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### **Research Article**

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# Infinite growth of solutions of second order complex differential equation

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**Abstract:** In this paper we study the growth of solutions of second order differential equation f'' + A(z)f' + B(z)f = 0. Under certain hypotheses, the non-trivial solution of this equation is of infinite order.

**Keywords:** Entire function, order, Complex differential equation

MSC: 30D20, 30D35, 34M05

# 1 Introduction and main results

In this article, we assume the reader is familiar with standard notations and basic results of Nevanlinna theory in the complex plane  $\mathbb{C}$ , see [1, 2]. Nevanlinna theory is an important tool to studying the complex differential equations, and there appears many results in this areas recent years. In this paper, the order of an entire function f is defined as

$$\rho(f) = \limsup_{r \to +\infty} \frac{\log^+ T(r, f)}{\log r} = \limsup_{r \to +\infty} \frac{\log^+ \log^+ M(r, f)}{\log r},$$

where  $\log^+ x = \max\{\log x, 0\}$  and M(r, f) denotes the maximum modulus of f on the circle |z| = r. Our main purpose is to consider the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, (1)$$

where A(z) and B(z) are entire functions. It's well known that all solutions of (1) are entire functions. If B(z) is transcendental and  $f_1, f_2$  are two linearly independent solutions of this equation, then at least one of  $f_1, f_2$  is of infinite order, see [3]. What conditions on A(z) and B(z) can guarantee that every solution  $f \neq 0$  of equation (1) is of infinite order? There has been many results on this subject. For example, we collect some results and give the following theorems:

**Theorem 1.1.** Let A(z) and B(z) be nonconstant entire functions, satisfying any one of the following additional hypotheses:

- 1.  $\rho(A) < \rho(B)$ , see [4];
- 2. A(z) is a polynomial and B(z) is transcendental, see [4];
- 3.  $\rho(B) < \rho(A) \le \frac{1}{2}$ , see [5],

then every solution  $f(\not\equiv 0)$  of equation (1) has infinite order.

**Theorem 1.2.** Let  $A(z) = e^{-z}$  and B(z) be nonconstant entire functions, satisfying any one of the following additional hypotheses:

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- 1. B(z) is a nonconstant polynomial, see [6];
- 2. B(z) is transcendental entire function with order  $\rho(B) \neq 1$ , see [7];
- 3.  $B(z) = h(z)e^{-dz}$ , where  $h(z)(\not\equiv 0)$  is an entire function with  $\rho(h) < 1$ ,  $d(\not\equiv 1)$  is a nonzero complex constant, see [8],

then every solution  $f(\not\equiv 0)$  of equation (1) has infinite order.

In the above theorem we can see that A(z) could not get the value zero which means zero is a deficient value of A(z). In general, if A(z) has a finite deficient value, we have the following collection theorem.

**Theorem 1.3.** Let A(z) be an finite order entire function with a finite deficient value and B(z) be a transcendental entire function, satisfying any one of the following additional hypotheses:

- 1.  $\mu(B) < 1/2$ , see [9]:
- 2.  $T(r, B) \sim \log M(r, B)$  as  $r \to \infty$  outside a set of finite logarithmic measure, see [10, Lemma 2.7], then every solution  $f(\not\equiv 0)$  of equation (1) has infinite order.

In this paper, we continue to study the above in order to find conditions which A(z), B(z) should satisfy to ensure that nontrivial solution of (1) has infinite order. Similar as in Theorem 1.2, we also assume zero is a Picard exceptional value of A(z), and the first main result is as follows.

**Theorem 1.4.** Suppose  $A(z) = e^{p(z)}$ , where p(z) is a nonconstant polynomial and B(z) is a transcendental entire function with nonzero finite order. If

$$\frac{1}{\rho(A)} + \frac{1}{\rho(B)} > 2$$
,

then every solution  $f(\not\equiv 0)$  of equation

$$f'' + e^{p(z)}f' + B(z)f = 0 (2)$$

is of infinite order.

**Remark 1.1.** It's clear that  $f(z) = e^z$  satisfies  $f'' + e^{-z}f' - (e^{-z} + 1)f = 0$ , here  $\rho(A) = \rho(B) = 1$ . Therefore, the inequality in the above theorem is sharp.

**Remark 1.2.** Simple calculation shows that the infinite order function  $f(z) = \exp\{e^z\}$  satisfies  $f'' + e^{-z}f' - (e^{2z} + e^z + 1)f = 0$ . But  $\rho(A)$  and  $\rho(B)$  do not satisfy the inequality. Thus, the converse problem of Theorem 1.4 is not true.

**Remark 1.3.** As in Theorem 1.3, if A(z) has a finite deficient value, does the conclusion still hold?

**Remark 1.4.** We adopt the method of Rossi [11] to prove this theorem. This method was also used by Cao [12] to affirm Brück conjecture provided the hyper-order of f(z) is equal to 1/2.

**Remark 1.5.** By the idea of reviewer, the conclusion of this theorem can also be obtained by a relevant result of [4, Theorem 7], but the proof in this paper is totally different from it.

In the following, we consider the case that B(z) is a nonconstant polynomial and p(z) is transcendental entire. Rewrite (1) as

$$e^{p(z)} = \frac{f''}{f'} + B(z)\frac{f}{f'}.$$
 (3)

Observe the order of both sides, we can deduce that the solution has infinite order. For the remaining case that p(z) and B(z) are both nonconstant polynomials, we have the following result.

**Theorem 1.5.** Suppose  $A(z) = e^{p(z)}$ , where p(z) is a polynomial with degree  $n \ge 2$  and B(z) = Q(z) is also a nonconstant polynomial with degree m. If m + 2 > 2n and  $n \nmid m + 2$ , then every solution  $f(\not\equiv 0)$  of equation

$$f'' + e^{p(z)}f' + Q(z)f = 0 (4)$$

is of infinite order.

**Remark 1.6.** The case when the degree of p(z) is one could be proved by a minor modification of the proof of Theorem 2 in [6], here the degree of Q(z) can be arbitrary positive integer.

# 2 Preliminary lemmas

**Lemma 2.1** ([13]). Let f be a transcendental meromorphic function. Let  $\alpha > 1$  be a constant, and k, j be integers satisfying  $k > j \ge 0$ . Then the following two statements hold:

1. There exists a set  $E_1 \subset (1, +\infty)$  which has finite logarithmic measure, and a constant K > 0, such that for all z satisfying  $|z| = r \notin E_1 \cup [0, 1]$ , we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le K \left[ \frac{T(\alpha r, f)}{r} (\log r)^{\alpha} \log T(\alpha r, f) \right]^{k-j}. \tag{5}$$

2. There exists a set  $E_2 \subset [0, 2\pi)$  which has zero linear measure, such that if  $\theta \in [0, 2\pi) \setminus E_2$ , then there is a constant  $R = R(\theta) > 0$  such that (5) holds for all z satisfying  $\arg z = \theta$  and  $|z| \ge R$ .

The following Lemma is proved in [11] by using [14, Theorem III.68]. We need some notations to state it. Let D be a region in  $\mathbb{C}$ . For each  $r \in \mathbb{R}^+$  set  $\theta_D^*(r) = \theta^*(r) = +\infty$  if the entire circle |z| = r lies in D. Otherwise, let  $\theta_D^*(r) = \theta^*(r)$  be the measure of all  $\theta$  in  $[0, 2\pi)$  such that  $re^{i\theta} \in D$ . As usual, we define the order  $\rho(u)$  of a function u subharmonic in the plane as

$$\rho(u) \coloneqq \limsup_{R \to +\infty} \frac{\log M(r, u)}{\log r},$$

where M(r, u) is the maximum modulus of subharmonic function u on a circle of radius r.

**Lemma 2.2** ([11]). Let u be a subharmonic function in  $\mathbb{C}$  and let D be an open component of  $\{z: u(z) > 0\}$ . Then

$$\rho(u) \ge \limsup_{R \to +\infty} \frac{\pi}{\log R} \int_{1}^{R} \frac{dt}{t\theta_{D}^{*}(t)}. \tag{6}$$

*Furthermore, given*  $\varepsilon > 0$ *, define*  $F = \{r : \theta_D^* \le \varepsilon \pi\}$ *. Then* 

$$\limsup_{R\to+\infty} \frac{1}{\log R} \int_{F\cap[1,R]} \frac{dt}{t} \le \varepsilon \rho(u). \tag{7}$$

**Lemma 2.3** ([11, 15]). Let  $l_1(t) > 0$ ,  $l_2(t) > 0$ ( $t \ge t_0$ ) be two measurable functions on  $(0, +\infty)$  with  $l_1(t) + l_2(t) \le (2 + \varepsilon)\pi$ , where  $\varepsilon > 0$ . If  $G \subseteq (0, +\infty)$  is any measurable set and

$$\pi \int_{G} \frac{dt}{t l_1(t)} \le \alpha \int_{G} \frac{dt}{t}, \qquad \alpha \ge \frac{1}{2}, \tag{8}$$

then

$$\pi \int_{G} \frac{dt}{t l_{2}(t)} \geq \frac{\alpha}{(2+\varepsilon)\alpha - 1} \int_{G} \frac{dt}{t}.$$
 (9)

**Lemma 2.4** ([6]). *Let S be the strip* 

$$z = x + iy$$
,  $x \ge x_0$ ,  $|y| \le 4$ .

Suppose that in S

$$Q(z) = a_n z^n + O(|z|^{n-2}),$$

where n is positive integer and  $a_n > 0$ . Then there exists a path  $\Gamma$  tending to infinity in S such that all solutions of y'' + Q(z)y = 0 tend to zero on  $\Gamma$ .

**Lemma 2.5** ([6]). Suppose that A(z) is analytic in a sector containing the ray  $\Gamma: re^{i\theta}$  and that as  $r \to \infty$ ,  $A(re^{i\theta}) = O(r^n)$  for some  $n \ge 0$ . Then all solutions of y'' + A(z)y = 0 satisfy

$$\log^+ |y(re^{i\theta})| = O(r^{(n+2)/2})$$

on  $\Gamma$ .

We recall the definition of an R-set; for reference, see [1]. Set  $B(z_n, r_n) = \{z : |z - z_n| < r_n\}$ . If  $\sum_{n=1}^{\infty} r_n < \infty$  and  $z_n \to \infty$ , then  $\bigcup_{n=1}^{\infty} B(z_n, r_n)$  is called an R-set. Clearly, the set  $\{|z| : z \in \bigcup_{n=1}^{\infty} B(z_n, r_n)\}$  is of finite linear measure. Moreover, by [1, Lemma 5.9], the set of angles  $\theta$  for which the ray  $re^{i\theta}$  meets infinitely many discs of a given R-set has linear measure zero.

**Lemma 2.6** ([1, Proposition 5.12]). Let f be a meromorphic function of finite order. Then there exists N = N(f) > 0 such that

$$\left| \frac{f'(z)}{f(z)} \right| = O(r^N) \tag{10}$$

holds outside of an R-set.

In the next Lemma, let  $p(z) = (\alpha + i\beta)z^n + \cdots + a_0$  be a polynomial with  $\alpha$ ,  $\beta$  real, and denote  $\delta(p, \theta) := \alpha \cos n\theta - \beta \sin n\theta$ .

**Lemma 2.7** ([1, Lemma 5.14]). Let p(z) be a polynomial of degree  $n \ge 1$ , and consider the exponent function  $A(z) := e^{p(z)}$  on a ray  $re^{i\theta}$ . Then we have

1. If  $\delta(p,\theta) > 0$ , there exists an  $r(\theta)$  such that  $\log |A(re^{i\theta})|$  is increasing on  $[r(\theta), +\infty)$  and

$$|A(re^{i\theta})| \ge \exp\left(\frac{1}{2}\delta(p,\theta)r^n\right)$$
 (11)

holds here;

2. If  $\delta(p,\theta) < 0$ , there exists an  $r(\theta)$  such that  $\log |A(re^{i\theta})|$  is decreasing on  $[r(\theta), +\infty)$  and

$$|A(re^{i\theta})| \le \exp\left(\frac{1}{2}\delta(p,\theta)r^n\right)$$
 (12)

holds here.

**Lemma 2.8** ([16, Theorem 7.3]). Phragmén-Lindelöf Theorem Let f(z) be an analytic function of  $z = re^{i\theta}$ , regular in a region D between rays making an angle  $\pi/\alpha$  at the origin and on the straight lines themselves. Suppose that  $|f(z)| \leq M$  on the lines and as  $r \to \infty$ ,  $f(z) = O(e^{r^{\beta}})$ , where  $\beta < \alpha$  uniformly. Then  $|f(z)| \leq M$  throughout D.

**Remark 2.1.** If  $f(z) \to a$  as  $z \to \infty$  along a straight line,  $f(z) \to b$  as  $z \to \infty$  along another straight line, and f(z) is analytic and bounded in the angle between, then a = b and  $f(z) \to a$  uniformly in the angle. The straight lines may be replaced by curves approaching  $\infty$ .

# 3 Proofs of theorems

*Proof of Theorem 1.4.* We assume that  $\rho(f) = \rho < +\infty$ , and would obtain the assertion by reduction to contradiction. By Lemma 2.1 and definition of the growth order, there exist constants K > 0,  $\beta > 1$  and  $C = C(\varepsilon)$  (depending on  $\varepsilon$ ) such that

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \le K \left( \frac{T(\beta r, f)}{\log r} (\log r)^{\beta} \log T(\beta r, f) \right)^{j} \le r^{C}, \quad j = 1, 2$$
(13)

holds for all  $r > r_0 = R(\theta)$  and  $\theta \notin J(r)$ , where J(r) is a set with zero linear measure. We may say  $m(J(r)) \le \varepsilon \pi$  where  $\varepsilon(>0)$  is given arbitrarily small. Fix  $\varepsilon > 0$  and let N be an integer such that  $N > C = C(\varepsilon)$ , and

$$\log M(2,B) < N\log 2. \tag{14}$$

Since B(z) is transcendental there exists  $z_0$ ,  $|z_0| > 2$ , such that  $\log |B(z_0)| > N \log |z_0|$ . Let  $D_1$  be the component of the set  $\{z : \log |B(z)| - N \log |z| > 0\}$  containing  $z_0$ . Clearly  $D_1$  is open and since (14) holds,  $\log |B(z)| - N \log |z|$  is subharmonic in  $D_1$  and identically zero on  $\partial D_1$ . Thus, if we define

$$u(z) = \begin{cases} \log |B(z)| - N \log |z|, & z \in D_1, \\ 0, & z \in \mathbb{C} \setminus D_1, \end{cases}$$

$$(15)$$

we have that u(z) is subharmonic in  $\mathbb{C}$  with

$$\rho(u) \le \rho(B). \tag{16}$$

Let  $D_2$  be an unbounded component of the set  $\{z : \log |e^{-p(z)}| > 0\}$ , such that if we define

$$v(z) = \begin{cases} \log |e^{-p(z)}|, & z \in D_2, \\ 0, & z \in \mathbb{C} \setminus D_2, \end{cases}$$

then v(z) is subharmonic in  $\mathbb{C}$  with

$$\rho(\mathbf{v}) = \rho(A)$$
.

Moreover, define  $D_3 := \{re^{i\theta} : \theta \in J(r)\}$ . For the above given  $\varepsilon$ , if  $(D_1 \cap D_2) \setminus D_3$  contains an unbounded sequence  $\{r_ne^{i\theta_n}\}$ , then we get from the above discussions that

$$|r_n^N < |B(r_n e^{i\theta_n})| \le \left| \frac{f''(r_n e^{i\theta_n})}{f(r_n e^{i\theta_n})} \right| + \left| e^{p(r_n e^{i\theta_n})} \right| \left| \frac{f'(r_n e^{i\theta_n})}{f(r_n e^{i\theta_n})} \right| \le 2r_n^{\mathcal{C}}, \tag{17}$$

and this clearly contradicts N > C for n large enough. Thus for arbitrary  $\varepsilon$ , we may assume that  $(D_1 \cap D_2) \setminus D_3$  is bounded, this implies that for  $r \ge r_1 \ge r_0$ ,

$$K_r := \{\theta : re^{i\theta} \in D_1 \cap D_2\} \subseteq J(r),$$

Obviously, we have

$$m(K_r) \leq \varepsilon \pi$$
. (18)

(We remark here that the proof of Theorem 1.4 would now follow easily from (6) and Lemma 2.3 if  $D_1$  and  $D_2$  were disjoint. As we shall see, (6), (13) and (18) imply that these sets are "essentially" disjoint. Define

$$l_{j}(t) = \begin{cases} 2\pi, & \text{if } \mathring{D}_{j}(t) = \infty, \\ \theta_{D_{i}}^{*}(t), & \text{otherwise,} \end{cases}$$
 (19)

for j = 1, 2. Since B(z) is transcendental, it follows  $D_1$  and  $D_2$  are unbounded open sets. Then we get  $l_1(t) > 0$ ,  $l_2(t) > 0$  for t sufficiently large, and

$$l_1(t) + l_2(t) \le 2\pi + \varepsilon\pi. \tag{20}$$

Set

$$\alpha := \limsup_{R \to \infty} \frac{\pi}{\log R} \int_{1}^{R} \frac{dt}{t l_1(t)}.$$
 (21)

By (21) and the fact  $l_1(t) \le 2\pi$ , we have

$$\alpha \geq \frac{1}{2\log R} \int_{1}^{R} \frac{dt}{t} = \frac{1}{2}.$$

Thus

$$\pi \int_{1}^{R} \frac{dt}{t l_{1}(t)} \leq \alpha \log R = \alpha \int_{1}^{R} \frac{dt}{t}.$$

Therefor, from Lemma 2.3 we obtain

$$\pi \int_{1}^{R} \frac{dt}{t l_{2}(t)} \geq \frac{\alpha}{(2+\varepsilon)\alpha-1} \int_{1}^{R} \frac{dt}{t} = \frac{\alpha}{(2+\varepsilon)\alpha-1} \log R,$$

and thus,

$$\limsup_{R \to +\infty} \frac{\pi}{\log R} \int_{1}^{R} \frac{dt}{t l_{2}(t)} \ge \frac{\alpha}{(2+\varepsilon)\alpha - 1}.$$
 (22)

Define the set

$$B_i := \{r : \theta_{D_i}^*(r) = +\infty\} \tag{23}$$

for j=1,2. If  $r\in B_1$  and  $r\geq r_1$ , then  $\theta_{D_2}^*(r)\leq \varepsilon\pi$  by (20). Thus  $B_1\subseteq \{r:\theta_{D_2}^*(r)\leq \varepsilon\pi\}$ . By Lemma 2.2 we have

$$\limsup_{R \to \infty} \frac{1}{\log R} \int_{B_1 \cap [1,R]} \frac{dt}{t} \le \varepsilon \rho(e^{p(z)}) = \varepsilon \rho(e^{-p(z)}). \tag{24}$$

The last equality follows by the first Nevanlinna theorem. Let  $\widetilde{B}_j = R^+ \setminus B_j$ , j = 1, 2. Then (6), (21) and (24) give

$$\rho(u) \ge \limsup_{R \to \infty} \frac{\pi}{\log R} \int_{1}^{R} \frac{dt}{t\theta_{D_{1}}^{*}(t)}$$

$$= \limsup_{R \to \infty} \frac{\pi}{\log R} \int_{\widetilde{B}_{1} \cap [1,R]} \frac{dt}{t\theta_{D_{1}}^{*}(t)}$$

$$= \limsup_{R \to \infty} \frac{1}{\log R} \left[ \pi \int_{1}^{R} \frac{dt}{tl_{1}(t)} - \frac{1}{2} \int_{B_{1} \cap [1,R]} \frac{dt}{t} \right]$$

$$\ge \alpha - \frac{\varepsilon \rho(e^{p(z)})}{2}, \tag{25}$$

which together with (16) and  $A(z) = e^{p(z)}$  shows

$$\rho(B) \ge \alpha - \frac{\varepsilon \rho(A)}{2}.\tag{26}$$

Applying the similar arguments as above to  $B_2$ , if  $r \in B_2$  and  $r \ge r_1$ , then  $\theta_{D_1(r)}^* \le \varepsilon \pi$ . Thus  $B_2 \subseteq \{r : \theta_{D_1(r)}^* \le \varepsilon \pi\}$ . Then we obtain also from Lemma 2.2 that

$$\limsup_{R \to \infty} \frac{1}{\log R} \int_{B_2 \cap [1,R]} \frac{dt}{t} \le \varepsilon \rho(u). \tag{27}$$

Combining (6), (22) with (27) we obtain

$$\rho(A) = \rho(e^{-p(z)}) \ge \limsup_{R \to \infty} \frac{\pi}{\log R} \int_{1}^{R} \frac{dt}{t\theta_{D_{2}}^{*}(t)}$$

$$= \limsup_{R \to \infty} \frac{\pi}{\log R} \int_{\widetilde{B_{2}} \cap [1,R]} \frac{dt}{t\theta_{D_{2}}^{*}(t)}$$

$$= \limsup_{R \to \infty} \frac{1}{\log R} \left[ \pi \int_{1}^{R} \frac{dt}{tl_{2}(t)} - \frac{1}{2} \int_{B_{2} \cap [1,R]} \frac{dt}{t} \right]$$

$$\ge \frac{\alpha}{(2+\varepsilon)\alpha - 1} - \frac{\varepsilon \rho(u)}{2}.$$
(28)

Since  $\frac{\alpha}{(2+\varepsilon)\alpha-1}$  is a monotone decreasing function of  $\alpha$ , inequalities (16), (26) and (28) give

$$\rho(A) \ge \frac{\rho(B) + \frac{\varepsilon \rho(A)}{2}}{(2 + \varepsilon)(\rho(B) + \frac{\varepsilon \rho(A)}{2}) - 1} - \frac{\varepsilon \rho(B)}{2}.$$
 (29)

Note that  $\varepsilon$  is positive arbitrary small and  $\rho(A)$ ,  $\rho(B)$  are finite, we obtain

$$\rho(A) \geq \frac{\rho(B)}{2\rho(B) - 1}.$$

That is,

$$\frac{1}{\rho(A)}+\frac{1}{\rho(B)}\leq 2,$$

which contradicts the assumption. Thus, every solution  $f(\not\equiv 0)$  of equation (2) is of infinite order.

*Proof of Theorem 1.5.* We assume that (4) has a solution f(z) with finite order. Set

$$f = y \exp\left\{-\frac{1}{2} \int_{0}^{z} e^{p(z)} dz\right\},\tag{30}$$

equation (4) can be transformed into

$$y'' + \left(Q(z) - \frac{1}{4}e^{2p(z)} - \frac{1}{2}e^{p(z)}p'(z)\right)y = 0.$$
(31)

By a translation we may assume that

$$Q(z) = a_m z^m + a_{m-2} z^{m-2} + \cdots, m > 2.$$
(32)

We define the critical ray for Q(z) as those ray  $re^{i\theta_j}$  for which

$$\theta_j = \frac{-\arg a_m + 2j\pi}{m+2},\tag{33}$$

where  $j = 0, 1, 2 \cdots, m + 1$ , and note that the substitution  $z = xe^{i\theta_j}$  transforms equation (31) into

$$\frac{d^2y}{dx^2} + (Q_1(x) + P_1(x))y = 0, (34)$$

where

$$Q_1(x) = \alpha_1 x^m + O(x^{m-2}), \ \alpha_1 > 0$$

and

$$P_1(x) = e^{2i\theta_j} \left( -\frac{1}{4} e^{2p(xe^{i\theta_j})} - \frac{1}{2} e^{p(xe^{i\theta_j})} p'(xe^{i\theta_j}) \right).$$

For the polynomial p(z) with degree n, set  $p(z) = (\alpha + i\beta)z^n + p_{n-1}(z)$  with  $\alpha$ ,  $\beta$  real, and denote  $\delta(p, \theta) := \alpha \cos n\theta - \beta \sin n\theta$ . The rays

$$\arg z = \theta_k = \frac{\arctan \frac{\alpha}{\beta} + k\pi}{n}, k = 0, 1, 2, \dots, 2n - 1$$

satisfying  $\delta(p, \theta_k) = 0$  can split the complex domain into 2n equal angular domains. Without loss of generality, denote these angle domains as

$$\Omega^{+} := \left\{ z = re^{i\theta} : 0 < r < +\infty, \frac{2i\pi}{n} < \theta < \frac{(2i+1)\pi}{n} \right\}, 
\Omega^{-} := \left\{ z = re^{i\theta} : 0 < r < +\infty, \frac{(2i+1)\pi}{n} < \theta < \frac{2(i+1)\pi}{n} \right\},$$
(35)

 $i=0,1,\cdots,n-1$ , where  $\delta(p,\theta)>0$  on  $\Omega^+$  and  $\delta(p,\theta)<0$  on  $\Omega^-$ . By Lemma 2.7, we obtain

$$|P_{1}(x)| \leq |e^{2p(xe^{i\theta_{j}})}| + |e^{p(xe^{i\theta_{j}})}||p'(xe^{i\theta_{j}})|$$

$$\leq \exp\{\delta(p,\theta)x^{n}\} + \exp\left\{\frac{1}{2}\delta(p,\theta)x^{n}\right\}O(x^{n-1}) \to 0$$
(36)

for  $xe^{i\theta_j} \in \Omega^-$  as  $x \to \infty$ , then by Lemma 2.4 and (34), for any critical line arg  $z = \theta_j$  lying in  $\Omega^-$  there exists a path  $\Gamma_{\theta_i}$  tending to infinity, such that arg  $z \to \theta_j$  on  $\Gamma_{\theta_i}$  while  $y(z) \to 0$  there. Moreover, by

$$\left| \exp\left\{ -\frac{1}{2} \int_{0}^{z} e^{p(z)} dz \right\} \right| \le \exp\left\{ \frac{1}{2} \left| \int_{0}^{z} e^{p(z)} dz \right| \right\} \le \exp\left\{ \frac{1}{2} r \exp\left\{ \frac{1}{2} \delta(p, \theta) r^{n} \right\} \right\} \to 1$$
 (37)

for  $z \in \Omega^-$  as  $r \to \infty$ , together with (30) we have  $f(z) \to 0$  along  $\Gamma_{\theta_i}$  tending to infinity.

Setting V = f'/f, equation (4) can be written as

$$V' + V^2 + e^{p(z)}V + Q(z) = 0. (38)$$

By Lemma 2.6, we have

$$|V'| + |V|^2 = O(|z|^N)$$
 (39)

outside an R-set U, where N is a positive constant. Moreover, if  $z = re^{i\phi} \in \Omega^+$  is such that the ray  $\arg z = \phi$  meets only finitely many discs of U we see that  $V = o(|z|^{-2})$  as z tends to infinity on this ray and hence f tends to a finite, nonzero limit. Applying this reasoning to a set of  $\phi$  outside a set of zero measure we deduce by the Phragmén-Lindelöf principle that without loss of generality, for any small enough given positive  $\varepsilon$ ,

$$f(re^{i\theta}) \to 1 \tag{40}$$

as  $r \to \infty$  with

$$z = re^{i\theta} \in \Omega_{\varepsilon}^{+} := \left\{ z = re^{i\theta} : 0 < r < +\infty, \frac{2i\pi}{n} + \varepsilon < \theta < \frac{(2i+1)\pi}{n} - \varepsilon \right\}. \tag{41}$$

For any  $z = re^{i\theta} \in \Omega^-$ , we have that  $\delta(p, \theta) < 0$ , and by Lemma 2.7 we have

$$\begin{aligned} \left| Q(z) - \frac{1}{4}e^{2p(z)} - \frac{1}{2}e^{p(z)}p'(z) \right| &\leq |Q(z)| + |e^{2p(z)}| + |e^{p(z)}||p'(z)| \\ &\leq O(r^m) + \exp\{\delta(p,\theta)r^n\} + \exp\left\{\frac{1}{2}\delta(p,\theta)r^n\right\}O(r^{n-1}) \end{aligned}$$

$$\leq O(r^m) \tag{42}$$

for sufficiently large r. Applying Lemma 2.5 to (31) and together with (42), v(z) satisfies

$$\log^{+}|y(re^{i\theta})| = O(r^{\frac{m+2}{2}}) \tag{43}$$

as  $r \to \infty$  for any  $z = re^{i\theta} \in \Omega^-$ . From (30) and (37), we have

$$\log^{+}|f(re^{i\theta})| = O(r^{\frac{m+2}{2}}) \tag{44}$$

as  $r \to \infty$  for any  $z = re^{i\theta} \in \Omega^-$ .

On the rays  $\arg z = \theta_k$  such that  $\delta(p,\theta_k) = 0$ , we have  $|e^{p(z)}| = |e^{p_{n-1}(z)}|$ . Consider the two cases  $\delta(p_{n-1},\theta_k) > 0$  or  $\delta(p_{n-1},\theta_k) < 0$ , by the same method above, we get  $f(z) \to 1$  or  $\log^+|f(z)| = O(r^{\frac{m+2}{2}})$ , respectively, on the ray  $\arg z = \theta_k$ . If  $\delta(p_{n-1},\theta_k) = 0$  also, repeating these arguments again. Finally, we deduce that either  $f(z) \to 1$  or  $\log^+|f(z)| = O(r^{\frac{m+2}{2}})$  on the rays  $\arg z = \theta_k$ ,  $k = 0, 1, \dots, 2n-1$ . Thus, (30), (40), (44) and the fact  $\varepsilon$  is arbitrary imply that, by the Phragmén-Lindelöf principle,

$$\sigma(f) \le \frac{m+2}{2}.\tag{45}$$

We claim that  $\frac{(2i+1)\pi}{n}$   $(i=0,1,\cdots,n-1)$  are critical rays for Q(z). For otherwise there exists a critical  $\theta_j$  for Q(z) in

$$\frac{(2i+1)\pi}{n} < \theta_j < \frac{(2i+1)\pi}{n} + \frac{2\pi}{m+2} \quad (i=0,1,\dots,n-1)$$

because m+2>2n. This implies the existence of an unbounded domain of angular measure at most  $\frac{2\pi}{m+2}+\varepsilon$ , bounded by a path on which  $f(z)\to 0$  and a ray on which  $f(z)\to 1$ . By the remark following Lemma 2.7 implies that  $\sigma(f)>\frac{m+2}{2}$ , contradicting (45). Then there exists a positive integer k satisfying  $\frac{2\pi}{n}=k\frac{2\pi}{m+2}$ , that is, m+2=kn, which contradicts n+m+2. Thus, we complete the proof.

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#### **Abbreviations**

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# References

- [1] Laine I., Nevanlinna theory and complex differential equations, Walter de Gruyter, Berlin, 1993.
- [2] Yang C. C., Yi H. X., Uniqueness theory of meromorphic functions, Kluwer Academic Publishers, Dordrecht, 2003.
- [3] Hille E., Ordinary differential equations in the complex domain, New York, Wiley, 1976.
- [4] Gundersen G., Finite order solutions of second order linear differential equations, Trans. Amer. Math. Soc. 1988, 305, 415-429.
- [5] Hellerstein S., Miles J., Rossi J., On the growth of solutions f'' + gf' + hf = 0, Trans. Amer. Math. Soc. 1991, 324, 693-706.
- [6] Langley J. K., On complex oscillation and a problem of Ozawa, Kodai Math. J., 1986, 9, 430-439.
- [7] Gundersen G., On the question of whether  $f'' + e^{-z}f' + B(z)f = 0$  can admit a solution  $f \neq 0$  of finite order, Pro. Roy. Soc. Edinburgh Sect. A, 1986, 102, 9-17.
- [8] Chen Z. X., The growth of solutions of  $f'' + e^{-z}f' + Q(z) = 0$  where the order (Q) = 1, Sci. China Ser A, 1991, 45, 290-300.
- [9] Wu P. C., Zhu J., On the growth of solutions to the complex differential equation f'' + Af' + Bf = 0, Sci. China Math., 2011, 54(5), 939-947.
- [10] Wang J., Chen Z. X., Limiting directions of Julia sets of entire solutions to complex differential equations, Acta Math. Sci, 2017, 37B(1), 97-107.
- [11] Rossi J., Second order differential equations with transcendental coefficients, Proc. Amer. Math. Soc. 1986, 97, 61-66.
- [12] Cao T. B., On the Brück conjecture, Bull. Aust. Math. Soc. 2016, 93(2), 248-259.
- [13] Gundersen G., Estimates for the logarithmic derivative of a meromorphic function, Plus Similar Estimates, J. London Math. Soc. 1988, 37(2), 88-104.
- [14] Tsuji M., Potential theory in modern function theory, Maruzen, Tokyo, 1959.
- [15] Eremenko A., Growth of entire and meromorphic functions on asymptotic curves, Sibirsk Mat. Zh. 1980, 21, 39-51; English transl. in Siberian Math. J. 1981, 673-683.
- [16] Holland A. S. B., Introduction to the theory of entire functions, Academic Press, New York-London, 1973.