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Lutfi Kalantan and Ibtesam Alshammari*

Epi-mild normality

Abstract: A space $(X, \mathcal{T})$ is called epi-mildly normal if there exists a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is Hausdorff (T_2) mildly normal. We investigate this property and present some examples to illustrate the relationships between epi-mild normality and other weaker kinds of normality.

In this paper, we introduce epi-mildly normal topological spaces. We investigate the property of epi-mild normality and present some examples to illustrate the relationships between epi-mild normality and other weaker kinds of normality. Throughout this paper, we denote an ordered pair by $\langle x, y \rangle$, the set of positive integers by $\mathbb{N}$, the set of rational numbers by $\mathbb{Q}$, the set of irrational numbers by $\mathbb{P}$, and the set of real numbers by $\mathbb{R}$. A T_4 space is a T_1 normal space, a Tychonoff $(T_{3\frac{1}{2}})$ space is a T_1 completely regular space, and a T_3 space is a T_1 regular space. We do not assume T_2 in the definition of compactness, countable compactness and paracompactness. For a subset A of a space X, int$A$ and $\overline{A}$ denote the interior and the closure of A, respectively. An ordinal γ is the set of all ordinals α such that $\alpha < \gamma$. The first infinite ordinal is ω_0 and the first uncountable ordinal is ω_1.

Definition 1. *A subset A of a space X is called closed domain [1], called also regularly closed, κ-closed, if $A = \overline{\text{int}A}$. A subset A of a space X is called open domain [1], called also regularly open, κ-open, if $A = \text{int}(\overline{A})$. A space X is called mildly normal [2], called also κ-normal [3], if for any two disjoint closed domains A and B of X there exist two disjoint open sets U and V of X such that $A \subseteq U$ and $B \subseteq V$.*

Definition 2. *A space $(X, \mathcal{T})$ is called epi-mildly normal if there exists a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_2 (Hausdorff) mildly normal.*

Note that if we require $(X, \mathcal{T}')$ to be just mildly normal in Definition 2 above, then any space will be epi-mildly normal as the indiscrete topology will refine. Also, if we require $(X, \mathcal{T}')$ to be T_1 mildly normal in Definition 2 above, then any T_1 space will be epi-mildly normal as the finite complement topology, see [4], will refine. It is clear from the definition that any T_2 mildly normal space is epi-mildly normal, just take the coarser topology equal the same topology. Observe that if $\mathcal{T}'$ and $\mathcal{T}$ are two topologies on X such that $\mathcal{T}'$ is coarser than $\mathcal{T}$ and $(X, \mathcal{T}')$ is T_i, $i \in \{0, 1, 2\}$, then so is $(X, \mathcal{T})$. So, we conclude the following.

Theorem 3. *Every epi-mildly normal space is T_2.*

Lutfi Kalantan: Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia, E-mail: LNKalantan@hotmail.com, lkalantan@kau.edu.sa

*Corresponding Author: Ibtesam Alshammari: Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia and Department of Mathematics, Faculty of Science, University of Hafr Al Batin, P.O.Box 1803, Hafr Al Batin 31991, Saudi Arabia, E-mail: iealshamri@hotmail.com, iealshamri@uohb.edu.sa

Recall that a topological space X is called completely Hausdorff, $T_{2\frac{1}{2}}$ [4] (called also Urysohn space [1]), if for each distinct elements $a, b \in X$ there exist two open sets U and V such that $a \in U$, $b \in V$, and $\overline{U} \cap \overline{V} = \varnothing$. A topological space $(X, \mathcal{T})$ is called submetrizable if there exists a metric d on X such that the topology $\mathcal{T}_d$ on X generated by d is coarser than $\mathcal{T}$ [5]. A topological space $(X, \mathcal{T})$ is called epinormal if there is a coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is T_4 [6]. For epinormality, we have something stronger.

Theorem 4. *Every epinormal space is completely Hausdorff.*

Proof. Let $(X, \mathcal{T})$ be any epinormal space. Let $\mathcal{T}'$ be a coarser topology on X such that $(X, \mathcal{T}')$ is T_4. We may assume that X has more than one element and pick distinct $a, b \in X$. Using T_2 of $(X, \mathcal{T}')$, choose $G, H \in \mathcal{T}'$ such that $a \in G$, $b \in H$, and $G \cap H = \varnothing$. Using regularity of $(X, \mathcal{T}')$, choose $U, V \in \mathcal{T}'$ such that $a \in U \subseteq \overline{U}^{\mathcal{T}'} \subseteq G$ and $b \in V \subseteq \overline{V}^{\mathcal{T}'} \subseteq H$. We have that $U, V \in \mathcal{T}$ and since $\overline{A}^{\mathcal{T}} \subseteq \overline{A}^{\mathcal{T}'}$ for any $A \subseteq X$, we get $\overline{U}^{\mathcal{T}} \cap \overline{V}^{\mathcal{T}} = \varnothing$. $\square$

Note that an epi-mildly normal space may not be completely Hausdorff and here is an example.

Example 5. *Let $X = \{\langle x, y \rangle : 0 \le y, x, y \in \mathbb{Q}\}$ and consider the irrational number $\sqrt{3}$. The Irrational Slop topology $\mathcal{IS}$ [4] on X is generated by the neighborhoods of the form $N_\epsilon(\langle x, y \rangle) = \{\langle x, y \rangle\} \cup B_\epsilon(x + \frac{y}{\sqrt{3}}) \cup B_\epsilon(x - \frac{y}{\sqrt{3}})\}$, where $B_\epsilon\langle \zeta \rangle = \{\langle r, 0 \rangle : r \in \mathbb{Q} \text{ and } | r - \zeta | < \epsilon\}$, which is a subset of the x-axis. Each $N_\epsilon(\langle x, y \rangle)$ consists of $\{\langle x, y \rangle\}$ plus two intervals on the rational x-axis centered at the two irrational points $x \pm \frac{y}{\sqrt{3}}$; the lines joining these points to $\langle x, y \rangle$ have slop $\pm\sqrt{3}$. Note that $(X, \mathcal{IS})$ is Hausdorff but not completely Hausdorff [4]. Moreover, $(X, \mathcal{IS})$ is mildly normal as the only disjoint closed domains are the ground set X and the empty set, hence it is epi-mildly normal.*

It is clear from the definitions that

$$\text{submetrizability} \implies \text{epinormality} \implies \text{epi-mild normality.}$$

The above implications are not reversible. $\omega_1 + 1$ is epinormal but not submetrizable [6]. For the second implication, the Irrational Slop Space is epi-mildly normal which is not epinormal because it is not completely Hausdorff. The question is whether there exist Tychonoff epi-mildly normal spaces which are not epinormal. We will give a partial answer in the class of minimal spaces below. Now, $(\mathbb{R}, \mathcal{T}_p)$, where $\mathcal{T}_p$ is the particular point topology, $p \in \mathbb{R}$ [4], is mildly normal because the only closed domains are $\varnothing$ and $\mathbb{R}$, but it is not epi-mildly normal because it is not T_2. Here is an example of a Tychonoff zero-dimensional scattered epi-mildly normal space which is not mildly normal. See also Example 9.

Example 6. *For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $p_n = \langle p, \frac{1}{n} \rangle \in \mathbb{R}^2$. For each $p \in \mathbb{P}$, fix a sequence $(p_n^\star)_{n \in \mathbb{N}}$ of rational numbers such that $p_n' = \langle p_n^\star, 0 \rangle \longrightarrow \langle p, 0 \rangle$, where the convergence is taken in $\mathbb{R}^2$ with its usual topology. For each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $A_n(\langle p, 0 \rangle) = \{p_k' : k \ge n\}$ and $B_n(\langle p, 0 \rangle) = \{p_k : k \ge n\}$. Now, for each $p \in \mathbb{P}$ and $n \in \mathbb{N}$, let $U_n(\langle p, 0 \rangle) = \{\langle p, 0 \rangle\} \cup A_n(\langle p, 0 \rangle) \cup B_n(\langle p, 0 \rangle)$.*
  Let $X = \{\langle x, 0 \rangle \in \mathbb{R}^2 : x \in \mathbb{R}\} \cup \{\langle p, \frac{1}{n} \rangle = p_n : p \in \mathbb{P} \text{ and } n \in \mathbb{N}\}$. For each $q \in \mathbb{Q}$, let $\mathcal{B}(\langle q, 0 \rangle) = \{\{\langle q, 0 \rangle\}\}$. For each $p \in \mathbb{P}$, let $\mathcal{B}(\langle p, 0 \rangle) = \{U_n(\langle p, 0 \rangle) : n \in \mathbb{N}\}$. For each $p \in \mathbb{P}$ and each $n \in \mathbb{N}$, let $\mathcal{B}(p_n) = \{\{p_n\}\}$. Denote by $\mathcal{T}$ the unique topology on X that has $\{\mathcal{B}(\langle x, 0 \rangle), \mathcal{B}(p_n) : x \in \mathbb{R}, p \in \mathbb{P}, n \in \mathbb{N}\}$ as its neighborhood system. Let $Z = \{\langle x, 0 \rangle : x \in \mathbb{R}\}$. That is, Z is the x-axis. Then $(Z, \mathcal{T}_Z) \cong (\mathbb{R}, \mathcal{RS})$, where $\mathcal{RS}$ is the Rational Sequence Topology, see [4]. Since Z is closed in X and $(\mathbb{R}, \mathcal{RS})$ is not normal, then X is not normal, but, since any basic open set is closed-and-open and X is T_1, then X is zero-dimensional, hence Tychonoff. Now, Let $E \subset \mathbb{P}$ and $F \subset \mathbb{P}$ be closed disjoint subsets that cannot be separated in $(\mathbb{R}, \mathcal{RS})$. Let $C = \cup\{B_1(\langle p, 0 \rangle) : p \in E\}$ and $D = \cup\{B_1(\langle p, 0 \rangle) : p \in F\}$. Then C and D are both open in $(X, \mathcal{T})$ and $\overline{C}$ and $\overline{D}$ are disjoint closed domains that cannot be separated, hence X is not mildly normal. But X is submetrizable by the usual metric, hence epi-mildly normal.

Note that the above example shows that epi-mild normality does not imply normality. Consider $\mathbb{R}$ with the left ray topology $\mathcal{L} = \{\varnothing, \mathbb{R}\} \cup \{(-\infty, x) : x \in \mathbb{R}\}$ [4]. It is normal because any two non-empty closed sets must intersect. But it is not epi-mildly normal because it is not T_2.

Theorem 7. *Epi-mild normality is a topological property*

Proof. Let $(X, \mathcal{T})$ be any epi-mildly normal space. Assume that $(X, \mathcal{T}) \cong (Y, \mathcal{S})$. Let $\mathcal{T}'$ be a coarser topology on X such that $(X, \mathcal{T}')$ is Hausdorff mildly normal space. Let $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ be a homeomorphism and define $\mathcal{S}'$ on Y by $\mathcal{S}' = \{f(U) : U \in \mathcal{T}'\}$. Then $\mathcal{S}'$ is a topology on Y coarser than $\mathcal{S}$ and $(Y, \mathcal{S}')$ is Hausdorff mildly normal. ☐

Epi-mild normality is an additive property.

Theorem 8. *The sum $\oplus_{\alpha \in \Lambda} X_\alpha$, where X_α is a space for each $\alpha \in \Lambda$, is epi-mildly normal if and only if all spaces X_α are epi-mildly normal.*

Proof. If the sum $X = \oplus_{\alpha \in \Lambda} X_\alpha$ is epi-mildly normal, then there exist $\mathcal{T}'$ topology on X, coarser than $\oplus_{\alpha \in \Lambda} \mathcal{T}_\alpha$ such that $(X, \mathcal{T}')$ is a Hausdorff mildly normal space. Since X_α is closed domain in X for each $\alpha \in \Lambda$, $(X_\alpha, \mathcal{T}'_\alpha)$, where $\mathcal{T}'_\alpha = \{U \cap X_\alpha : U \in \mathcal{T}'\}$, is a Hausdorff mildly normal space. Thus all spaces X_α are epi-mildly normal as $(X_\alpha, \mathcal{T}'_\alpha)$ is coarser topology than $(X_\alpha, \mathcal{T}_\alpha)$. Conversely, if all the X_α's are epi-mildly normal, then there exists a topology $\mathcal{T}'_\alpha$ on X_α for each $\alpha \in \Lambda$, coarser than $\mathcal{T}_\alpha$ such that $(X_\alpha, \mathcal{T}'_\alpha)$ is a Hausdorff mildly normal space. Since Hausdorffness is additive [1], then $(X, \oplus_{\alpha \in \Lambda} \mathcal{T}'_\alpha)$ is a Hausdorff space. On the other hand, mild normality is an additive property because each factor is open-and-closed in X and the intersection of any closed domain in X with each factor X_α will be a closed domain in X_α. Therefore, X are epi-mildly normal as $\oplus_{\alpha \in \Lambda} \mathcal{T}'_\alpha$ is coarser topology than $\oplus_{\alpha \in \Lambda} \mathcal{T}_\alpha$. ☐

Recall that a topology $\mathcal{T}$ on a non-empty set X is said to be minimal Hausdorff if $(X, \mathcal{T})$ is Hausdorff and there is no Hausdorff topology on X strictly coarser than $\mathcal{T}$, see [7, 8]. In [7], it was proved that "if the product space is minimal Hausdorff, then each factor is minimal Hausdorff". In [9], the converse of the previous statement was proved. Namely, "the product of minimal Hausdorff spaces is minimal Hausdorff". Intuitively, the product of two epi-mildly normal spaces may not be epi-mildly normal. If X and Y are both minimal Hausdorff mildly normal spaces whose product $X \times Y$ is not mildly normal, then $X \times Y$ cannot be epi-mildly normal. We have not been able to find such two spaces yet. As far as we know from the literature, the only example of two linearly ordered topological spaces whose product is not mildly normal was given in [10]. This space turns out to be epi-mildly normal. Here is the example.

Example 9. *We will define a Hausdorff compact linearly ordered space Y such that $\omega_1 \times Y$ is epi-mildly normal. Let $\{y_n : n < \omega_0\}$ be a countably infinite set such that $\{y_n : n < \omega_0\} \cap (\omega_1 + 1) = \varnothing$. Let $Y = \{y_n : n < \omega_0\} \cup (\omega_1 + 1)$. Let $\mathcal{T}$ be the topology on Y generated by the following neighborhood system: For an $\alpha \in \omega_1$, a basic open neighborhood of α is the same as in ω_1 with its usual order topology. For $n \in \omega_0$, a basic open neighborhood of y_n is $\{y_n\}$. A basic open neighborhood of ω_1 is of the form $(\alpha, \omega_1] \cup \{y_n : n \geq k\}$ where $\alpha < \omega_1$ and $k \in \omega_0$. In other words, $\{y_n : n < \omega_0\}$ is a sequence of isolated points which converges to ω_1. Note that if we define an order $<$ on Y as follows: For each $n \in \omega_0$, $\omega_1 < y_{n+1} < y_n$, and $<$ on $\omega_1 + 1$ is the same as the usual order on $\omega_1 + 1$, then $(Y, \mathcal{T})$ is a linearly ordered topological space. It was shown in [10] that $(Y, \mathcal{T})$ is a Hausdorff compact space, hence it is mildly normal. Also, it is well known that ω_1 is a Hausdorff normal space and hence mildly normal. But $\omega_1 \times Y$ is not mildly normal [10]. We will show that $\omega_1 \times Y$ is epi-mildly normal. Define a topology $\mathcal{V}$ on ω_1 generated by the following neighborhood system: Each non-zero element $\beta < \omega_1$ will have the same open neighborhood as in the usual ordered topology in ω_1. Each open neighborhood of 0 is of the form $U = (\beta, \omega_1) \cup \{0\}$ where $\beta < \omega_1$. Simply, the idea is to move the minimal element 0 to the top and make it the maximal element. Then $\mathcal{V}$ is coarser than the usual ordered topology on ω_1 and $(\omega_1, \mathcal{V})$ is a Hausdorff compact space because it is homeomorphic to $\omega_1 + 1$. Thus $(\omega_1, \mathcal{V}) \times (Y, \mathcal{T})$ is T_2 compact, hence T_4 and the*

product topology $\mathcal{V} \times \mathcal{T}$ is coarser than $\mathcal{T}_0 \times \mathcal{T}$, where $\mathcal{T}_0$ is the usual order topology defined on ω_1. Therefore, $\omega_1 \times Y$ is epi-mildly normal.

Here is a case when the product of two epi-mildly normal spaces will be epi-mildly normal.

Theorem 10. *If X is epi-mildly normal countably compact and M is Hausdorff paracompact first countable, then $X \times M$ is epi-mildly normal.*

Proof. Let $(X, \mathcal{T})$ be any epi-mildly normal countably compact space. Then there exists coarser topology $\mathcal{T}'$ on X such that $(X, \mathcal{T}')$ is Hausdorff mildly normal space. Since $(X, \mathcal{T})$ is countably compact, $(X, \mathcal{T}')$ is countably compact. Hence $(X, \mathcal{T}') \times M$ is Hausdorff mildly normal, by [10, Theorem 2.9]. Thus $X \times M$ is epi-mildly normal. □

Corollary 11. *If X is epi-mildly normal countably compact and M is metrizable, then $X \times M$ is epi-mildly normal.*

Let us go back to the question: "Is there a Tychonoff epi-mildly normal space which is not epinormal?¨ We answer this in the class of minimal Tychonoff spaces [7]. Let $(X, \mathcal{T})$ be any minimal Tychonoff epi-mildly normal space. The theorem: "All minimal completely regular spaces are compact", [7, 3.3], gives that $(X, \mathcal{T})$ is compact, hence T_4, thus epinormal. So, we get the following theorem.

Theorem 12. *In the class of minimal Tychonoff spaces, any epi-mildly normal space is T_4.*

So, the above question remains open. Observe that in [10, 1.4], using the continuum hypothesis (CH), a Mrówka space which is mildly normal, hence epi-mildly normal, was constructed. This Mrówka space turns out to be epinormal. Indeed, we show that any Mrówka space is epinormal, hence epi-mildly normal. Recall that two countably infinite sets are said to be *almost disjoint* [11] if their intersection is finite. Call a subfamily of $[\omega_0]^{\omega_0} = \{ A \subset \omega_0 : A \text{ is infinite} \}$ a *mad family* [11] on ω_0 if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Let $\mathcal{A}$ be a pairwise almost disjoint subfamily of $[\omega_0]^{\omega_0}$. The Mrówka space $\Psi(\mathcal{A})$ is defined as follows: The underlying set is $\omega_0 \cup \mathcal{A}$, each point of ω_0 is isolated, and a basic open neighborhood of $W \in \mathcal{A}$ has the form $\{W\} \cup (W \smallsetminus F)$, with $F \in [\omega_0]^{<\omega_0} = \{B \subseteq \omega_0 : B \text{ is finite}\}$. It is well known that there exists an almost disjoint family $\mathcal{A} \subset [\omega_0]^{\omega_0}$ such that $|\mathcal{A}| > \omega_0$ and the Mrówka space $\Psi(\mathcal{A})$ is a Tychonoff, separable, first countable, and locally compact space which is neither countably compact, paracompact, nor normal. And $\mathcal{A}$ is a *mad* family if and only if $\Psi(\mathcal{A})$ is pseudocompact [12]. Let us recall the following definition from [13].

Definition 13. *A topological space X is called C_2-paracompact if there exist a Hausdorff paracompact space Y and a bijective function $f : X \longrightarrow Y$ such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$.*

In [13], the following easy proved theorem was given. "If X is a C_2-paracompact Fréchet space and $f : X \longrightarrow Y$ is any witness of the C_2-paracompactness of X, then f is continuous".

Theorem 14. *Any C_2-paracompact Fréchet space is epinormal.*

Proof. Let $(X, \mathcal{T})$ be any C_2-paracompact Fréchet space. If $(X, \mathcal{T})$ is normal, we are done. Assume that $(X, \mathcal{T})$ is not normal. Let $(Y, \mathcal{T}')$ be a T_2 paracompact space and $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{T}')$ be a bijective function such that the restriction $f \restriction_A : A \longrightarrow f(A)$ is a homeomorphism for each compact subspace $A \subseteq X$. Since X is Fréchet, f is continuous. Define $\mathcal{T}^\star = \{f^{-1}(U) : U \in \mathcal{T}'\}$. It is clear that $\mathcal{T}^\star$ is a topology on X coarser than $\mathcal{T}$ such that $f : (X, \mathcal{T}^\star) \longrightarrow (Y, \mathcal{T}')$ is continuous. If $W \in \mathcal{T}^\star$, then W is of the form $W = f^{-1}(U)$ where $U \in \mathcal{T}'$. So, $f(W) = f(f^{-1}(U)) = U$ which gives that f is open, hence homeomorphism. Thus $(X, \mathcal{T}^\star)$ is T_4. Therefore, $(X, \mathcal{T})$ is epinormal. □

Theorem 15. *Any Mrówka space $\Psi(\mathcal{A})$ is epinormal.*

Proof. For an almost disjoint family $\mathcal{A}$, the Mrówka space $\Psi(\mathcal{A})$ is C_2-paracompact, being locally compact, see [13] and [1, 3.3.D]. $\Psi(\mathcal{A})$ is also Fréchet being first countable. We conclude that such a Mrówka space is epinormal. $\qquad\square$

Here is another application of Theorem 14. The space in the next example, due to Urysohn, see [7], is a famous example of a minimal Hausdorff space which is not compact.

Example 16. *Let $X = \{ a_{ij}, b_{ij}, c_i, a, b : i \in \mathbb{N}, j \in \mathbb{N} \}$ where all these elements are assumed to be distinct. Define the following neighborhood system on X:*

$$\text{For each } i, j \in \mathbb{N}, \ a_{ij} \text{ is isolated and } b_{ij} \text{ is isolated}.$$

$$\text{For each } i \in \mathbb{N}, \ \mathcal{B}(c_i) = \{ V^n(c_i) = \cup_{j \geq n}\{a_{ij}, b_{ij}, c_i\} : n \in \mathbb{N} \}.$$

$$\mathcal{B}(a) = \{ V^n(a) = \cup_{j \in \mathbb{N}} \cup_{i \geq n} \{a_{ij}, a\} : n \in \mathbb{N} \}.$$

$$\mathcal{B}(b) = \{ V^n(b) = \cup_{j \in \mathbb{N}} \cup_{i \geq n} \{b_{ij}, b\} : n \in \mathbb{N} \}.$$

Let us denote the unique topology on X generated by the above neighborhood system by $\mathcal{T}$. Then $\mathcal{T}$ is minimal Hausdorff and $(X, \mathcal{T})$ is not compact [7].

Claim. *$(X, \mathcal{T})$ is not mildly normal.*

Proof of Claim. Let $G = \{ a_{ij} : i \text{ is odd}, j \in \mathbb{N} \}$ and $H = \{ b_{ij} : i \text{ is even}, j \in \mathbb{N} \}$. Then G and H are both open. Thus $E = \overline{G}$ and $F = \overline{H}$ are closed domains. But $E = \overline{G} = G \cup \{c_i : i \text{ is odd}\} \cup \{a\}$ and $F = \overline{H} = H \cup \{c_i : i \text{ is even}\} \cup \{b\}$. Thus $E \cap F = \varnothing$. Any open set containing b will meet any open set containing the set $\{c_i : i \text{ is odd}\}$. Thus E and F cannot be separated by disjoint open sets. Therefore, $(X, \mathcal{T})$ is not mildly normal and Claim is proved. We conclude that $(X, \mathcal{T})$ is not epi-mildly normal. Hence it cannot be epinormal. So, by Theorem 14, X cannot be paracompact. $\qquad\square$

Recall that a topological space X is called almost compact [14] if each open cover of X has a finite subfamily the closures of whose members covers X. A space X is called nearly compact [14] if each open cover of X has a finite subfamily the interiors of the closures of whose members covers X. A space X is said to be an almost regular if for any closed domain subset A and any $x \notin A$, there exist two disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. A technique which is useful in the theory of coarser topologies is the semiregularization. The topology on X generated by the family of all open domains is denoted by $\mathcal{T}_s$. The space $(X, \mathcal{T}_s)$ is called the semi regularization of X. A space $(X, \mathcal{T})$ is semiregular if $\mathcal{T} = \mathcal{T}_s$. A space X is H-closed [1] if X is closed in every Hausdorff space in which X can be embedded. It is clear that if X is completely Hausdorff space H-closed, then X is epi-mildly normal.

Theorem 17. *Every Hausdorff nearly compact space is epinormal (hence epi-mildly normal).*

Proof. Let $\mathcal{T}_s$ be the semiregularization of $\mathcal{T}$. Since $(X, \mathcal{T})$ is a Hausdorff nearly compact space, $\mathcal{T}_s$ is a compact Hausdorff space [15]. Hence $(X, \mathcal{T}_s)$ is a T_4 space. Therefore X is epinormal space. $\qquad\square$

Since the semiregularization of a nearly compact space is compact, we conclude the following Corollary.

Corollary 18. *For each $\alpha \in \Lambda$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a Hausdorff nearly compact space. Then $\prod_{\alpha \in \Lambda}(X_\alpha, \mathcal{T}_\alpha)$ is epi-mildly normal.*

Theorem 19. *If $(X, \mathcal{T})$ is almost regular almost compact and $\mathcal{T}_s$ is T_1, then $(X, \mathcal{T})$ is epi-mildly normal.*

Proof. Since $(X, \mathcal{T})$ is almost regular, $(X, \mathcal{T}_s)$ is regular [15]. Hence $(X, \mathcal{T}_s)$ is T_3. Moreover, the coarser topology of almost compact is almost compact. So, $\mathcal{T}_s$ is almost compact. But every almost regular almost compact is mildly normal [2]. Thus $\mathcal{T}_s$ is Hausdorff mildly normal. Therefore $(X, \mathcal{T})$ is epi-mildly normal. $\qquad\square$

We need the following Lemma from [15] to prove the next theorem.

Lemma 20. *Let $(Y, \mathcal{U})$ be a regular space. If $f : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{U})$ is continuous, then $f : (X, \mathcal{T}_s) \longrightarrow (Y, \mathcal{U})$ is continuous.*

Theorem 21. *If $(X, \mathcal{T})$ is T_2 mildly normal, then $(X, \mathcal{T}_s)$ is mildly normal (hence epi-mildly normal).*

Proof. Let A and B be two disjoint closed domains in semiregularization of X. Hence A and B are closed domains in $(X, \mathcal{T})$ [1, 1.7.8(b)]. Since $(X, \mathcal{T})$ is mildly normal, there exists a continuous function $f : (X, \mathcal{T}) \longrightarrow (I, \mathcal{U}_I)$ such that $f(a) = 0$, for each $a \in A$, and $f(b) = 1$, for each $b \in B$. Since I is regular, $f : (X, \mathcal{T}_s) \longrightarrow (I, \mathcal{U}_I)$ is continuous by Lemma 20. Thus $\mathcal{T}_s$ is mildly normal [2].   □

The following problems are still open:

1. Is epi-mild normality hereditary?
    Observe that the space X in Example 16 can be embedded in another Hausdorff space by the following theorem:"A Hausdorff space can be embedded as a closed subspace of a minimal Hausdorff space", [16], see also [17]. But there is no reason to guarantee that the larger space is mildly normal or at least epi-mildly normal. Also, there is a theorem:"A Hausdorff space can be densely embedded in a minimal Hausdorff space if and only if the space is semiregular". [18]. For the same reason, as previous, we cannot apply it even if we modify X to make it semiregular without lossing its minimality.
2. Is a β-normal [19] epi-mildly normal space normal?

References

[1]   Engelking R., General Topology, 1977, PWN, Warszawa.
[2]   Singal M.K., Singal A.R., Mildly normal spaces, Kyungpook Math. J., 1973, 13, 27-31.
[3]   Shchepin E.V., Real valued functions and spaces close to normal, Sib. J. Math., 1972, 13, 1182-1196.
[4]   Steen L., Seebach JA., Counterexamples in Topology, 1978, New York: Springer.
[5]   Gruenhage G., Generalized metric spaces, In: Handbook of Set-theoretic Topology, K.Kunen and J.Vaughan, eds., North-Holland, Amsterdam, 1984, 423-501.
[6]   Kalantan L., AlZahrani S., Epinormality, J. Nonlinear Sci. Appl., 2016, 9, 5398-5402.
[7]   Berri M.P., Minimal topological spaces, Trans. Amer. Math. Soc., 1963, 108, 97-105.
[8]   Bourbaki N., Topologie general, 1951, Paris: Actualites Sci. Ind., nos. Hermann, 858-1142.
[9]   Ikenage S., Product of minimal topology spaces, Proc. Jap. Acad., 1964, 40, 329-331.
[10] Kalantan L., Results about κ-normality, Topol. Appl., 2002, 125, 47-62.
[11] Van Douwen E.K., The integers and topology, In: Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, 111-167.
[12] Mrówka S., On completely regular spaces, Fundamenta Mathematicae, 1954, 41, 105-106.
[13] Kalantan L., Saeed M.M., Alzumi H., C-paracompactness and C_2-paracompactness, Turk. J. Math., in press.
[14] Lambrinos P., On almost compact and nearly compact spaces, Rendiconti del Circolo Matematico di Palermo, 1975, 24, 14-18.
[15] Mrsevic M. et al, On Semi-Rregularization Topologes, Austral. Math. Soc., 1985, 38, 40-54.
[16] Strecker G., Wattel E., On semi-regular and minimal Hausdorff embedding, Proc. Roy. Neth. Acad. Sci., 1960, 7, 234-237.
[17] Porter J.R., Thomas J.D., On H-closed and minimal Hausdorff spaces, Trans. Amer. Math. Soc., 1969, 138, 159-170.
[18] Banaschewski B., Über Hausdorffsche-minimale, Erweiterungen von Räumen, Arch. Math., 1961, 12, 355-365.
[19] Arhangel'skii A., Ludwig L., On α-normal and β-normal spaces, Comment. Math. Univ. Carolin., 2001, 42, 507-519.