t \nsubseteq Q_{t'} \Longrightarrow \exists s \in S, f \nsubseteq Q_{(s,t')}$ and $\overline{P}_{(s,t)} \nsubseteq F$.

Similarly, (f, F) is called injective if it satisfies the condition

(INJ) For $s, \hat{s} \in S$, and $t \in \mathcal{L}$ with $f \nsubseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(\hat{s},t)} \nsubseteq F \Longrightarrow P_s \nsubseteq Q_{\hat{s}}$.

We now recall the notion of ditopology on texture spaces.

Definition 2.5 ([14]). *A pair* (τ, κ) *of subsets of* \Im *is said to be a ditopology on a texture space* (S, \Im) , *if* $\tau \subseteq \Im$ *satisfies:*

- (1) $S, \emptyset \in \tau$.
- (2) $G_1, G_2 \in \tau$ implies $G_1 \cap G_2 \in \tau$ and
- (3) $G_{\alpha} \in \tau$, $\alpha \in I$ implies $\bigvee_{\alpha} G_{\alpha} \in \tau$,

and $\kappa \subseteq \Im$ satisfies:

- (1) $S, \emptyset \in \kappa$.
- (2) $F_1, F_2 \in \kappa$ implies $F_1 \cup F_2 \in \kappa$ and
- (3) $F_{\alpha} \in \kappa$, $\alpha \in I$ implies $\bigcap F_{\alpha} \in \kappa$,

where the members of τ are called open sets and members of κ are closed sets. Also τ is called topology, κ is called cotopology and (τ, κ) is called ditopology. If (τ, κ) is a ditopology on (S, \Im) then (S, \Im, τ, κ) is called a ditopological texture space.

Note that in general we assume no relation between the open and closed sets in ditopology. In case of complemented texture space (S, \Im, σ) , τ and κ are connected by the relation $\kappa = \sigma(\tau)$, where σ is a complementation on (S, \Im) , that is an inclusion reversing involution $\sigma: \Im \to \Im$, then we call (τ, κ) a complemented ditopology on (S, \Im) . A complemented ditopological texture space is denoted by $(S, \Im, \sigma, \tau, \kappa)$. In this case we have $\sigma(\overline{A}) = (\sigma(A))^\circ$ and $\sigma(A^\circ) = \overline{(\sigma(A))}$, where ()° denotes the interior and $\overline{(O)}$ denotes the closure. Recall that for a ditopology (τ, κ) on (S, \Im) , for $A \in \Im$ the closure of A for the ditopology (τ, κ) is denoted by $\overline{(A)}$ and defined by

$$(\overline{A}) = \bigcap \{F \in \kappa : A \subseteq F\},$$

and the interior of A is denoted by $(A)^{\circ}$ and defined by

$$(A)^{\circ} = \bigvee \{G \in \tau : G \subseteq A\}$$

For terms not defined here, the reader is referred to see [6, 13, 15].

The idea of semi-open sets in topological spaces was first introduced by Norman Levine in 1963 in [16]. § Dost extended this concept of semi-open sets from topological spaces to ditopological texture spaces in 2012 in [17].

It is known from [17] that in a ditopological texture space (S, \Im, τ, κ) :

- 1. $A \in \Im$ is semi-open if and only if there exists a set $G \in O(S)$ such that $G \subseteq A \subseteq \overline{G}$.
- 2. $B \in \Im$ is semi-closed if and only if there exists a set $F \in C(S)$ such that $(F)^{\circ} \subseteq B \subseteq F$.
- 3. $O(S) \subseteq SO(S)$ and $C(S) \subseteq SC(S)$. The collection of all semi-open (resp. semi-closed) sets in \Im is denoted by $SO(S, \Im, \tau, \kappa)$ or simply SO(S) (resp. $SC(S, \Im, \tau, \kappa)$ or simply SC(S)). SR(S) is the collection of all the semi-regular sets in S. A set A is semi-regular if A is semi-open as well as semi-closed in S.
- 4. Arbitrary join of semi-open sets is semi-open.
- 5. Arbitrary intersection of semi-closed sets is semi-closed.

If A is semi-open in ditopoloical texture space (S, \Im, τ, κ) then its complement may not be *semi-closed*. Every open set is semi-open, whereas a semi-open set may not be open. The intersection of two semi-open sets may not be semi-open, but intersection of an open set and a semi-open set is always semi-open.

In general there is no connection between the semi-open and semi-closed sets, but in case of complemented ditopological texture space $(S, \Im, \sigma, \tau, \kappa)$, $A \in \Im$ is semi-open if and only if $\sigma(A)$ is semi-closed. Where $()_{\circ}$ denotes the semi-Interior and () denotes the semi-closure.

Definition 2.6 ([18]). Let (S, \Im, τ, κ) be a ditopological texture space and $A \in \Im$. We define:

(i) The semi-closure (\underline{A}) of A under (τ , κ) by

$$(\underline{A}) = \bigcap \{B : B \in SC(S), and A \subseteq B\}$$

(ii) The semi-interior (A) $_{\circ}$ of A under (τ, κ) by

$$(A)_{\circ} = \bigvee \{B : B \in SO(S), and B \subseteq A\}.$$

Lemma 2.7 ([18]). Let (S, \Im, τ, κ) be a ditopological texture space. A set $A \in \Im$ is called:

- (a) semi-open if and only if $A \subseteq \overline{(A^{\circ})}$
- *(b)* semi-closed if and only if $(\overline{A})^{\circ} \subset A$.

A diffunction $(f, F): (S, \Im_1, \tau_S, \kappa_S) \to (T, \Im_2, \tau_T, \kappa_T)$ is:

- (i) continuous[17]; if $F^{\leftarrow}(G) \in \tau_S$ where $G \in \tau_T$;
- (ii) cocontinuous [17]; if $f^{\leftarrow}(K) \in \kappa_S$ where $K \in \kappa_T$;
- (iii) bicontinuous [17]; if it is continuous and cocontinuous.

Definition 2.8. [17] Let $(S_i, \Im_i, \tau_i, \kappa_i)$, i = 1, 2 be ditoplogical texture spaces. A diffunction $(f, F) : (S_1, \Im_1) \to (S_2, \Im_2)$ is said to be:

- (i) semi-continuous (semi-irresolute) if for each open (resp. semi-open) set $A \in \Im_2$, the inverse image $F^{\leftarrow}(A) \in \Im_1$ is a semi-open set.
- (ii) semi-cocontinuous (semi-co-irresolute) if for each closed (resp. semi-closed) set $B \in \Im_2$, the inverse image $f \in B \in \Im_1$ is a semi-closed set.
- (iii) semi-bicontinuous if it semi-continuous and semi-cocontinuous.
- (iv) semi-bi-irresolute if it is semi-irresolute and semi-co-irresolute

Throughout this paper a space *S* is an infinite ditopological texture space (S, \Im , τ , κ) on which no separation axioms are assumed unless otherwise stated.

3 Selection properties of ditopological texture spaces

In 2017 Kočinac and özçağ [19] introduced Hurewicz type properties in texture and ditopological texture spaces as a continuation of the studies of selection properties of texture structures initiated in 2015 in [14]. In this section, semi-Hurewicz selection properties of ditopological texture spaces are defined and studied.

Definition 3.1. Let (τ, κ) be a ditopology on the texture space (S, \Im) and take $A \in \Im$. The family $\{G_\alpha : \alpha \in \nabla\}$ is said to be semi-open cover of A if $G_{\alpha} \in SO(S)$ for all $\alpha \in \nabla$, and $A \subseteq \vee_{\alpha \in \nabla} G_{\alpha}$. Dually, we may speak of a semi-closed cocover of A, namely a family $\{\mathcal{F}_{\alpha}: \alpha \in \nabla\}$ with $\mathcal{F}_{\alpha} \in SC(S)$ satisfying $\cap_{\alpha \in \nabla} \mathcal{F}_{\alpha} \subseteq A$.

Let sO denote the collection of all semi-open covers of a ditopological texture space (S, \Im, τ, κ) . Note that the class of semi-open covers contains the class of open covers of the ditopological texture space (S, \Im, τ, κ) .

Definition 3.2. Let (S, \Im, τ, κ) be a ditopological texture space and A be a subset of S.

- 1. A is said to have the semi-Hurewicz property (or s-Hurewicz property) for the ditopological texture space if for each sequence $(U_n)_{n\in\mathbb{N}}$ of semi-open covers of A there is a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$, \mathcal{V}_n is a finite subset of \mathcal{U}_n and $\bigvee_{n\in\mathbb{N}}\bigcap_{m>n}(\bigcup\mathcal{V}_m)$ is a cover of A. We say that (S,\Im,τ,κ) is s-Hurewicz if the set *S* is s-Hurewicz. This property is denoted by $U_{fin}(sO,sO)$.
- 2. A is said to have the co-semi-Hurewicz property (or co-s-Hurewicz property) for the ditopological texture space if for each sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of semi-closed cocovers of A there is a sequence $(\kappa_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, κ_n is a finite subset of \mathfrak{T}_n and $\bigcap_{n \in \mathbb{N}} \bigvee_{m > n} (\bigcap \kappa_m) \subseteq A$. We say that (S, \Im, τ, κ) is co-s-Hurewicz if the set \emptyset is co-s-Hurewicz. This property is denoted by $U_{cfin}(s\mathbb{C},s\mathbb{C})$, where $s\mathbb{C}$ is the family of all semi-closed cocovers of \emptyset .

Every s-Hurewicz ditopological texture space (S, \Im, τ, κ) is a Hurewicz ditopological texture space and co-s-Hurewicz ditopological texture space (S, \Im, τ, κ) is co-Hurewicz ditopological texture space. Converses are not true in general.

Definition 3.3 ([20]). Let (τ, κ) be a ditopology on the texture space (S, \Im) and take $A \in \Im$.

(1) A is said to be s-compact if whenever $\{G_{\alpha}: \alpha \in \nabla\}$ is semi-open cover of A, there is a finite subset ∇_0 of ∇ , with $A \subseteq \vee_{\alpha \in \nabla_0} G_{\alpha}$.

The ditopological texture space (S, \Im, τ, κ) is s-compact if S is s-compact. Every s-compact space in the ditopologcal texture space is compact but not conversely.

(2) A is said to be s-cocompact if whenever $\{\mathcal{F}_{\alpha}: \alpha \in \nabla\}$ is a semi-closed cocover of A, there is a finite subset ∇_0 of ∇ , with $\cap_{\alpha \in \nabla_0} \mathfrak{F}_\alpha \subseteq A$. In particular, the ditopolgical texture space (S, \Im, τ, κ) is s-cocompact if \emptyset is s-cocompact. It is clear that s-cocompact ditopological texture space is cocompact.

In general s-compactness and s-cocompactness are independent.

Definition 3.4. A ditopological texture space (S, \Im, τ, κ) is said to be σ -s-compact (resp. σ -s-cocompact) if there is a sequence $(A_n : n \in \mathbb{N})$ of s-compact (s-cocompact) subsets of S such that $\vee_{n \in \mathbb{N}} A_n = S$ (resp. $\cap_{n\in\mathbb{N}}A_n=\emptyset$).

Theorem 3.5. Let (S, \Im, τ, κ) be a ditopological texture space.

- 1. If (S, \Im, τ, κ) is σ -s-compact, then (S, \Im, τ, κ) has the s-Hurewicz property.
- 2. If (S, \Im, τ, κ) is σ -s-cocompact, then (S, \Im, τ, κ) has the co-s-Hurewicz property.

Proof. (1). Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of semi-open covers of S. Since S is σ -s-compact therefore it can be represented in the form $S = \bigvee_{i \in \mathbb{N}} A_i$, where each A_i is s-compact $A_i \subseteq A_{i+1}$ for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose a finite set $\mathcal{V}_i \subseteq \mathcal{U}_i$ such that $A_i \subseteq \vee \mathcal{V}_i = \cup \mathcal{V}_i$. Then the sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ shows that S is s-Hurewicz.

(2). Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of semi-closed cocovers of \emptyset . We have $\emptyset = \cap_{i\in\mathbb{N}}A_i$, where each A_i is s-cocompact and $A_i \supseteq A_{i+1}$, $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose a finite subset $\kappa_i \subseteq \mathcal{F}_i$ such that $\cap \kappa_i \subseteq A_i$. Then $\cap_{i\in\mathbb{N}}\vee_{m>i}\cap\kappa_m\subseteq\cap_{i\in\mathbb{N}}A_i=\emptyset$, which means that S is co-s-Hurewicz.

For complemented ditopological texture spaces we have:

Theorem 3.6. Let (S, \Im, σ) be a texture with the complementation σ and let (τ, κ) be a complemented ditopology on (S, \Im, σ) . Then $S \in \mathcal{U}_{fin}(s\mathbb{O}, s\mathbb{O})$ if and only if $\emptyset \in \mathcal{U}_{cfin}(s\mathbb{C}, s\mathbb{C})$.

Proof. Let $S \in \mathcal{U}_{fin}(s\mathcal{O}, s\mathcal{O})$ and let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a sequence of semi-closed cocovers of \emptyset . Then $(\sigma(\mathcal{F}_n) = \{\sigma(F) : F \in \mathcal{F}_n\})$ and $(\sigma(\mathcal{F}_n)_{n \in \mathbb{N}})$ is a sequence of semi-open covers of S. Since $S \in \mathcal{U}_{fin}(s\mathcal{O}, s\mathcal{O})$, there is a sequence $(\mathcal{V}_n)_{n \in \mathbb{N}}$ of finite sets such that for each n, $\mathcal{V}_n \subseteq \sigma(\mathcal{F}_n)$ and $\vee_{n \in \mathbb{N}} \cap_{m > n} (\vee \mathcal{V}_m) = S$. We have $(\sigma(\mathcal{V}_n) : n \in \mathbb{N})$ is a sequence of finite sets, and also

$$\emptyset = \sigma(S) = \sigma(\bigvee_{n \in \mathbb{N}} \bigcap_{m > n} (\vee \mathcal{V}_m)) = \bigcap_{n \in \mathbb{N}} \bigvee_{m > n} (\cap \sigma(\mathcal{V}_m))$$

Hence $\emptyset \in \mathcal{U}_{cfin}(s\mathbb{C}, s\mathbb{C})$.

Conversely let $\emptyset \in \mathcal{U}_{cfin}(s\mathbb{C}, s\mathbb{C})$ and $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a sequence of semi-open covers of S. Then $(\sigma(\mathcal{U}_n) = \{\sigma(U) : U \in \mathcal{U}_n\})$ and $(\sigma(\mathcal{U}_n)_{n \in \mathbb{N}})$ is a sequence of semi-closed cocovers of S. Since S is a sequence S, there is a sequence S is a sequence S in S

$$S = \sigma(\emptyset) = \sigma(\bigcap_{n \in \mathbb{N}^{m > n}} (\cap \kappa_m)) = \bigvee_{n \in Nm > n} (\vee \sigma(\kappa_m))$$

Hence $S \in \mathcal{U}_{fin}(s\mathfrak{O}, s\mathfrak{O})$.

Example 3.7. *There is a ditopological texture space which is s-Hurewicz, but not s-compact.*

Let $(\mathbb{R}, \Re, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ be the real line with the texture $\Re = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$, topology $\tau_{\mathbb{R}} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. This ditopological texture space is not compact by (Example 2.3 [19]) and hence not s-compact because the (open and hence) semi-open cover $\mathbb{U} = \{(-\infty, n) : n \in \mathbb{N}\}$ does not contain a finite subcover, nor the texture space is s-cocompact because its (closed and hence) semi-closed cocover $\{(-\infty, n] : n \in \mathbb{N}\}$ does not contain a finite cocover. But $(\mathbb{R}, \Re, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is s-Hurewicz and co-s-Hurewicz. Let us prove that this space is s-Hurewicz. Let $(\mathbb{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of \mathbb{R} . We note that semi-open sets can be of the form $(-\infty, r)$ and $(-\infty, r]$. Write $\mathbb{R} = \cup \{(-\infty, n] : n \in \mathbb{N}\}$. For each n, \mathbb{U}_n is a semi-open cover of \mathbb{R} , hence there is some $r_n \in \mathbb{R}$ such that $(-\infty, r_n] \subseteq (-\infty, n] \in \mathbb{U}_n$. Then the collection $\{(-\infty, r_n] : n \in \mathbb{N}\}$ shows that $(\mathbb{R}, \Re, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is s-Hurewicz.

Evidently we have the following diagram:

Theorem 3.8. Let (S, \Im, σ) be a texture space with complementation σ and let (τ, κ) be a complemented ditopology on (S, \Im, σ) . Then for $K \in \kappa$ with $K \neq S$, K is s-Hurewicz if and only if G is co-s-Hurewicz for some $G \in \tau$ and $G \neq \emptyset$.

Proof. (⇒) Let G ∈ τ with $G ≠ \emptyset$. Let $(\mathcal{F}_n)_{n ∈ \mathbb{N}}$ be a sequence of semi-closed cocovers of G. Set K = σ(G) and we obtain K ∈ κ with K ≠ S. K is s-Hurewicz, for $(σ(\mathcal{F}_n))_{n ∈ \mathbb{N}}$ the sequence of semi-open covers of K there is a sequence $(\mathcal{V}_n)_{n ∈ \mathbb{N}}$ of finite sets such that for each n, $\mathcal{V}_n ⊆ σ(\mathcal{F}_n)$ and $\vee_{n ∈ \mathbb{N}} \cap_{m > n} (∪\mathcal{V}_m)$ is semi-open cover of K. Thus $(σ(\mathcal{V}_n))_{n ∈ \mathbb{N}}$ is a sequence of finite sets such that for each n, $σ(\mathcal{V}_n) ⊆ \mathcal{F}_n$ and $σ(\bigvee_{n ∈ \mathbb{N}} \cap_{m > n} (∪\mathcal{V}_m)) = \bigcap_{n ∈ \mathbb{N}} \bigvee_{m > n} (\cap σ(\mathcal{V}_m)) ⊆ G$ which gives G is co-s-Hurewicz.

 (\Leftarrow) $K \in \kappa$ with $K \neq S$. Let $(\mathfrak{U}_n)_{n \in \mathbb{N}}$ be a sequence of semi-open covers of K. Since $K = \sigma(G)$ but G is co-s-Hurewicz, so for $(\sigma(\mathfrak{U}_n))_{n \in \mathbb{N}}$ the sequence of semi-closed cocovers of G there is a sequence $(\kappa_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, κ_n is a finite subset of $(\sigma(\mathfrak{U}_n)$ and $\bigcap_{n \in \mathbb{N}} \vee_{m > n} (\bigcap \kappa_m) \subseteq G$. Thus $\sigma(\kappa_n)$ is a sequence such that for each $n \in \mathbb{N}$, κ_n is a finite subset of \mathfrak{U}_n and $\sigma(\bigcap_{n \in \mathbb{N}} \vee_{m > n} (\bigcap \kappa_m)) = \bigvee_{n \in \mathbb{N}} \bigcap_{m > n} (\bigcup \mathcal{V}_m) \supseteq K$ which gives K is s-Hurewicz.

4 Operations in ditopological texture subspaces

Let (S, \Im) be a texture space and $A \in \Im$. The texturing $\Im_A = \{A \cap K : K \in \Im\}$ of A is called the induced texture on A [21], and (A, \Im_A) is called a *principal subtexture* of (S, \Im) .

Definition 4.1 ([20]). Let (S, \Im, τ, κ) be a ditopological texture space and (A, \Im_A) be a principle subtexture of (S, \Im) for $A \in \Im$. Then $(A, \Im_A, \tau_A, \kappa_A)$ is a subspace of a ditopological texture space (S, \Im, τ, κ) , where $\tau_A = \{A \cap G : G \in \tau\}$ and $\kappa_A = \{A \cap K : K \in \kappa\}$.

Note that if $A \in \tau$, then A is said to be open subspace and if $A \in \kappa$, then A is closed subspace.

Lemma 4.2 ([20]). Let $(A, \Im_A, \tau_A, \kappa_A)$ be a subspace of a ditopological texture space (S, \Im, τ, κ) and $\mathcal{B} \subseteq A$ then:

- (1) \mathcal{B} is τ_A open if and only if $\mathcal{B} = A \cap G$ for some τ -open set G.
- (2) \mathcal{B} is κ_A closed if and only if $\mathcal{B} = A \cap F$ for some κ -closed set F.

Theorem 4.3. Let $(S, \Im, \sigma, \tau, \kappa)$ be a complemented ditopological texture space. If S is s-Hurewicz and $A \in SR(S)$, then $(A, \Im_A, \tau_A, \kappa_A)$ is also s-Hurewicz.

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of semi-open covers of A. Then for each $n\in\mathbb{N}$, $V_n=U_n\cup\{\sigma(A)\}$ is semi-open cover of S. By hypothesis there are finite families $W_n\subseteq V_n$, $n\in\mathbb{N}$, such that $S=\vee_{n\in\mathbb{N}}\cap_{m>n}\cup W_m$. Set $\mathcal{H}_n=W_n\setminus\{\sigma(A)\}$, $n\in\mathbb{N}$. Then for each $n\in\mathbb{N}$, \mathcal{H}_n is a finite subset of U_n and $\mathcal{H}_n\subseteq V_n\cap \mathcal{H}_n$, i.e., \mathcal{H}_n is semi-Hurewicz.

Theorem 4.4. Let $(S, \Im, \sigma, \tau, \kappa)$ be a complemented ditopological texture space. If S is co-s-Hurewicz and $A \in SR(S)$, then $(A, \Im_A, \tau_A, \kappa_A)$ is also co-s-Hurewicz.

Proof. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of semi-closed cocovers of A. Then for each $n\in\mathbb{N}$, $\mathcal{K}_n=\mathcal{F}_n\cup\{\sigma(A)\}$ is a semi-closed cocover of S. But by hypothesis S is co-s-Hurewicz therefore, there are finite families $W_n\subseteq\mathcal{K}_n$, for each $n\in\mathbb{N}$, such that $\cap_{n\in\mathbb{N}}\vee_{m>n}\cap W_m\subseteq S$. Set $\mathcal{G}_n=W_n\setminus\{\sigma(A)\}$, $n\in\mathbb{N}$. Then for each $n\in\mathbb{N}$, \mathcal{G}_n is a finite subset of \mathcal{F}_n and $\cap_{n\in\mathbb{N}}\vee_{m>n}(\cap\mathcal{G}_m)\subseteq A$, i.e., A is co-s-Hurewicz. □

Example 4.5. If a complemented ditopological texture space S is s-Hurewicz, and $A \in SR(S)$ then the subtexture S_A is also s-Hurewicz.

Let \mathbb{R} be the real line textured by $\mathfrak{R} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$, with complemented ditopology $(\tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ such that $\tau_{\mathbb{R}} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and $\kappa_{\mathbb{R}} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Where $SR(\mathbb{R}) = \{\{(-\infty, r) : r \in \mathbb{R}\} \cup \{(-\infty, r] : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}\}$. This ditopological texture space is s-Hurewicz and co-s-Hurewicz by Example 3.7. Let $\mathbb{Y} = \{(-\infty, r] : r \in \mathbb{R}\}$, then (\mathbb{Y}, \mathbb{Y}) is an induced subtexture of (\mathbb{R}, \mathbb{R}) with texturing $\mathbb{Y} = \{\mathbb{Y} \cap A \mid A \in \mathbb{R}\} = \{(-\infty, s) : s \in \mathbb{Y}\} \cup \{(-\infty, s] : s \in \mathbb{Y}\} \cup \{\mathbb{Y}, \emptyset\}$, topology $\tau_{\mathbb{Y}} = \{G \cap \mathbb{Y} : G \in \tau\} = \{(-\infty, s) : s \in \mathbb{Y}\} \cup \{\mathbb{Y}, \emptyset\}$ and cotopology $\kappa_{\mathbb{Y}} = \{K \cap \mathbb{Y} : K \in \kappa\} = \{(-\infty, s] : s \in \mathbb{Y}\} \cup \{\mathbb{Y}, \emptyset\}$. Thus $(\mathbb{Y}, \mathbb{Y}, \tau_{\mathbb{Y}}, \kappa_{\mathbb{Y}})$ is a ditopological texture space.

Now we have to show that \mathbb{Y} is s-Hurewicz. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of semi-open covers of \mathbb{Y} . Represent \mathbb{Y} as $\cup \{(-\infty, n) : n \in \mathbb{N}\}$. For each n, \mathcal{U}_n is a semi-open cover of \mathbb{Y} , hence there is some $r_n \in \mathbb{R}$ such

that $(-\infty, r_n) \subseteq (-\infty, n) \in \mathcal{U}_n$. Then the collection $\{(-\infty, r_n) : n \in \mathbb{N}\}$ shows that subtexture $(\mathbb{Y}, \mathcal{Y}, \tau_{\mathbb{Y}}, \kappa_{\mathbb{Y}})$ is s-Hurewicz.

Lemma 4.6 ([17]). Let $(S_j, \Im_j, \tau_j, \kappa_j)$, j = 1, 2, be ditopological texture space. (f, F) be a diffunction between them if:

- (1) The following statements are equivalent:
- (a) (f, F) is semi-continuous.
- $(b) (F^{\flat}(A))^{\circ} \subseteq F^{\flat} (A_{\circ}) \forall A \in \mathfrak{I}_1.$
- $(c) f \leftarrow (B^{\circ}) \subseteq (f \leftarrow (B))_{\circ} \forall B \in \mathfrak{F}_2.$
- (2) The following statements are equivalent:
- (a) (f, F) is semi-cocontinuous.
- $(b) f \rightarrow (\underline{A}) \subseteq \overline{(f \rightarrow (A))} \, \forall \, A \in \Im_1.$
- $(c) (F^{\leftarrow}(B)) \subseteq F^{\leftarrow}(\overline{B}) \forall B \in \Im_1.$

Definition 4.7. Let $(S_i, \Im_i, \tau_i, \kappa_i)$, i = 1, 2 be ditoplogical texture spaces. A diffunction $(f, F) : (S_1, \Im_1) \rightarrow (S_2, \Im_2)$ is said to be:

- (i) s-continuous; if $F^{\leftarrow}(A) \in O(S_1)$ for each $A \in SO(S_2)$.
- (ii) s-cocontinuous; if $f \leftarrow (B) \in C(S_1)$ for each $B \in SC(S_2)$.
- (iii) s-bicontinuous; if it both s-continuous and s-cocontinuous.

Lemma 4.8 ([17]). *Let* (f, F) : $(S_1, \Im_1, \tau_1, \kappa_1) \rightarrow (S_2, \Im_2, \tau_2, \kappa_2)$ *be a diffunction.*

- (1) if (f, F) is cocontinuous and coclosed then $F \leftarrow (\overline{B}) = (\overline{F} \leftarrow (B))$; for all $B \in \Im_2$.
- (2) if (f, F) is continuous and open then $F \leftarrow (B^{\circ}) = (F \leftarrow (B))^{\circ}$; for all $B \in \Im_2$.

Lemma 4.9 ([17]). Let (f, F): $(S_1, \Im_1, \tau_1, \kappa_1) \rightarrow (S_2, \Im_2, \tau_2, \kappa_2)$ be a diffunction.

- (1) if (f, F) is cocontinuous and coclosed then (f, F) is semi-irresolute.
- (2) if (f, F) is continuous and open then (f, F) is semi-co-irresolute.

Theorem 4.10. Let $(f, F): (S_1, \Im_1, \tau_1, \kappa_1) \to (S_2, \Im_2, \tau_2, \kappa_2)$ be a semi-continuous diffunction between the ditopological texture spaces. If $A \in \Im_1$ is s-Hurewicz, then $f \to (A) \in \Im_2$ is Hurewicz.

Proof. Let $(\mathcal{V}_n)_{n\in\mathbb{N}}$ be a sequence of τ_2 -open covers of $f^{\rightarrow}(A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) along with semi-continuity of (f, F), for each n, we have

$$A \subseteq F \leftarrow (f \rightarrow (A)) \subseteq F \leftarrow (\vee \mathcal{V}_n) = \vee F \leftarrow (\mathcal{V}_n)$$

so that each $F^{\leftarrow}(\mathcal{V}_n)$ is a τ_1 -semi-open cover of A. As A is s-Hurewicz, therefore, for each n, there exist finite subsets $\mathcal{W}_n \subseteq \mathcal{V}_n$ such that $A \subseteq \vee_{n \in \mathbb{N}} \cap_{m \geq n} (\cup F^{\leftarrow}(\mathcal{W}_m))$. Again by Lemma 2.3 (2b) and Lemma 2.1 (2) we have

$$f^{\rightarrow}(A) \subseteq f^{\rightarrow}(\vee_{n\in\mathbb{N}}\cap_{m>n}(\cup F^{\leftarrow}(\mathcal{W}_m))$$

$$\Rightarrow \vee_{n \in \mathbb{N}} \cap_{m > n} \cup (f \xrightarrow{f} F \leftarrow (\mathcal{W}_m)) \subseteq \vee_{n \in \mathbb{N}} \cap_{m > n} \cup \mathcal{W}_m$$

This proves that $f^{\rightarrow}(A)$ is a Hurewicz space.

Theorem 4.11. Let $(f, F): (S_1, \Im_1, \tau_1, \kappa_1) \to (S_2, \Im_2, \tau_2, \kappa_2)$ be a semi-irresolute difunction between the ditopological texture spaces. If $A \in \Im_1$ is s-Hurewicz, then $f^{\to}(A) \in \Im_2$ is also s-Hurewicz.

Proof. Let $(\mathcal{V}_n)_{n\in\mathbb{N}}$ be a sequence of τ_2 -semi-opoen covers of $f^{\rightarrow}(A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) and semi-irresoluteness of (f, F), for each n, we have

$$A \subseteq F \stackrel{\leftarrow}{} (f \stackrel{\rightarrow}{} (A)) \subseteq F \stackrel{\leftarrow}{} (\vee \mathcal{V}_n) = \vee F \stackrel{\leftarrow}{} (\mathcal{V}_n)$$

so that each $F^{\leftarrow}(\mathcal{V}_n)$ is a τ_1 –semi-open cover of A. As A is s-Hurewicz, therefore, for each n, there exist finite sets $\mathcal{W}_n \subseteq \mathcal{V}_n$ such that $A \subseteq \bigvee_{n \in \mathbb{N}} \cap_{m > n} (\bigcup F^{\leftarrow}(\mathcal{W}_m))$. Again by Lemma 2.3 (2b) and Lemma 2.1 (2) we have

$$f^{\rightarrow}(A) \subset f^{\rightarrow}(\vee_{n \in \mathbb{N}} \cap_{m > n} (\cup F^{\leftarrow}(\mathcal{W}_m)))$$

$$\implies \bigvee_{n\in\mathbb{N}} \cap_{m>n} \cup (f^{\rightarrow}F^{\leftarrow}(\mathcal{W}_m)) \subseteq \bigvee_{n\in\mathbb{N}} \cap_{m>n} \cup \mathcal{W}_m$$

This proves that $f^{\rightarrow}(A)$ is s-Hurewicz.

Theorem 4.12. Let $(S_1, \Im_1, \tau_1, \kappa_1)$ and $(S_2, \Im_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a cocontinuous diffunction between them. If $A \in \Im_1$ is co-Hurewicz, then $F \to (A) \in \Im_2$ is also co-Hurewicz.

Proof. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of κ_2 -closed cocovers of $F^{\rightarrow}(A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) with cocontinuity of (f, F), for each n, we have

$$\cap \mathcal{F}_n \subset F^{\rightarrow}(A) \Rightarrow f^{\leftarrow}(\cap \mathcal{F}_n) \subset f^{\leftarrow}(F^{\rightarrow}(A)) \subset A \Rightarrow \cap f^{\leftarrow}(\mathcal{F}_n) \subset A$$

This gives each $f \in (\mathcal{F}_n)$ is a κ_1 -closed cocover of A. As A is co-Hurewicz, therefore, for each n there exist finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n \in \mathbb{N}} \bigvee_{m > n} \bigcap_{n \in \mathbb{N}} \bigvee_{m > n} \bigcap_{m \in \mathbb{N}} \bigvee_{m \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigvee_{m > n} \bigcap_{m \in \mathbb{N}} \bigvee_{m \in \mathbb{N}} \bigvee_{m \in \mathbb{N}} \bigcap_{m \in$

$$\bigcap_{n\in\mathbb{N}}\vee_{m>n}\left(\cap f\overset{\leftarrow}{(}\mathfrak{K}_{m})\right)\subseteq A\Rightarrow F\overset{\rightarrow}{(}\cap_{n\in\mathbb{N}}\vee_{m>n}\cap f\overset{\leftarrow}{(}\mathfrak{K}_{m}))\subseteq F\overset{\rightarrow}{(}A)$$

This implies
$$\cap_{n\in\mathbb{N}} \vee_{m>n} \cap (F^{\rightarrow}f^{\leftarrow}(\mathfrak{X}_m)) \subseteq F^{\rightarrow}(A) \Rightarrow \cap_{n\in\mathbb{N}} \vee_{m>n} \cap \mathfrak{X}_m \subseteq F^{\rightarrow}(A)$$

This proves that $F^{\rightarrow}(A)$ is co-Hurewicz.

Theorem 4.13. Let $(S_1, \Im_1, \tau_1, \kappa_1)$ and $(S_2, \Im_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a semi-cocontinuous difunction between them. If $A \in \Im_1$ is co-s-Hurewicz, then $F \to (A) \in \Im_2$ is co-Hurewicz.

Proof. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of κ_2 -closed cocovers of $F^{\rightarrow}(A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) with semi-cocontinuity of (f, F), for each n, we have

$$\cap \mathcal{F}_n \subseteq F \xrightarrow{\rightarrow} (A) \Rightarrow f \xleftarrow{\leftarrow} (\cap \mathcal{F}_n) \subseteq f \xleftarrow{\leftarrow} (F \xrightarrow{\rightarrow} (A)) \subseteq A \Rightarrow \cap f \xleftarrow{\leftarrow} (\mathcal{F}_n) \subseteq A$$

This gives that $f \leftarrow (\mathcal{F}_n)$ is a κ_1 –semi-closed cocover of A. As A is co-s-Hurewicz, therefore, for each n, there exist finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n \in \mathbb{N}} \vee_{m > n} \cap f \leftarrow (\mathcal{K}_m) \subseteq A$. Again by Lemma 2.3 (2b) and Lemma 2.1 (2) we have

$$\cap_{n\in\mathbb{N}}\vee_{m>n}(\cap f^{\leftarrow}(\mathfrak{X}_m))\subseteq A\Rightarrow F^{\rightarrow}(\cap_{n\in\mathbb{N}}\vee_{m>n}(\cap f^{\leftarrow}(\mathfrak{X}_m)))\subseteq F^{\rightarrow}(A)$$

This gives
$$\cap_{n\in\mathbb{N}} \vee_{m>n} \cap (F \xrightarrow{f} (\mathfrak{K}_m)) \subseteq F \xrightarrow{} (A) \Rightarrow \cap_{n\in\mathbb{N}} \vee_{m>n} \cap \mathfrak{K}_m \subseteq F \xrightarrow{} (A)$$

This proves that $F^{\rightarrow}(A)$ is co-Hurewicz.

Theorem 4.14. Let $(S_1, \Im_1, \tau_1, \kappa_1)$ and $(S_2, \Im_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a semi-co-irresolute diffunction between them. If $A \in \Im_1$ is co-s-Hurewicz, then $F \to (A) \in \Im_2$ is co-s-Hurewicz.

Proof. Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of κ_2 -semi-closed cocovers of $F \to (A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) with semi-co-irresoluteness of (f, F), for each n, we have

$$\cap \mathcal{F}_n \subset F^{\rightarrow}(A) \Rightarrow f^{\leftarrow}(\cap \mathcal{F}_n) \subset f^{\leftarrow}(F^{\rightarrow}(A)) \subset A \Rightarrow \cap f^{\leftarrow}(\mathcal{F}_n) \subset A$$

This gives $f \leftarrow (\mathcal{F}_n)$ is a κ_1 –semi-closed cocover of A. As A is co-s-Hurewicz, therefore, for each n, there exist finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n \in \mathbb{N}} \bigvee_{m > n} \bigcap f \leftarrow (\mathcal{K}_m) \subseteq A$. Again by Lemma 2.3 (2b) and Lemma 2.1 (2) we have

$$\bigcap_{n\in\mathbb{N}}\vee_{m>n}\left(\cap f\stackrel{\leftarrow}{(}\mathfrak{K}_{m})\right)\subseteq A\Rightarrow F^{\rightarrow}\left(\bigcap_{n\in\mathbb{N}}\vee_{m>n}\cap f\stackrel{\leftarrow}{(}\mathfrak{K}_{m})\right)\subseteq F^{\rightarrow}(A)$$

This implies
$$\cap_{n\in\mathbb{N}} \vee_{m>n} \cap (F^{\rightarrow}f^{\leftarrow}(\mathfrak{X}_m)) \subseteq F^{\rightarrow}(A) \Rightarrow \cap_{n\in\mathbb{N}} \vee_{m>n} \cap \mathfrak{X}_m \subseteq F^{\rightarrow}(A)$$

This proves that $F^{\rightarrow}(A)$ is co-s-Hurewicz.

Theorem 4.15. Let $(f, F): (S_1, \Im_1, \tau_1, \kappa_1) \to (S_2, \Im_2, \tau_2, \kappa_2)$ be a s-continuous difunction between the ditopological texture spaces. If $A \in \Im_1$ is Hurewicz, then $f \to (A) \in \Im_2$ is s-Hurewicz.

Proof. Let $(\mathcal{V}_n)_{n\in\mathbb{N}}$ be a sequence of τ_2 semi-open covers of $f^{\rightarrow}(A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) with s-continuity of (f, F), for each n, we have

$$A \subseteq F \stackrel{\leftarrow}{} (f \stackrel{\rightarrow}{} (A)) \subseteq F \stackrel{\leftarrow}{} (\vee \mathcal{V}_n) = \vee F \stackrel{\leftarrow}{} (\mathcal{V}_n),$$

This gives that $F^{\leftarrow}(\mathcal{V}_n)$ is a τ_1 -open cover of A. As A is Hurewicz, therefore, for each n, there exist finite sets $\mathcal{W}_n \subseteq F^{\leftarrow}(\mathcal{V}_n)$ such that $A \subseteq \vee_{n \in \mathbb{N}} \cap_{m > n} (\cup F^{\leftarrow}(\mathcal{W}_m))$. Again by Lemma 2.3 (2b) and Lemma 2.1 (2) we have

$$f \rightarrow (A) \subset f \rightarrow (\vee_{n \in \mathbb{N}} \cap_{m > n} (\cup F \leftarrow (\mathcal{W}_m)))$$

$$\implies \bigvee_{n \in \mathbb{N}} \cap_{m > n} \cup (f \xrightarrow{\rightarrow} F \xleftarrow{\leftarrow} (\mathcal{W}_m)) \subset \bigvee_{n \in \mathbb{N}} \cap_{m > n} \cup \mathcal{W}_m$$

This proves that $f^{\rightarrow}(A)$ is s-Hurewicz.

Theorem 4.16. Let $(S_1, \Im_1, \tau_1, \kappa_1)$ and $(S_2, \Im_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be an *s*-cocontinuous difunction between them. If $A \in \Im_1$ is co-Hurewicz, then $F \to (A) \in \Im_2$ is co-s-Hurewicz.

Proof. Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of κ_2 -semi-closed cocovers of $F^{\rightarrow}(A)$. Then by Lemma 2.3 (2a) and Lemma 2.1 (2) and s-cocontinuity of (f, F), for each n, we have

$$\cap \mathfrak{F}_n \subseteq F^{\rightarrow}(A) \Rightarrow f^{\leftarrow}(\cap \mathfrak{F}_n) \subseteq f^{\leftarrow}(F^{\rightarrow}(A)) \subseteq A \Rightarrow \cap f^{\leftarrow}(\mathfrak{F}_n) \subseteq A$$

This gives that $f^{\leftarrow}(\mathcal{F}_n)$ is a κ_1 -closed cocover of A. As A is co-Hurewicz, therefore, for each n, there exist finite sets $\mathcal{K}_n \subseteq \mathcal{F}_n$ such that $\bigcap_{n \in \mathbb{N}} \bigcap f^{\leftarrow}(\mathcal{K}_n) \subseteq A$. Again by Lemma 2.3 (2b) and Lemma 2.1 (2) we have

$$\bigcap_{n\in\mathbb{N}}\vee_{m>n}\left(\cap f\stackrel{\leftarrow}{(}\mathfrak{K}_{m})\right)\subseteq A\Rightarrow F\stackrel{\rightarrow}{(}\cap_{n\in\mathbb{N}}\vee_{m>n}\cap f\stackrel{\leftarrow}{(}\mathfrak{K}_{m}))\subseteq F\stackrel{\rightarrow}{(}A)$$

This implies
$$\cap_{n\in\mathbb{N}} \vee_{m>n} \cap (F^{\rightarrow}f^{\leftarrow}(\mathfrak{X}_m)) \subseteq F^{\rightarrow}(A) \Rightarrow \cap_{n\in\mathbb{N}} \vee_{m>n} \cap \mathfrak{X}_m \subseteq F^{\rightarrow}(A)$$

This proves $F^{\rightarrow}(A)$ is co-s-Hurewicz.

5 Semi-stability in ditopological texture spaces

Definition 5.1 ([14]). *A ditopology* (τ, κ) *on* (S, \Im) *is said to be:*

- (i) Stable; if for every $K \in \kappa$ with $K \neq S$ is compact.
- (ii) Co-stable; if for every $G \in \tau$ with $G \neq \emptyset$ is cocompact.

A ditopological texture space (S, \Im, τ, κ) is called dicompact if it is compact, cocompact, stable and co-stable.

Definition 5.2 ([22]). A ditopology (τ, κ) on (S, \Im) is said to be:

- (i) Semi-stable; if for all $F \in SC(S)$ with $F \neq S$ is semi-compact.
- (ii) Semi-co-stable; if for every $G \in SO(S)$ with $G \neq \emptyset$ is semi-cocompact.

A ditopological texture space (S, \Im, τ, κ) is called s-dicompact if it is semi-compact, semi-cocompact, semi-stable and semi-co-stable.

Proposition 5.3 ([22]). *For a ditopological texture space* (S, \Im, τ, κ)

- (1) semi-stable \Rightarrow stable
- (2) semi-co-stable \Rightarrow co-stable.

Proof. It follows directly from the fact that $O(S) \subseteq SO(S)$ and $C(S) \subseteq SC(S)$.

Definition 5.4 ([19]). A ditopological texture space (S, \Im, τ, κ) is said to be:

- (i) *H-stable*; if for all $F \in \kappa$ with $F \neq S$ is Hurewicz.
- (ii) *H-co-stable*; if for every $G \in \tau$ with $G \neq \emptyset$ is co-Hurewicz.

A ditopological texture space (S, \Im, τ, κ) is called di-Hurewicz if it is Hurewicz, co-Hurewicz, H-stable and H-co-stable.

Definition 5.5. A ditopological texture space (S, \Im, τ, κ) is said to be:

(i) sH-stable; if for all $F \in SC(S)$ with $F \neq S$ is s-Hurewicz.

(ii) sH-co-stable; if for every $G \in SO(S)$ with $G \neq \emptyset$ is co-s-Hurewicz.

The following example shows that the ditopological texture spaces are sH-stable and sH-co-stable respectively.

Example 5.6. (1) Let $\mathbb{B} = (0, 1]$ with texturing $\beta = \{(0, b] : b \in \mathbb{B}\}$, $\tau_{\mathbb{B}} = \beta$, $\kappa_{\mathbb{B}} = \{\mathbb{B}, \emptyset\}$. Where $SO(\mathbb{B}) = \tau$ and $SC(\mathbb{B}) = \kappa$. Then (τ, κ) is sH-stable because the only semi-closed set different from \mathbb{B} is \emptyset , and this set is trivially semi-compact and hence s-Hurewicz.

(2) Dually let $\mathbb{B} = (0, 1]$, $\beta = \{(0, b] : b \in \mathbb{B}\}$, $\tau_{\mathbb{B}} = \{\mathbb{B}, \emptyset\}$ $\kappa_{\mathbb{B}} = \beta$, where $SO(\mathbb{B}) = \tau$ and $SC(\mathbb{B}) = \kappa$. Then (τ, κ) is sH-co-stable because the only semi-open set different from \emptyset is \mathbb{B} , and this set is semi-cocompact and hence co-s-Hurewicz.

A ditopological texture space (S, \Im, τ, κ) is called s-di-Hurewicz if it is s-Hurewicz, s-co-Hurewicz, sH-stable and sH-co-stable.

Theorem 5.7. Let $(f, F): (S_1, \Im_1, \tau_1, \kappa_1) \to (S_2, \Im_2, \tau_2, \kappa_2)$ be bicontinuous surjective diffunction between ditopological texture spaces. If $(S_1, \Im_1, \tau_1, \kappa_1)$ is sH-stable then $(S_2, \Im_2, \tau_2, \kappa_2)$ is also sH-stable.

Proof. Let $K \in \kappa_2$ be such that $K \neq S_2$. Since (f, F) is cocontinuous we have that $f^{\leftarrow}(K) \in \kappa_1$. Now we first show that $f^{\leftarrow}(K) \neq S_1$. On the contrary suppose that $f^{\leftarrow}(K) = S_1$, since (f, F) is surjective so we have $f^{\leftarrow}(S_2) = S_1$ which by [13] $f^{\leftarrow}(S_2) \subseteq f^{\leftarrow}(K)$. Since (f, F) is surjective, then by Corollary 2.33(1 ii) in [13] implies $S_2 \subseteq K$. This is contradiction so $f^{\leftarrow}(K) \neq S_1$.

Since $(S_1, \Im_1, \tau_1, \kappa_1)$ is sH-stable so $f^{\leftarrow}(K)$ is s-Hurewicz set. Continuity of (f, F) and Theorem 12 implies $f^{\rightarrow}(f^{\leftarrow}(K))$ is s-Hurewicz in $(S_2, \Im_2, \tau_2, \kappa_2)$. By [13], Corollary 2.33(1) the latter set is equal to K, and hence the proof.

Theorem 5.8. Let $(f, F): (S_1, \Im_1, \tau_1, \kappa_1) \to (S_2, \Im_2, \tau_2, \kappa_2)$ be bicontinuous surjective diffunction between ditopological texture spaces. If $(S_1, \Im_1, \tau_1, \kappa_1)$ is sH-costable then $(S_2, \Im_2, \tau_2, \kappa_2)$ is also sH-costable.

Proof. Let $G \in \tau_2$ be such that $G \neq \emptyset_2$. Since (f, F) is continuous we have that $F^{\leftarrow}(G) \in \tau_1$. Now we first show that $F^{\leftarrow}(G) \neq \emptyset_1$. Suppose that it is not true. $f^{\leftarrow}(K) = S_1$, Since (f, F) is surjective so we have $F^{\leftarrow}(\emptyset_2) = \emptyset_1$ which by [13] $F^{\leftarrow}(\emptyset_2) \subseteq F^{\leftarrow}(G)$. Since (f, F) is surjective, then by Corollary 2.33(1 ii) in [13] implies $\emptyset_2 \subseteq G$. This contradiction shows $F^{\leftarrow}(G) \neq \emptyset_1$.

Since $(S_1, \Im_1, \tau_1, \kappa_1)$ is sH-stable so $F^{\leftarrow}(G)$ is co-s-Hurewicz set. Continuity of (f, F) and Theorem 12 implies $F^{\rightarrow}(F^{\leftarrow}(G))$ is co-s-Hurewicz in $(S_2, \Im_2, \tau_2, \kappa_2)$. By [13], Corollary 2.33(1) the latter set is equal to G, and hence the proof.

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