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Hongxing Wang* and Jianlong Chen

Weak group inverse

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Abstract: In this paper, we introduce the weak group inverse (called as the WG inverse in the present paper) for square complex matrices of an arbitrary index, and give some of its characterizations and properties. Furthermore, we introduce two orders: one is a pre-order and the other is a partial order, and derive several characterizations of the two orders. The paper ends with a characterization of the core EP order using WG inverses.

Keywords: Group inverse, Weak group inverse, WG order, CE partial order, Core-EP decomposition

MSC: 15A09, 15A57, 15A24

1 Introduction

In this paper, we use the following notations. The symbol $\mathbb{C}_{m,n}$ is the set of $m \times n$ matrices with complex entries; A^*, $\mathcal{R}(A)$ and $\mathrm{rk}\,(A)$ represent the *conjugate transpose*, *range space* (or *column space*) and *rank* of $A \in \mathbb{C}_{m,n}$, respectively. Let $A \in \mathbb{C}_{n,n}$ be singular, the smallest positive integer k satisfying $\mathrm{rk}\left(A^{k+1}\right) = \mathrm{rk}\left(A^k\right)$ is called the *index* of A and is denoted by $\mathrm{Ind}(A)$. The index of a non-singular matrix A is 0 and the index of a null matrix is 1. The symbol $\mathbb{C}_n^{\mathrm{CM}}$ stands for a set of $n \times n$ matrices of index less than or equal to 1. The *Moore-Penrose inverse* of $A \in \mathbb{C}_{m,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying the equations:

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA,$$

and is denoted as $X = A^\dagger$; P_A stands for the orthogonal projection $P_A = AA^\dagger$. A matrix X such that $AXA = A$ is called a generalized inverse of A. The *Drazin inverse* of $A \in \mathbb{C}_{n,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the equations

$$(6^k)\ XA^{k+1} = A^k,\ \ (2)\ XAX = X,\ \ (5)\ AX = XA,$$

and is usually denoted as $X = A^D$, where $k = \mathrm{Ind}(A)$. In particular, when $A \in \mathbb{C}_n^{\mathrm{CM}}$, the matrix X is called the *group inverse* of A, and is denoted as $X = A^\#$ (see [1]). The *core inverse* of $A \in \mathbb{C}_n^{\mathrm{CM}}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying

$$AX = AA^\dagger,\ \ \mathcal{R}(X) \subseteq \mathcal{R}(A)$$

and is denoted as $X = A^{\textcircled{\#}}$ [2]. When $A \in \mathbb{C}_n^{\mathrm{CM}}$, we call it a core invertible (or group invertible) matrix.

Several generalizations of the core inverse have been introduced, for example, the DMP inverse[3] the BT inverse[4] and the core-EP inverse[5], etc. Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}\,(A) = k$. The *DMP inverse* of A is $A^{d,\dagger} = A^D AA^\dagger$ [3]. The *BT inverse* of A is $A^\diamond = \left(A^2A^\dagger\right)^\dagger$ [4, Definition 1]. The *core-EP inverse* of A is

***Corresponding Author: Hongxing Wang:** School of Mathematics, Southeast University, Nanjing, 210096, China and School of Science, Guangxi Key Laboratory of Hybrid Computation and IC Design Analysis, Guangxi University for Nationalities, Nanning, 530006, China, E-mail: winghongxing0902@163.com
Jianlong Chen: School of Mathematics, Southeast University, Nanjing, 210096, China, E-mail: jlchen@seu.edu.cn

$A^{\oplus} = A^k\left(\left(A^*\right)^k A^{k+1}\right)^- \left(A^*\right)^k$ [5, Theorem 3.5 and Remark 2]. It is evident that $A^{\circledast} = A^{\diamond} = A^{d,\dagger} = A^{\oplus}$ in case of $A \in \mathbb{C}_n^{\text{CM}}$. More results on the core inverse and related problems can be seen in [6–10].

Furthermore, it is known that the index of a group invertible matrix is less than or equal to 1, that is, a matrix is core invertible if and only if it is group invertible. Although the generalizations of the core inverse have attracted huge attention, the generalizations of group inverse have not received the same kind of attention. Therefore, it is of interest to inquire whether one can do something similar to the group inverse and that too by using some matrix decompositions as a tool as it has been used to study generalizations of core inverse.

In this paper, our main tool is the core-EP decomposition. By using this decomposition, we introduce a generalization of the group inverse for square matrices of an arbitrary index. We also give some of its characterizations, properties and applications.

2 Preliminaries

In this section, we present some preliminary results.

Lemma 2.1 ([1]). *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$. Then*

$$A^D = A^k\left(A^{k+1}\right)^{\#}. \tag{1}$$

The following decomposition is attributed to Hartwig and Spindelböck[11] and is called Hartwig-Spindelböck decomposition

Lemma 2.2 ([11, Hartwig-Spindelböck Decomposition]). *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{rk}(A) = r$. Then there exists a unitary matrix U such that*

$$A = U\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{2}$$

where $\Sigma = diag\left(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t}\right)$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy $KK^* + LL^* = I_r$.

Furthermore, A is core invertible if and only if K is non-singular, [2]. When $A \in \mathbb{C}_n^{\text{CM}}$, it is easy to check that

$$A^{\oplus} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{3}$$

$$A^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^*, \tag{4}$$

where $T = \Sigma K$ and $S = \Sigma L$.

The core-nilpotent decomposition of a square matrix is widely used in matrix theory [1, 12] and just to remind ourselves it is given as:

Lemma 2.3 ([12, Core-nilpotent Decomposition]). *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, then A can be written as the sum of matrices $\widehat{A}_1$ and $\widehat{A}_2$, i.e. $A = \widehat{A}_1 + \widehat{A}_2$, where*

$$\widehat{A}_1 \in \mathbb{C}_n^{\text{CM}},\ \widehat{A}_2^k = 0 \text{ and } \widehat{A}_1\widehat{A}_2 = \widehat{A}_2\widehat{A}_1 = 0.$$

Very similar to core-nilpotent decomposition is the core-EP decomposition of a square matrix of arbitrary index and was introduced by Wang [13]. We record it as:

Lemma 2.4 ([13, Core-EP Decomposition]). *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, then A can be written as the sum of matrices A_1 and A_2, i.e. $A = A_1 + A_2$, where*

(i) $A_1 \in \mathbb{C}_n^{\text{CM}}$;
(ii) $A_2^k = 0$;
(iii) $A_1^* A_2 = A_2 A_1 = 0$.
Here one or both of A_1 and A_2 can be null.

Lemma 2.5 ([13]). *Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in Lemma 2.4. Then there exists a unitary matrix U such that*

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \quad A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{5}$$

where T is non-singular, and N is nilpotent. Furthermore, the core-EP inverse of A is

$$A^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*. \tag{6}$$

3 WG inverse

In this section, we apply the core-EP decomposition to introduce a generalized group inverse (i.e. the WG inverse) and consider some characterizations of the generalized inverse.

3.1 Definition and properties of the WG inverse

Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$, and consider the system of equations [1]

$$(2') \ AX^2 = X, \quad (3^c) \ AX = A^{\textcircled{\dagger}}A. \tag{7}$$

Theorem 3.1. *The system of equations (7) is consistent and has a unique solution*

$$X = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \tag{8}$$

Proof. Let $A \in \mathbb{C}_{n,n}$ with $\text{Ind}(A) = k$. Since $A^{\textcircled{\dagger}} = A^k\left((A^*)^k A^{k+1}\right)^- (A^*)^k$, $\mathcal{R}\left(A^{\textcircled{\dagger}}A\right) \subseteq \mathcal{R}(A)$. Therefore, (3^c) is consistent. Let A be as in (5). From (6), we obtain

$$\left(A^{\textcircled{\dagger}}\right)^2 A = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* \tag{9}$$

and

$$A\left(\left(A^{\textcircled{\dagger}}\right)^2 A\right) = A^{\textcircled{\dagger}}A, \tag{10}$$

that is, $\left(A^{\textcircled{\dagger}}\right)^2 A$ is a solution to (3^c).

It is obvious that $(2')$ is consistent. Applying (9), we have

$$A\left(\left(\left(A^{\textcircled{\dagger}}\right)^2 A\right)^2\right) = \left(A^{\textcircled{\dagger}}\right)^2 A, \tag{11}$$

that is, $\left(A^{\textcircled{\dagger}}\right)^2 A$ is a solution to $(2')$.

Therefore, from (9), (10) and (11), we derive that (7) is consistent and (8) is a solution of (7).

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1 Since $A^{\textcircled{\dagger}}A$ is core invertible, we use the symbol 3^c in (7).

Furthermore, suppose that both X and Y satisfy (7), then

$$X = AX^2 = A^{\textcircled{\dagger}}AX = A^{\textcircled{\dagger}}A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}AY = AY^2 = Y,$$

that is, the solution to the system of equations (7) is unique. □

Definition 3.2. *Let $A \in \mathbb{C}_{n,n}$ be a matrix of index k. The WG inverse of A, denoted as $A^{\textcircled{w}}$, is defined to be the solution to the system (7) .*

Remark 3.3. *When $A \in \mathbb{C}_n^{\mathrm{CM}}$, we have $A^{\textcircled{w}} = A^{\#}$.*

Remark 3.4. *In [14, Definition 1], the notion of weak Drazin inverse was given: let $A \in \mathbb{C}_{n,n}$ and $\mathrm{Ind}(A) = k$, then X is a weak Drazin inverse of A if X satisfies (6^k). Applying (8), it is easy to check that the WG inverse $A^{\textcircled{w}}$ is a weak Drazin inverse of A.*

Remark 3.5. *Let $A \in \mathbb{C}_{n,n}$. Applying Theorem 3.1, it is easy to check $A^{\textcircled{w}}AA^{\textcircled{w}} = A^{\textcircled{w}}$ and $\mathcal{R}\left(A^{\textcircled{w}}\right) = \mathcal{R}\left(A^k\right)$.*

More details about the weak Drazin inverse can be seen in [14–16].

In the following example, we explain that the WG inverse is different from the Drazin, DMP, core-EP and BT inverses.

Example 3.6. *Let* $A = \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}$. *It is easy to check that* $\mathrm{Ind}(A) = 2$, *the Moore-Penrose inverse* $A^{\dagger}$ *and the Drazin inverse* A^D *are*

$$A^{\dagger} = \begin{bmatrix} 0.5&0&0&0 \\ 0&1&-1&0 \\ 0.5&0&0&0 \\ 0&0&1&0 \end{bmatrix} \text{ and } A^D = \begin{bmatrix} 1&0&1&1 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix},$$

the DMP inverse $A^{d,\dagger}$ *and the BT inverse* $A^{\diamond}$ *are*

$$A^{d,\dagger} = A^DAA^{\dagger} = \begin{bmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \text{ and } A^{\diamond} = \left(A^2A^{\dagger}\right)^{\dagger} = \begin{bmatrix} 0.5&0&0&0 \\ 0&1&0&0 \\ 0.5&0&0&0 \\ 0&0&0&0 \end{bmatrix},$$

and the core-EP inverse $A^{\textcircled{\dagger}}$ *and the WG inverse* $A^{\textcircled{w}}$ *are*

$$A^{\textcircled{\dagger}} = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix} \text{ and } A^{\textcircled{w}} = \begin{bmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}.$$

3.2 Characterizations of the WG inverse

Theorem 3.7. *Let $A \in \mathbb{C}_{n,n}$ be as in (5). Then*

$$A^{\textcircled{w}} = A_1^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^*. \tag{12}$$

Proof. Let $A = \widehat{A}_1 + \widehat{A}_2$ be the core-nilpotent decomposition of $A \in \mathbb{C}_{n,n}$. Then $A^D = \widehat{A}_1^{\#}$. Applying Lemma 2.4, (5) and (8), we derive (12). □

Theorem 3.8. *Let* $A \in \mathbb{C}_{n,n}$ *with* $\operatorname{Ind}(A) = k$. *Then*

$$A^{\textcircled{w}} = \left(AA^{\textcircled{\dagger}}A\right)^{\#} = \left(A^{\textcircled{\dagger}}\right)^2 A = \left(A^2\right)^{\textcircled{\dagger}} A. \tag{13}$$

Proof. Let A be as in (5). Then

$$\begin{aligned} AA^{\textcircled{\dagger}}A &= U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*, \\ \left(A^{\textcircled{\dagger}}\right)^2 &= \left(U\begin{bmatrix} T^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^*\right)^2 = U\begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix}U^*, \\ \left(A^2\right)^{\textcircled{\dagger}} &= \left(U\begin{bmatrix} T^2 & TS + SN \\ 0 & N^2 \end{bmatrix}U^*\right)^{\textcircled{\dagger}} = U\begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix}U^*. \end{aligned}$$

It follows from Theorem 3.7 that

$$\begin{aligned} \left(AA^{\textcircled{\dagger}}A\right)^{\#} &= \left(U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*\right)^{\#} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{w}}, \\ \left(A^{\textcircled{\dagger}}\right)^2 A = \left(A^2\right)^{\textcircled{\dagger}} A &= U\begin{bmatrix} T^{-2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{w}}. \end{aligned}$$

Therefore, we obtain (13). □

Theorem 3.9. *Let* $A \in \mathbb{C}_{n,n}$ *with* $\operatorname{Ind}(A) = k$. *Then*

$$A^{\textcircled{w}} = A^k\left(A^{k+2}\right)^{\textcircled{\#}} A = \left(A^2 P_{A^k}\right)^{\dagger} A. \tag{14}$$

Proof. Let A be as in (5). Then

$$A^k = U\begin{bmatrix} T^k & \Phi \\ 0 & 0 \end{bmatrix}U^*, \tag{15}$$

where $\Phi = \sum\limits_{i=1}^{k} T^{i-1}SN^{k-i}$. It follows that

$$\begin{aligned} A^k\left(A^{k+2}\right)^{\textcircled{\#}} A &= U\begin{bmatrix} T^k & \Phi \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^{-(k+2)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* \\ &= U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{w}}, \end{aligned} \tag{16}$$

$$P_{A^k} = A^k\left(A^k\right)^{\dagger} = U\begin{bmatrix} I_{\operatorname{rk}(A^k)} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

$$\left(A^2 P_{A^k}\right)^{\dagger} A = U\begin{bmatrix} T^2 & 0 \\ 0 & 0 \end{bmatrix}^{\dagger}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^* = A^{\textcircled{w}}. \tag{17}$$

Therefore, we have (14). □

It is known that the Drazin inverse is one generalization of the group inverse. We will see the similarities and differences between the Drazin inverse and the WG inverse from the following corollaries.

Corollary 3.10. *Let* $A \in \mathbb{C}_{n,n}$ *with* $\operatorname{Ind}(A) = k$. *Then*

$$\operatorname{rk}\left(A^{\textcircled{w}}\right) = \operatorname{rk}\left(A^D\right) = \operatorname{rk}\left(A^k\right).$$

It is well known that $\left(A^2\right)^D = \left(A^D\right)^2$, but the same is not true for the WG inverse. Applying the core-EP decomposition (5) of A, we have

$$A^2 = U\begin{bmatrix} T^2 & TS + SN \\ 0 & N^2 \end{bmatrix} U^* \tag{18}$$

and

$$\left(A^2\right)^{ⓦ} = U\begin{bmatrix} T^{-2} & T^{-4}\left(TS + SN\right) \\ 0 & 0 \end{bmatrix} U^*, \ \left(A^{ⓦ}\right)^2 = U\begin{bmatrix} T^{-2} & T^{-3}S \\ 0 & 0 \end{bmatrix} U^*. \tag{19}$$

Therefore, $\left(A^2\right)^{ⓦ} = \left(A^{ⓦ}\right)^2$ if and only if $T^{-4}\left(TS + SN\right) = T^{-3}S$. Since T is invertible, we derive the following Corollary 3.11.

Corollary 3.11. *Let $A \in \mathbb{C}_{n,n}$ be as in (5). Then $\left(A^2\right)^{ⓦ} = \left(A^{ⓦ}\right)^2$ if and only if $SN = 0$.*

The commutativity is one of the main characteristics of the group inverse. The Drazin inverse too has the characteristic. It is of interest to inquire whether the same is true or not for the WG inverse. Applying the core-EP decomposition (5) of A, we have

$$AA^{ⓦ} = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U\begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*; \tag{20a}$$

$$A^{ⓦ}A = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix} U^* = U\begin{bmatrix} I & T^{-1}S + T^{-2}SN \\ 0 & 0 \end{bmatrix} U^*. \tag{20b}$$

Therefore, we have the following Corollary 3.12.

Corollary 3.12. *Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in (5). Then $AA^{ⓦ} = A^{ⓦ}A$ if and only if $SN = 0$.*

Corollary 3.13. *Let $A \in \mathbb{C}_{n,n}$ with $\mathrm{Ind}(A) = k$, the core-EP decomposition of A be as in (5) and $SN = 0$. Then*

$$A^{ⓦ} = A^D = \left(A^{k+1}\right)^{\textcircled{\#}} A^k = \left(A^{t+1}\right)^{\textcircled{\dagger}} A^t,$$

where t is an arbitrary positive integer.

Proof. Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in (5).

By applying $SN = 0$ and $\mathrm{Ind}(A) = k$, we have

$$A^{k-1} = U\begin{bmatrix} T^{k-1} & T^{k-2}S \\ 0 & N^{k-1} \end{bmatrix} U^*, \ A^k = U\begin{bmatrix} T^k & T^{k-1}S \\ 0 & 0 \end{bmatrix} U^*, \ A^{k+1} = U\begin{bmatrix} T^{k+1} & T^kS \\ 0 & 0 \end{bmatrix} U^*.$$

It follows from applying (1), (4) and (6) that

$$\left(A^{k+1}\right)^{\#} = \left(A^{k+1}\right)^{\textcircled{\#}} = U\begin{bmatrix} T^{-(k+1)} & T^{-(k+2)}S \\ 0 & 0 \end{bmatrix} U^*,$$

$$A^D = \left(A^{k+1}\right)^{\#} A^k = U\begin{bmatrix} T^{-(k+1)} & T^{-(k+2)}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^k & T^{k-1}S \\ 0 & 0 \end{bmatrix} U^*$$

$$= U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = A^{ⓦ}.$$

Therefore, $A^{ⓦ} = A^D = \left(A^{k+1}\right)^{\textcircled{\#}} A^k$.

Let t be an arbitrary positive integer. By applying $SN = 0$, we have

$$A^t = U\begin{bmatrix} T^t & T^{t-1}S \\ 0 & N^t \end{bmatrix} U^*, \ A^{t+1} = U\begin{bmatrix} T^{t+1} & T^tS \\ 0 & N^{t+1} \end{bmatrix} U^*.$$

It follows from Lemma 2.5 that

$$\left(A^{t+1}\right)^{\circledcirc} = U\begin{bmatrix} T^{-(t+1)} & 0 \\ 0 & 0 \end{bmatrix}U^*,$$

$$\left(A^{t+1}\right)^{\circledcirc} A^t = U\begin{bmatrix} T^{-(t+1)} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^t & T^{t-1}S \\ 0 & N^t \end{bmatrix}U^* = A^{\circledS}, \tag{21}$$

Therefore, we derive $A^{\circledS} = \left(A^{t+1}\right)^{\circledcirc} A^t$, in which t is an arbitrary positive integer. □

4 Two orders

Recall the definitions of the *minus partial order*, *sharp partial order*, *Drazin order* and *core-nilpotent partial order* [12] :

$$A \overset{-}{\leq} B: \ A, B \in \mathbb{C}_{m,n}, \ \operatorname{rk}(B - A) = \operatorname{rk}(B) - \operatorname{rk}(A), \tag{22}$$

$$A \overset{\#}{\leq} B: \ A, B \in \mathbb{C}_n^{\text{CM}}, \ A^2 = AB = BA, \tag{23}$$

$$A \overset{D}{\leq} B: \ A, B \in \mathbb{C}_{n,n}, \ \widehat{A}_1 \overset{\#}{\leq} \widehat{B}_1, \tag{24}$$

$$A \overset{\#,-}{\leq} B: \ A, B \in \mathbb{C}_{n,n}, \ \widehat{A}_1 \overset{\#}{\leq} \widehat{B}_1 \text{ and } \widehat{A}_2 \overset{-}{\leq} \widehat{B}_2, \tag{25}$$

in which $A = \widehat{A}_1 + \widehat{A}_2$ and $B = \widehat{B}_1 + \widehat{B}_2$ are the core-nilpotent decompositions of A and B, respectively. Similarly, in this section, we apply the core-EP decomposition to introduce two orders: one is the WG order and the other is the CE partial order.

4.1 WG order

Consider the binary relation:

$$A \overset{\text{WG}}{\leq} B : A, B \in \mathbb{C}_{n,n}, \ \ if \ A_1 \overset{\#}{\leq} B_1, \tag{26}$$

in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B, respectively.

Reflexivity of the relation is obvious. Suppose $A \overset{\text{WG}}{\leq} B$ and $B \overset{\text{WG}}{\leq} C$, in which $A = A_1 + A_2$, $B = B_1 + B_2$ and $C = C_1 + C_2$ are the core-EP decompositions of A, B and C, respectively. Then $A_1 \overset{\#}{\leq} B_1$ and $B_1 \overset{\#}{\leq} C_1$. Therefore $A_1 \overset{\#}{\leq} C_1$. It follows from (26) that $A \overset{\text{WG}}{\leq} C$.

Example 4.1. *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Although $A \overset{\text{WG}}{\leq} B$ and $B \overset{\text{WG}}{\leq} A$, $A \neq B$. Therefore, the anti-symmetry of the binary operation (26) does not hold in general.

Therefore, we have the following Theorem 4.2.

Theorem 4.2. *The binary relation (26) is a pre-order. We call this pre-order the weak-group (WG for short) order.*

Remark 4.3. *The WG order coincides with the sharp partial order on $\mathbb{C}_n^{\text{CM}}$.*

We give below two examples to show that WG order is different from Drazin order and that either of two orders does not imply the other order.

Example 4.4. *Let A and B be as in Example 4.1. We have*

$$A^D = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $A \overset{\text{WG}}{\leq} B$.

Since $A^D A \neq A^D B$, *we derive* $A \overset{D}{\nleq} B$. *Therefore, the WG order does not imply the Drazin order.*

Example 4.5. *Let*

$$\widehat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \widehat{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A = P\widehat{A}P^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B = P\widehat{B}P^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A_2 = 0, \; B_1 = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

in which $A = A_1 + A_2$ *and* $B = B_1 + B_2$ *are the core-EP decompositions of A and B, respectively. Then* $A \overset{D}{\leq} B$ *and* $A_1 \overset{\#}{\nleq} B_1$. *Therefore, the Drazin order does not imply the WG order.*

It is well known that $A \overset{D}{\leq} B \Rightarrow A^2 \overset{D}{\leq} B^2$, but the same is not true for the WG order as the following example shows:

Example 4.6. *Let A and B be as in Example 4.1, then*

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B^2 = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We derive $A^2 \overset{\text{WG}}{\nleq} B^2$. *Therefore,* $A \overset{\text{WG}}{\leq} B \nRightarrow A^2 \overset{\text{WG}}{\leq} B^2$.

Theorem 4.7. *Let* $A, B \in \mathbb{C}_{n,n}$. *Then* $A \overset{\text{WG}}{\leq} B$ *if and only if there exists a unitary matrix* $\widehat{U}$ *such that*

$$A = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix} \widehat{U}^*, \tag{27a}$$

$$B = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1 T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1 S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2 \end{bmatrix} \widehat{U}^*, \tag{27b}$$

where T *and* T_1 *are invertible,* $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ *and* N_2 *are nilpotent.*

Proof. Assume $A \overset{\text{WG}}{\leq} B$. Let $A = A_1 + A_2$ and $B = B_1 + B_2$ be the core-EP decompositions of A and B, A_1 and A_2 be as given in (5), and partition

$$U^* B_1 U = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}. \tag{28}$$

Applying (12) gives

$$A_1 A_1^{\#} = U \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*;$$

$$B_1 A_1^{\#} = U \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} B_{11}T^{-1} & B_{11}T^{-2}S \\ B_{21}T^{-1} & B_{21}T^{-2}S \end{bmatrix} U^*.$$

Since $A \overset{\text{WG}}{\leq} B$, $A_1 \overset{\#}{\leq} B_1$. It follows from $A_1 A_1^{\#} = B_1 A_1^{\#}$ that

$$B_{11} = T \text{ and } B_{21} = 0. \tag{29}$$

By applying (12) and (29), we have

$$A_1^{\#} A_1 = U \begin{bmatrix} I & T^{-1}S \\ 0 & 0 \end{bmatrix} U^*,$$

$$A_1^{\#} B_1 = U \begin{bmatrix} I & T^{-1}B_{12} + T^{-2}SB_{22} \\ 0 & 0 \end{bmatrix} U^*.$$

It follows from $A_1^{\#} A_1 = A_1^{\#} B_1$ that

$$T^{-1}\left(S - T^{-1}SB_{22} - B_{12}\right) = 0.$$

Therefore,

$$B_{12} = S - T^{-1}SB_{22}, \tag{30}$$

in which B_{22} is an arbitrary matrix of an appropriate size. From (29) and (30), we obtain

$$B_1 = U \begin{bmatrix} T & S - T^{-1}SB_{22} \\ 0 & B_{22} \end{bmatrix} U^*. \tag{31}$$

Since B_1 is core invertible and T is non-singular, B_{22} is core invertible. Let the core-EP decomposition of B_{22} be as

$$B_{22} = U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^*, \tag{32}$$

where T_1 is invertible. Denote

$$\widehat{U} = U \begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix}.$$

It is easy to see that $\widehat{U}$ is a unitary matrix. Let SU_1 be partitioned as follows:

$$SU_1 = \left[\widehat{S}_1 \; \widehat{S}_2\right],$$

where the number of columns of $\widehat{S}_1$ coincides with the size of the square matrix T_1. Then

$$A_1 = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^* \tag{33}$$

and

$$B_1 = U \begin{bmatrix} T & S - T^{-1} S B_{22} \\ 0 & U_1 \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} U_1^* \end{bmatrix} U^*$$

$$= U \begin{bmatrix} I & 0 \\ 0 & U_1 \end{bmatrix} \begin{bmatrix} T & SU_1 - T^{-1} SU_1 U_1^* B_{22} U_1 \\ 0 & \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U_1^* \end{bmatrix} U^*$$

$$= \widehat{U} \begin{bmatrix} T & \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix} - T^{-1} \begin{bmatrix} \widehat{S}_1 & \widehat{S}_2 \end{bmatrix} \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \\ 0 & \begin{bmatrix} T_1 & S_1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \widehat{U}^*$$

$$= \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1} \widehat{S}_1 T_1 & \widehat{S}_2 - T^{-1} \widehat{S}_1 S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^*. \tag{34}$$

From (26), (33) and (34), we derive (27a) and (27b). □

4.2 CE partial order

Consider the binary relation:

$$A \overset{\text{CE}}{\leq} B : A, B \in \mathbb{C}_{n,n},\ A_1 \overset{\#}{\leq} B_1 \text{ and } A_2 \overset{-}{\leq} B_2, \tag{35}$$

in which $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B, respectively.

Definition 4.8. *Let $A, B \in \mathbb{C}_{n,n}$. We say that A is below B under the core-EP-minus (CE for short) order if A and B satisfy the binary relation (35).*

When A is below B under the CE order, we write $A \overset{\text{CE}}{\leq} B$.

Remark 4.9. *According to (26) and (35) we derive that the CE order implies the WG order, that is,*

$$A \overset{\text{CE}}{\leq} B \;\Rightarrow\; A \overset{\text{WG}}{\leq} B. \tag{36}$$

Furthermore,

$$A \overset{\text{CE}}{\leq} B \Leftrightarrow A \overset{\text{WG}}{\leq} B \text{ and } A_2 \overset{-}{\leq} B_2. \tag{37}$$

Theorem 4.10. *The CE order is a partial order.*

Proof. Reflexivity is trivial.

Let $A \overset{\text{CE}}{\leq} B$, $B \overset{\text{CE}}{\leq} C$ and $A = A_1 + A_2$, $B = B_1 + B_2$ and $C = C_1 + C_2$ are the core-EP decompositions of A, B and C, respectively. Then $A_1 \overset{\#}{\leq} B_1$, $B_1 \overset{\#}{\leq} C_1$ and $A_2 \overset{-}{\leq} B_2$, $B_2 \overset{-}{\leq} C_2$. Therefore $A_1 \overset{\#}{\leq} C_1$ and $A_2 \overset{-}{\leq} C_2$. It follows from Definition 4.8 that $A \overset{\text{CE}}{\leq} C$.

If $A \overset{\text{CE}}{\leq} B$ and $B \overset{\text{CE}}{\leq} A$, Then $A_1 = B_1$ and $A_2 = B_2$, that is, $A = B$. □

Theorem 4.11. *Let $A, B \in \mathbb{C}_{n,n}$. Then $A \overset{\text{CE}}{\leq} B$ if and only if there exists a unitary matrix $\widehat{U}$ satisfying*

$$A = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & N_{22} \end{bmatrix} \widehat{U}^*, \tag{38a}$$

$$B = \widehat{U}\begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1 T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1 S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2 \end{bmatrix}\widehat{U}^*, \tag{38b}$$

where T and T_1 are invertible, N_{22} and N_2 are nilpotent, and $N_{22} \overset{-}{\le} N_2$.

Proof. Let $A \overset{\text{CE}}{\le} B$, and $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B, respectively. Then $A_1 \overset{\#}{\le} B_1$ and $A_2 \overset{-}{\le} B_2$. By applying Lemma 2.5, Theorem 4.7 and $A_1 \overset{\#}{\le} B_1$, we have

$$A_1 = \widehat{U}\begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\widehat{U}^*, \ A_2 = \widehat{U}\begin{bmatrix} 0 & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix}\widehat{U}^*,$$

$$B_1 = \widehat{U}\begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1 T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1 S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & 0 \end{bmatrix}\widehat{U}^*, \ B_2 = \widehat{U}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix}\widehat{U}^*,$$

where $\widehat{U}$, T, T_1, $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}$ and N_2 are as in Theorem 4.7.

Since $A_2 \overset{-}{\le} B_2$, we have $\text{rk}(B_2 - A_2) = \text{rk}(B_2) - \text{rk}(A_2)$, that is,

$$\text{rk}\left(\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right) = \text{rk}(N_2) - \text{rk}\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right). \tag{39}$$

In addition, it is easy to check that

$$\begin{aligned} \text{rk}(N_2) - \text{rk}\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right) &\le \text{rk}(N_2) - \text{rk}(N_{22}) \\ &\le \text{rk}(N_2 - N_{22}) \le \text{rk}\left(\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right). \end{aligned} \tag{40}$$

Applying (39) to (40) we obtain

$$\text{rk}(N_{22}) = \text{rk}\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right) \tag{41}$$

$$\text{rk}(N_2) - \text{rk}(N_{22}) = \text{rk}(N_2 - N_{22}). \tag{42}$$

Therefore, we obtain

$$N_{22} \overset{-}{\le} N_2. \tag{43}$$

Since $N_{22} \overset{-}{\le} N_2$, there exist nonsingular matrices P and Q such that

$$N_{22} = P\begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}Q, \quad N_2 = P\begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}Q,$$

where D_1 and D_2 are nonsingular, (see [12, Theorem 3.7.3]). It follows that

$$\text{rk}(N_{22}) = \text{rk}(D_1) \ \text{ and } \ \text{rk}(N_2) - \text{rk}(N_{22}) = \text{rk}(D_2). \tag{44}$$

Denote

$$N_{12} = \begin{bmatrix} M_{12} & M_{13} & M_{14} \end{bmatrix}Q \ \text{ and } \ N_{21} = P\begin{bmatrix} M_{21} \\ M_{31} \\ M_{41} \end{bmatrix}. \tag{45}$$

Then

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} = \begin{bmatrix} I_{\mathrm{rk}(N_{11})} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} N_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & D_1 & 0 & 0 \\ M_{31} & 0 & 0 & 0 \\ M_{41} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\mathrm{rk}(N_{11})} & 0 \\ 0 & Q \end{bmatrix}$$

and

$$\mathrm{rk}\left(\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right) = \mathrm{rk}\,(D_1) + \mathrm{rk}\left(\begin{bmatrix} M_{13} & M_{14} \end{bmatrix}\right) + \mathrm{rk}\left(\begin{bmatrix} M_{31} \\ M_{41} \end{bmatrix}\right) + \mathrm{rk}\left(N_{11} - M_{12}D_1^{-1}M_{21}\right)$$

It follows from (44) and (41) that

$$M_{13} = 0,\ M_{14} = 0,\ M_{31} = 0 \text{ and } M_{41} = 0. \tag{46}$$

Therefore,

$$\begin{bmatrix} -N_{11} & -N_{12} \\ -N_{21} & N_2 - N_{22} \end{bmatrix} = \begin{bmatrix} I_{\mathrm{rk}(N_{11})} & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} -N_{11} & -M_{12} & 0 & 0 \\ -M_{21} & 0 & 0 & 0 \\ 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{\mathrm{rk}(N_{11})} & 0 \\ 0 & Q \end{bmatrix}.$$

By applying (41), (44) and $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \overset{-}{\leq} \begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix}$, we derive that

$$\begin{aligned} \mathrm{rk}\left(\begin{bmatrix} 0 & 0 \\ 0 & N_2 \end{bmatrix} - \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\right) &= \mathrm{rk}\left(\begin{bmatrix} N_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix}\right) + \mathrm{rk}\,(D_2) \\ &= \mathrm{rk}\,(N_2) - \mathrm{rk}\,(N_{22}) \\ &= \mathrm{rk}\,(D_2)\,. \end{aligned}$$

Therefore, $\begin{bmatrix} N_{11} & M_{12} \\ M_{21} & 0 \end{bmatrix} = 0$, that is, $N_{11} = 0$, $M_{12} = 0$ and $M_{21} = 0$. By applying (45) and (46), we obtain $N_{11} = 0$, $N_{12} = 0$ and $N_{21} = 0$. So, we obtain (38a) and (38b).

Let A and B be of the forms as given in (38a) and (38b). It is easy to check that $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B, respectively, and

$$A_1 = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 & \widehat{S}_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^*,\ A_2 = \widehat{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_{22} \end{bmatrix} \widehat{U}^*;$$

$$B_1 = \widehat{U} \begin{bmatrix} T & \widehat{S}_1 - T^{-1}\widehat{S}_1 T_1 & \widehat{S}_2 - T^{-1}\widehat{S}_1 S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & 0 \end{bmatrix} \widehat{U}^*,\ B_2 = \widehat{U} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & N_2 \end{bmatrix} \widehat{U}^*.$$

It follows from (23) and $N_{22} \overset{-}{\leq} N_2$ that $A_1 \overset{\#}{\leq} B_1$ and $A_2 \overset{-}{\leq} B_2$. Therefore, $A \overset{\mathrm{CE}}{\leq} B$. □

Remark 4.12. *In Ex. 4.5, it is easy to check that $A \overset{\#,-}{\leq} B$. Since $A_1 \overset{\#}{\nleq} B_1$, we have $A \overset{\mathrm{CE}}{\nleq} B$. Therefore, the core-nilpotent partial order does not imply the CE partial order.*

Corollary 4.13. *Let $A, B \in \mathbb{C}_{n,n}$. If $A \overset{\mathrm{CE}}{\leq} B$, then $A \overset{-}{\leq} B$.*

Proof. Let $A, B \in \mathbb{C}_{n,n}$. Then A and B have the forms as given in Theorem 4.11. According to $A \overset{\mathrm{CE}}{\leq} B$, we have $N_{22} \overset{-}{\leq} N_2$, that is,

$$\mathrm{rk}\,(N_2 - N_{22}) = \mathrm{rk}\,(N_2) - \mathrm{rk}\,(N_{22})\,. \tag{47}$$

Since T and T_1 are invertible, it follows that

$$\begin{aligned}
\mathrm{rk}(B) &= \mathrm{rk}(T) + \mathrm{rk}(T_1) + \mathrm{rk}(N_2);\\
\mathrm{rk}(A) &= \mathrm{rk}(T) + \mathrm{rk}(N_{22});\\
\mathrm{rk}(B-A) &= \mathrm{rk}\left(\begin{bmatrix} 0 & -T^{-1}\widehat{S}_1T_1 & -T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2-N_{22}\end{bmatrix}\right)\\
&= \mathrm{rk}\left(\begin{bmatrix} I_{\mathrm{rk}(T)} & T^{-1}\widehat{S}_1 & 0 \\ 0 & I_{\mathrm{rk}(T_1)} & 0 \\ 0 & 0 & I_{n-\mathrm{rk}(T)-\mathrm{rk}(T_1)}\end{bmatrix}\begin{bmatrix} 0 & -T^{-1}\widehat{S}_1T_1 & -T^{-1}\widehat{S}_1S_1 \\ 0 & T_1 & S_1 \\ 0 & 0 & N_2-N_{22}\end{bmatrix}\right)\\
&= \mathrm{rk}\left(\begin{bmatrix} T_1 & S_1 \\ 0 & N_2-N_{22}\end{bmatrix}\right) = \mathrm{rk}\left(\begin{bmatrix} T_1 & 0 \\ 0 & N_2-N_{22}\end{bmatrix}\right)\\
&= \mathrm{rk}(T_1) + \mathrm{rk}(N_2-N_{22}).
\end{aligned} \tag{48}$$

Therefore, by applying (22), (47) and (48) we derive $\mathrm{rk}(B-A) = \mathrm{rk}(B) - \mathrm{rk}(A)$, that is, $A \overset{-}{\leq} B$. □

5 Characterizations of the core-EP order

As is noted in [13], the core-EP order is given:

$$A \overset{\textcircled{\dagger}}{\leq} B: \ A, B \in \mathbb{C}_{n,n}, A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B \text{ and } AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}. \tag{49}$$

Some characterizations of the core-EP order are given in [13].

Lemma 5.1 ([13]). *Let $A, B \in \mathbb{C}_{n,n}$ and $A \overset{\textcircled{\dagger}}{\leq} B$. Then there exists a unitary matrix U such that*

$$A = U\begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22}\end{bmatrix}U^*, \quad B = U\begin{bmatrix} T_1 & T_2 & S_1 \\ 0 & T_3 & S_2 \\ 0 & 0 & N_2\end{bmatrix}U^*, \tag{50}$$

where $\begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22}\end{bmatrix}$ and N_2 are nilpotent, T_1 and T_3 are non-singular .

Theorem 5.2. *Let $A, B \in \mathbb{C}_{n,n}$. Then $A \overset{\textcircled{\dagger}}{\leq} B$ if and only if*

$$AA^{\textcircled{w}} = BA^{\textcircled{w}} \text{ and } A^*A^{\textcircled{w}} = B^*A^{\textcircled{w}}. \tag{51}$$

Proof. Let A be as given in (5), and denote

$$U^*BU = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4\end{bmatrix}. \tag{52}$$

By applying (20a) and

$$BA^{\textcircled{w}} = U\begin{bmatrix} B_1 & B_2 \\ B_3 & B_4\end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0\end{bmatrix}U^* = U\begin{bmatrix} B_1T^{-1} & B_1T^{-2}S \\ B_3T^{-1} & B_3T^{-2}S\end{bmatrix}U^*,$$

we have $AA^{\textcircled{w}} = BA^{\textcircled{w}}$ if and only if

$$B_1 = T \text{ and } B_3 = 0.$$

It follows that

$$A^*A^{\textcircled{w}} = U\begin{bmatrix} T^* & 0 \\ S^* & N^* \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*T^{-1} & T^*T^{-2}S \\ S^*T^{-1} & S^*T^{-2}S \end{bmatrix}U^*,$$
$$B^*A^{\textcircled{w}} = U\begin{bmatrix} T^* & 0 \\ B_2^* & B_4^* \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^* = U\begin{bmatrix} T^*T^{-1} & T^*T^{-2}S \\ B_2^*T^{-1} & B_2^*T^{-2}S \end{bmatrix}U^*.$$

Therefore, $AA^{\textcircled{w}} = BA^{\textcircled{w}}$ and $A^*A^{\textcircled{w}} = B^*A^{\textcircled{w}}$ if and only if

$$B_1 = T, B_3 = 0, B_2 = S, \text{ and } B_4 \text{ is arbitrary,} \tag{53}$$

that is, A and B satisfy $AA^{\textcircled{w}} = BA^{\textcircled{w}}$ and $A^*A^{\textcircled{w}} = B^*A^{\textcircled{w}}$ if and only if there exists a unitary matrix U such that

$$A = U\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^*, \quad B = U\begin{bmatrix} T & S \\ 0 & B_4 \end{bmatrix}U^*, \tag{54}$$

where N is nilpotent, T is non-singular and B_4 is arbitrary. Therefore, by applying Lemma 5.1, we derive the characterization (51) of the core-EP order. □

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