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Orlicz difference bodies

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Abstract: In this paper, by the Orlicz-Minkowski combinations of convex bodies, we define the general Orlicz difference bodies and study their properties. Furthermore, we obtain the extreme values of the general Orlicz difference bodies and their polars.

Keywords: Difference body, Orlicz-Minkowski combination, General Orlicz difference body, Extreme value

MSC: 52A20, 52A40, 52A39

1 Introduction

A convex body in $\mathbb{R}^n$ is a compact convex subset with non-empty interior. Let $\mathcal{K}^n$ denote the set of all convex bodies in $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroids are at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$, $\mathcal{K}^n_c$ and $\mathcal{K}^n_{os}$, respectively. Let B denote the standard Euclidean unit ball in $\mathbb{R}^n$ and write S^{n-1} for its surface. In addition, we write $V(K)$ for the n-dimensional volume of a body K. Especially, $V(B) = \omega_n$.

If $K \in \mathcal{K}^n$, then its support function h_K=$h(K, \cdot)$:$\mathbb{R}^n \to (-\infty, +\infty)$ is defined by (see [1, 2])

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

From (1.1), we easily know that: If $A \in GL(n)$, then (see [1])

$$h(AK, u) = h(K, A'u), \quad \text{for all } u \in S^{n-1}. \tag{1.2}$$

Here, $GL(n)$ denotes the set of all general (non-singular) affine transformations and A' denotes the transpose of A. Further, from (1.2), it is easy to get that $h(-K, u) = h(K, -u)$, for any $u \in S^{n-1}$.

For $K, L \in \mathcal{K}^n$, and $\lambda, \mu \geq 0$ (not both zero), the Minkowski linear combination, $\lambda K + \mu L \in \mathcal{K}^n$, of K and L is defined by (see [1, 2])

$$h(\lambda K + \mu L, \cdot) = \lambda h(K, \cdot) + \mu h(L, \cdot), \tag{1.3}$$

where $\lambda K = \{\lambda x : x \in K\}$.

Taking $\lambda = \mu = 1/2$, $L = -K$ in (1.3), then the difference body, ΔK, of $K \in \mathcal{K}^n$ is given by (see [1])

$$\Delta K = \frac{1}{2}K + \frac{1}{2}(-K). \tag{1.4}$$

Obviously, ΔK is an origin-symmetric convex body.

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For the difference body, we know that (see [1]): *If $K \in \mathcal{K}^n$, then*

$$V(\Delta K) \geq V(K),$$

with equality if and only if K is centrally symmetric.

A recent extension of the Brunn-Minkowski theory is the Orlicz-Brunn-Minkowski theory, which was first launched by Lutwak, Yang and Zhang ([3, 4]) with affine isoperimetric inequalities for Orlicz centroid and projection bodies. In addition, Gardner, Hug and Weil ([5]) built the foundation and provided a general framework for the Orlicz-Brunn-Minkowski theory. In [6], Xi, Jin and Leng obtained the beautiful Orlicz-Brunn-Minkowski inequality, which can be viewed as a generalization of the classical Brunn-Minkowski inequality. Corresponding to Orlicz-Brunn-Minkowski theory, Zhu, Zhou and Xu ([7]) also established the dual Orlicz-Brunn-Minkowski theory. For the recent development of the Orlicz theory, see also [8–29].

Let Φ denote the set of convex and strictly increasing functions $\varphi : [0, \infty) \to [0, +\infty)$ and $\varphi(0) = 0$. In 2014, Gardner, Hug and Weil (see [5], also see [6]) defined the Orlicz-Minkowski combination: Let $\varphi \in \Phi$ satisfy $\varphi(1) = 1$, for $K, L \in \mathcal{K}_o^n$, $\alpha, \beta \geq 0$ (not both zero), define the Orlicz-Minkowski combination $+_\varphi(K, L, \alpha, \beta)$ by

$$h_{+_\varphi(K,L,\alpha,\beta)}(u) = \inf\left\{\lambda > 0 : \alpha\varphi\left(\frac{h_K(u)}{\lambda}\right) + \beta\varphi\left(\frac{h_L(u)}{\lambda}\right) \leq \varphi(1)\right\}. \tag{1.5}$$

Notice that since the function $z \to \alpha\varphi(\frac{h_K(u)}{z}) + \beta\varphi(\frac{h_L(u)}{z})$ is strictly decreasing, we have

$$h_{+_\varphi(K,L,\alpha,\beta)}(u) = \lambda_u,$$

if and only if

$$\alpha\varphi\left(\frac{h_K(u)}{\lambda_u}\right) + \beta\varphi\left(\frac{h_L(u)}{\lambda_u}\right) = \varphi(1). \tag{1.6}$$

From definition (1.5), we first give the concept of Orlicz difference body as follows:

For $K \in \mathcal{K}_o^n$, define the Orlicz difference body $\Delta_\varphi K$ of K by

$$h_{\Delta_\varphi K}(u) = \inf\left\{\lambda > 0 : \frac{1}{2}\varphi\left(\frac{h_K(u)}{\lambda}\right) + \frac{1}{2}\varphi\left(\frac{h_{-K}(u)}{\lambda}\right) \leq \varphi(1)\right\}. \tag{1.7}$$

From (1.7), it is easy to conclude that $\Delta_\varphi K \in \mathcal{K}_{os}^n$, and $K \in \mathcal{K}_{os}^n$ implies $\Delta_\varphi K = K$. If $\varphi(t) = t$ in (1.7), then by (1.4), we have $\Delta_\varphi K = \Delta K$.

By (1.7), for $K \in \mathcal{K}_o^n$ and real $\tau \in [-1, 1]$, we also define the general Orlicz difference body, $\Delta_\varphi^\tau K$, of K by

$$h_{\Delta_\varphi^\tau K}(u) = \inf\left\{\lambda > 0 : f_1(\tau)\varphi\left(\frac{h_K(u)}{\lambda}\right) + f_2(\tau)\varphi\left(\frac{h_{-K}(u)}{\lambda}\right) \leq \varphi(1)\right\}. \tag{1.8}$$

Here,

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \quad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)} \tag{1.9}$$

for $\tau \in [-1, 1]$. Obviously, by (1.9), functions $f_1(\tau)$ and $f_2(\tau)$ satisfy

$$f_1(\tau) + f_2(\tau) = 1, \tag{1.10}$$

$$f_1(-\tau) = f_2(\tau). \tag{1.11}$$

From (1.7), (1.8) and (1.9), we easily know that if $\tau = 0$, then $\Delta_\varphi^0 K = \Delta_\varphi K$, and if $\tau = \pm 1$, then $\Delta_\varphi^{+1} K = K$, $\Delta_\varphi^{-1} K = -K$.

Remark 1.1. *For $p \geq 1$, let $\varphi(t) = t^p$ in (1.8), then $\Delta_\varphi^\tau K$ implies the general L_p-difference body $\Delta_p^\tau K$ which is given by Wang and Ma (see [30]). In particular, for $p = 1$, we write $\Delta_p^\tau K = \Delta^\tau K$ and call $\Delta^\tau K$ the general difference body of K.*

The aim of this paper is to study the general Orlicz difference bodies and their polars. First, we obtain the following inclusion relationship between general difference bodies $\Delta^\tau K$ and general Orlicz difference bodies $\Delta^\tau_\varphi K$.

Theorem 1.2. *If $K \in \mathcal{K}^n_o$, $\varphi \in \Phi$, then for any $\tau \in [-1, 1]$, we have*

$$\Delta^\tau K \subseteq \Delta^\tau_\varphi K, \tag{1.12}$$

with equality either if and only if $K \in \mathcal{K}^n_{os}$ or φ is a linear function.

Obviously, by (1.12) we immediately get the following corollary.

Corollary 1.3. *If $K \in \mathcal{K}^n_o$, $\tau \in [-1, 1]$, and $\varphi \in \Phi$, then*

$$V(\Delta^\tau_\varphi K) \geq V(\Delta^\tau K),$$

with equality either if and only if $K \in \mathcal{K}^n_{os}$ or φ is a linear function.

Next, the extreme value of the general Orlicz difference bodies is also obtained:

Theorem 1.4. *If $K \in \mathcal{K}^n_o$, $\tau \in [-1, 1]$, and $\varphi \in \Phi$, then*

$$V(\Delta^\tau_\varphi K) \geq V(K). \tag{1.13}$$

If $K \notin \mathcal{K}^n_{os}$, equality holds in (1.13) if and only if $\tau = \pm 1$; if $K \in \mathcal{K}^n_{os}$, then (1.13) becomes an equality.

Let $\Delta^{\tau,*}_\varphi K$ denote the polar body of general Orlicz difference body $\Delta^\tau_\varphi K$, the following result gives the extreme value of the volume, $V(\Delta^{\tau,*}_\varphi K)$, of $\Delta^{\tau,*}_\varphi K$.

Theorem 1.5. *If $K \in \mathcal{K}^n_o$, $\tau \in [-1, 1]$, and $\varphi \in \Phi$, then*

$$V(\Delta^{\tau,*}_\varphi K) \leq V(K^*). \tag{1.14}$$

If $K \notin \mathcal{K}^n_{os}$, equality holds in (1.14) if and only if $\tau = \pm 1$; When $K \in \mathcal{K}^n_{os}$, inequality (1.14) becomes an identity.

According to the fact that the Orlicz-Minkowski combination $+_\varphi(K, L, \alpha, \beta) \in \mathcal{K}^n_o$ (see [6]), we easily deduce that $\Delta^\tau_\varphi K \in \mathcal{K}^n_o$. Using inequality (1.13) and the Blaschke-Santaló inequality for convex bodies, we have

Theorem 1.6. *If $K \in \mathcal{K}^n_o$, $\varphi \in \Phi$, then for any $\tau \in [-1, 1]$, we have*

$$V(K)V(\Delta^{\tau,c}_\varphi K) \leq \omega_n^2, \tag{1.15}$$

with equality if and only if K is an ellipsoid centered at the origin. Here $\Delta^{\tau,c}_\varphi K = (\Delta^\tau_\varphi K)^c$, and for $Q \in \mathcal{K}^n_o$, $Q^c = (Q - \mathrm{cent}Q)^*$ and $\mathrm{cent}Q$ denotes the centroid of Q.

Obviously, when $\varphi(t) = t^p$ $(p \geq 1)$, it is easily checked that Theorems 1.2–1.6 reduce to the results of general L_p-difference body $\Delta^\tau_p K$ (see [30]).

This paper is organized as follows. In Section 2, we list some basic notions that will be indispensable to the proofs of our results. In Section 3, several elementary properties of the general Orlicz difference bodies are listed. In Section 4, the proofs of Theorems 1.2–1.6 are completed.

2 Basic notions

2.1 Radial functions and the polar of convex bodies

If K is a compact star-shaped (with respect to the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \to [0,+\infty)$, is defined by (see [1])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous, then K will be called a star body (with respect to the origin). Denote by $\mathcal{S}_o^n$ the set of star bodies (about the origin) in $\mathbb{R}^n$. When $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$, we say that two star bodies K and L are dilates (of each other).

For a non-empty subset $E \subseteq \mathbb{R}^n$, the polar set E^* of E is defined by (see [1, 2])

$$E^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, y \in E\}. \tag{2.1}$$

From definition (2.1), we easily get that for $K \in \mathcal{K}_o^n$,

$$h(K,u) = \frac{1}{\rho(K^*,u)}, \quad u \in S^{n-1}. \tag{2.2}$$

The well-known Blaschke-Santaló inequality for convex bodies has the following representation ([31]):

Theorem 2.1. *If $K \in \mathcal{K}_c^n$, then*

$$V(K)V(K^*) \leq \omega_n^2, \tag{2.3}$$

with equality if and only if K is an ellipsoid centered at the origin.

Remark 2.2. *For $Q \in \mathcal{K}^n$, let $centQ \in intQ$ denote the centroid of Q. Associated with each $Q \in \mathcal{K}^n$ is a point $s = San(Q) \in intK$, called the Santaló point of Q, defined as the unique point $s \in intQ$, such that $Cent((-s + Q)^*) = 0$. Let $\mathcal{K}_s^n$ denote the set of convex bodies having their Santaló point at the origin. Thus, $Q \in \mathcal{K}_s^n$ if and only if $Q^* \in \mathcal{K}_c^n$.*

2.2 The Orlicz mixed volume

Using the Orlicz-Minkowski combination $+_\varphi(\alpha,\beta,K,L)$, Gardner, Hug and Weil ([5], also see [6]) defined the following Orlicz mixed volume: For $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$, the Orlicz mixed volume of K and L is defined by

$$V_\varphi(K,L) = \frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{h_L(u)}{h_K(u)}\right) h_K(u)\,dS_K(u).$$

Associated with the definition of Orlicz mixed volume, Gardner, Hug and Weil ([5], also see [6]) presented the following Orlicz-Minkowski inequality.

Theorem 2.3. *If $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$, then*

$$V_\varphi(K,L) \geq V(K)\varphi\left(\frac{V(L)^{\frac{1}{n}}}{V(K)^{\frac{1}{n}}}\right). \tag{2.4}$$

If φ is strictly convex, then equality holds either if and only if K and L are dilates or $L = \{o\}$.

2.3 Dual Orlicz mixed volume

In [17], Ma and Wang made a further study on Orlicz theory. They introduced the notion of Orlicz radial combination as follows: For $K, L \in \mathcal{S}_o^n$, $\varphi \in \Phi$, the Orlicz radial combination, $\alpha \circ K \widetilde{+}_{-\varphi} \beta \circ L$, of K and L is defined by (see [17])

$$\rho(\alpha\circ K\widetilde{+}_{-\varphi}\beta\circ L,u)=\sup\left\{\lambda>0:\alpha\varphi\left(\frac{\lambda}{\rho(K,u)}\right)+\beta\varphi\left(\frac{\lambda}{\rho(L,u)}\right)\le 1\right\}$$

for all $u\in S^{n-1}$. Further, Ma and Wang in [17] gave the corresponding definition of dual Orlicz mixed volume $\widetilde{V}_{-\varphi}(K,L)$:

$$-\frac{n}{\varphi_r'(1)}\widetilde{V}_{-\varphi}(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}_{-\varphi}\varepsilon\circ L)-V(K)}{\varepsilon}.$$

Here $\varphi_r'(1)$ denote the right derivative of φ at 1.

Based on the above definition, an important integral representation of the dual Orlicz mixed volume was obtained immediately in [17].

Theorem 2.4. *Suppose $\varphi\in\Phi$ and $\varphi(1)=1$. If $K,L\in\mathcal{S}_o^n$, then*

$$\widetilde{V}_{-\varphi}(K,L)=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{\rho_K(u)}{\rho_L(u)}\right)\rho_K^n(u)dS(u).\tag{2.5}$$

Obviously, by (2.5) we get

$$\begin{aligned}\widetilde{V}_{-\varphi}(K,K)&=\frac{1}{n}\int_{S^{n-1}}\varphi\left(\frac{\rho_K(u)}{\rho_K(u)}\right)\rho_K^n(u)dS(u)\\&=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(u)dS(u)=V(K).\end{aligned}\tag{2.6}$$

For dual Orlicz mixed volume $\widetilde{V}_{-\varphi}(K,L)$, using the similar argument of Orlicz-Minkowski inequality (2.4), the corresponding dual Orlicz-Minkowski inequality can be stated as follows (see [17]):

Theorem 2.5. *If $\varphi\in\Phi$, $K,L\in\mathcal{S}_o^n$, then*

$$\widetilde{V}_{-\varphi}(K,L)\ge V(K)\varphi\left(\frac{V(K)^{\frac{1}{n}}}{V(L)^{\frac{1}{n}}}\right).\tag{2.7}$$

Equality holds if and only if K and L are dilates.

3 Basic properties of general Orlicz difference bodies

In this section, we will list some properties of general Orlicz difference bodies, which are essential for the proofs of our Theorems.

Lemma 3.1. *Let $K\in\mathcal{K}_o^n$, $\varphi\in\Phi$, if $A\in GL(n)$, then for any $\tau\in[-1,1]$, we have*

$$\Delta_\varphi^\tau AK=A\Delta_\varphi^\tau K.$$

Proof. By (1.2) and (1.8), this gives the desired result. □

Lemma 3.2. *If $K\in\mathcal{K}_o^n$, $\tau\in[-1,1]$, and $\varphi\in\Phi$, then*

$$\Delta_\varphi^\tau(-K)=\Delta_\varphi^{-\tau}K=-\Delta_\varphi^\tau K.$$

Proof. For all $u\in S^{n-1}$, by (1.8) and (1.11), and together with $h(-K,u)=h(K,-u)$, Lemma 3.2 can be easily proved. □

Lemma 3.3. *If $K\in\mathcal{K}_o^n$, $\tau\in[-1,1]$, $\tau\neq 0$ and $\varphi\in\Phi$, then $\Delta_\varphi^\tau K=\Delta_\varphi^{-\tau}K$ if and only if $K\in\mathcal{K}_{os}^n$.*

Proof. By (1.8) and (1.11), and notice that (1.6), it is easy to get the desired result. Contrarily, if $K \in \mathcal{K}_{os}^n$, together with Lemma 3.2, we immediately get $\Delta_\varphi^\tau K = \Delta_\varphi^{-\tau} K$ for all $u \in S^{n-1}$. □

From Lemma 3.3, we obtain immediately the following result.

Corollary 3.4. *Let $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$. If $K \notin \mathcal{K}_{os}^n$, then for any $\tau \in [-1, 1]$, we have $\Delta_\varphi^\tau K = \Delta_\varphi^{-\tau} K$ if and only if $\tau = 0$.*

By Lemma 3.2, we see that if $\Delta_\varphi^\tau K = \Delta_\varphi^{-\tau} K$, then $\Delta_\varphi^\tau K = -\Delta_\varphi^\tau K$. This means $\Delta_\varphi^\tau K \in \mathcal{K}_{os}^n$. This and Lemma 3.3 yield

Corollary 3.5. *Let $K \in \mathcal{K}_o^n$, $\tau \in [-1, 1]$ and $\varphi \in \Phi$. If $\tau \neq 0$, then $\Delta_\varphi^\tau K \in \mathcal{K}_{os}^n$ if and only if $K \in \mathcal{K}_{os}^n$.*

Lemma 3.6. *Suppose $K \in \mathcal{K}_{os}^n$, $\tau \in [-1, 1]$ and $\varphi \in \Phi$, then*

$$\Delta_\varphi^\tau K = K. \tag{3.1}$$

Proof. Since $K \in \mathcal{K}_{os}^n$, from (1.8), (1.10) and (1.6), we easily can get

$$\varphi\left(\frac{h_K(u)}{h_{\Delta_\varphi^\tau K}(u)}\right) = \varphi(1).$$

The strictly increasing property of convex function φ shows that

$$h_{\Delta_\varphi^\tau K}(u) = h_K(u)$$

for all $u \in S^{n-1}$. This gives (3.1). □

Lemma 3.6 immediately infers

Corollary 3.7. *Suppose $K, L \in \mathcal{K}_{os}^n$, $\tau \in [-1, 1]$ and $\varphi \in \Phi$, then*

$$\Delta_\varphi^\tau K = \Delta_\varphi^\tau L \iff K = L.$$

4 Results and proofs

In this section, we will give the proofs of Theorems 1.2–1.6. First, the following Orlicz-Brunn-Minkowski inequality (see [5, 6]) is useful and indispensable for the proofs of our results.

Lemma 4.1. *If $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$ and $\alpha, \beta > 0$, then we have*

$$\alpha\varphi\left(\frac{V(K)^{\frac{1}{n}}}{V(+_\varphi(\alpha, \beta, K, L))^{\frac{1}{n}}}\right) + \beta\varphi\left(\frac{V(L)^{\frac{1}{n}}}{V(+_\varphi(\alpha, \beta, K, L))^{\frac{1}{n}}}\right) \leq \varphi(1). \tag{4.1}$$

with equality if K and L are dilates. If φ is strictly convex, equality holds if and only if K and L are dilates of each other.

Proof of Theorem 1.2. Notice that the function φ is a strictly increasing convex function on $[0, +\infty)$, and together with (1.3) and (1.6) we get for all $u \in S^{n-1}$,

$$\varphi(1) = f_1(\tau)\varphi\left(\frac{h_K(u)}{h_{\Delta_\varphi^\tau K}(u)}\right) + f_2(\tau)\varphi\left(\frac{h_{-K}(u)}{h_{\Delta_\varphi^\tau K}(u)}\right)$$

$$\geq \varphi\left(f_1(\tau)\frac{h_K(u)}{h_{\Delta_\varphi^\tau K}(u)} + f_2(\tau)\frac{h_{-K}(u)}{h_{\Delta_\varphi^\tau K}(u)}\right)$$

$$= \varphi\left(\frac{h_{\Delta^{\tau}K}(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right). \tag{4.2}$$

Therefore, we have for all $u \in S^{n-1}$,

$$h_{\Delta^{\tau}K}(u) \leq h_{\Delta^{\tau}_{\varphi}K}(u).$$

i.e.,

$$\Delta^{\tau}K \subseteq \Delta^{\tau}_{\varphi}K.$$

So we obtain (1.12).

Now, we discuss the case of equality about (1.12). Using the increasing property of convex function φ, we have $\frac{h_K(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)} = \frac{h_{-K}(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}$, namely, $K = -K$. That is to say, $K \in \mathcal{K}^n_{os}$. On the contrary, if $K \in \mathcal{K}^n_{os}$, from Lemma 3.6 and definition (1.3), it is easy to see that $\Delta^{\tau}K = \Delta^{\tau}_{\varphi}K$.

When φ is not strictly convex, the equality holds in (4.2) means φ must be a linear function. If φ is linear, we will show that

$$\begin{aligned}
\varphi(1) &= f_1(\tau)\varphi\left(\frac{h_K(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right) + f_2(\tau)\varphi\left(\frac{h_{-K}(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right)\\
&= \varphi\left(f_1(\tau)\frac{h_K(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right) + \varphi\left(f_2(\tau)\frac{h_{-K}(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right)\\
&= \varphi\left(f_1(\tau)\frac{h_K(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)} + f_2(\tau)\frac{h_{-K}(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right)\\
&= \varphi\left(\frac{h_{\Delta^{\tau}K}(u)}{h_{\Delta^{\tau}_{\varphi}K}(u)}\right).
\end{aligned}$$

Thus, $h_{\Delta^{\tau}K}(u) = h_{\Delta^{\tau}_{\varphi}K}(u)$. It can concluded that the equality holds in (1.12).

By the above discussions, this gives the proof of Theorem 1.2. □

Lemma 4.2. *If $K, L \in \mathcal{K}^n_o$, $\varphi \in \Phi$. Then for any $0 < \alpha < 1$, one gets*

$$V(+_{\varphi}(\alpha, 1-\alpha, K, L)) \geq V(K)^{\alpha}V(L)^{1-\alpha}. \tag{4.3}$$

Equality holds if and only if $K = L$.

Proof. For the sake of brevity, let $K_{\varphi} = +_{\varphi}(\alpha, 1-\alpha, K, L)$. Notice that $\varphi \in \Phi$, so, we know that φ is convex and strictly increasing, from the Orlicz-Brunn-Minkowski inequality (4.1) and the harmonic-geometric-arithmetic mean (HG-AM) inequality (see [32], p.515), we have

$$\begin{aligned}
\varphi(1) &\geq \alpha\varphi\left(\frac{V(K)^{\frac{1}{n}}}{V(K_{\varphi})^{\frac{1}{n}}}\right) + (1-\alpha)\varphi\left(\frac{V(L)^{\frac{1}{n}}}{V(K_{\varphi})^{\frac{1}{n}}}\right)\\
&\geq \varphi\left(\alpha\frac{V(K)^{\frac{1}{n}}}{V(K_{\varphi})^{\frac{1}{n}}} + (1-\alpha)\frac{V(L)^{\frac{1}{n}}}{V(K_{\varphi})^{\frac{1}{n}}}\right)\\
&\geq \varphi\left(\frac{V(K)^{\frac{\alpha}{n}}V(L)^{\frac{1-\alpha}{n}}}{V(K_{\varphi})^{\frac{1}{n}}}\right).
\end{aligned}$$

Therefore, we get

$$1 \geq \frac{V(K)^{\frac{\alpha}{n}}V(L)^{\frac{1-\alpha}{n}}}{V(K_{\varphi})^{\frac{1}{n}}},$$

i.e.

$$V(+_{\varphi}(\alpha, 1-\alpha, K, L)) \geq V(K)^{\alpha}V(L)^{1-\alpha}.$$

From (4.1), and the characteristic of convex function φ, together with the equality condition of HG-AM inequality, we know that equality holds in (4.3) if and only if $K = L$. □

Proof of Theorem 1.4. Since $\tau \in (-1, 1)$, thus $0 < f_1(\tau) < 1$. Let $\alpha = f_1(\tau)$ in (4.3), we have

$$V(\Delta_\varphi^\tau K) \geq V(K)^{f_1(\tau)} V(-K)^{1-f_1(\tau)} = V(K).$$

This gives inequality (1.13).

By Lemma 4.1, if $\tau \in (-1, 1)$ (i.e. $\tau \neq \pm 1$), then equality holds in (1.13) if and only if $K = -K$, thus, $K \in \mathcal{K}_{os}^n$. So we see that if $K \notin \mathcal{K}_{os}^n$, then equality holds in (1.13) if and only if $\tau = \pm 1$. □

Proof of Theorem 1.5. According to the equality (1.6), we know

$$f_1(\tau)\varphi\left(\frac{h_K(u)}{h_{\Delta_\varphi^\tau K}(u)}\right) + f_2(\tau)\varphi\left(\frac{h_{-K}(u)}{h_{\Delta_\varphi^\tau K}(u)}\right) = \varphi(1).$$

This and (2.2) yield

$$f_1(\tau)\varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{K^*}(u)}\right) + f_2(\tau)\varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{(-K)^*}(u)}\right) = \varphi(1). \tag{4.4}$$

Observing $(-K)^* = -K^*$, then (4.4) can be written as

$$f_1(\tau)\varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{K^*}(u)}\right) + f_2(\tau)\varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{-K^*}(u)}\right) = \varphi(1). \tag{4.5}$$

Using (2.6), (2.5), (4.5) and notice that $\varphi(1) = 1$, then

$$\begin{aligned}
V(\Delta_\varphi^{\tau,*} K) &= \frac{1}{n}\int_{S^{n-1}} \rho_{\Delta_\varphi^{\tau,*} K}^n(u)\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \varphi(1)\rho_{\Delta_\varphi^{\tau,*} K}^n(u)\,dS(u) \\
&= \frac{1}{n}\int_{S^{n-1}} \left[f_1(\tau)\varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{K^*}(u)}\right) + f_2(\tau)\varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{-K^*}(u)}\right)\right]\rho_{\Delta_\varphi^{\tau,*} K}^n(u)\,dS(u) \\
&= f_1(\tau)\frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{K^*}(u)}\right)\rho_{\Delta_\varphi^{\tau,*} K}^n(u)\,dS(u) \\
&\quad + f_2(\tau)\frac{1}{n}\int_{S^{n-1}} \varphi\left(\frac{\rho_{\Delta_\varphi^{\tau,*} K}(u)}{\rho_{-K^*}(u)}\right)\rho_{\Delta_\varphi^{\tau,*} K}^n(u)\,dS(u) \\
&= f_1(\tau)\widetilde{V}_{-\varphi}(\Delta_\varphi^{\tau,*} K, K^*) + f_2(\tau)\widetilde{V}_{-\varphi}(\Delta_\varphi^{\tau,*} K, -K^*).
\end{aligned}$$

This and the dual Orlicz-Minkowski inequality (2.7) yield

$$V(\Delta_\varphi^{\tau,*} K) \geq f_1(\tau)V(\Delta_\varphi^{\tau,*} K)\varphi\left(\frac{V(\Delta_\varphi^{\tau,*} K)^{\frac{1}{n}}}{V(K^*)^{\frac{1}{n}}}\right) + f_2(\tau)V(\Delta_\varphi^{\tau,*} K)\varphi\left(\frac{V(\Delta_\varphi^{\tau,*} K)^{\frac{1}{n}}}{V(-K^*)^{\frac{1}{n}}}\right).$$

Therefore, by (1.10) we get that

$$1 = \varphi(1) \geq (f_1(\tau) + f_2(\tau))\varphi\left(\frac{V(\Delta_\varphi^{\tau,*} K)^{\frac{1}{n}}}{V(K^*)^{\frac{1}{n}}}\right) = \varphi\left(\frac{V(\Delta_\varphi^{\tau,*} K)^{\frac{1}{n}}}{V(K^*)^{\frac{1}{n}}}\right).$$

For the function φ, it is strictly increasing on $[0, +\infty)$. So, we have

$$V(\Delta_\varphi^{\tau,*} K) \leq V(K^*).$$

This is just inequality (1.14).

About the condition of equality in (1.14), if $K \in \mathcal{K}_{os}^n$, from Lemma 3.6, we know that (1.14) is evidently true. If $K \notin \mathcal{K}_{os}^n$, associated with the condition of equality in (2.7), we know that equality holds in (1.14) if and only if $\Delta_\varphi^{\tau,*} K$ and K^*, $\Delta_\varphi^{\tau,*} K$ and $-K^*$ both are dilates. These together with $V(\Delta_\varphi^{\tau,*} K) = V(K^*)$ give that $\Delta_\varphi^\tau K = \pm K$, i.e., $\tau = \pm 1$. □

Proof of Theorem 1.6. Since $\Delta_\varphi^\tau K \in \mathcal{K}_o^n$, thus $\Delta_\varphi^\tau K - c \in \mathcal{K}_c^n$, where c is the centroid of $\Delta_\varphi^\tau K$. Because of $\Delta_\varphi^{\tau,c} K = (\Delta_\varphi^\tau K)^c = (\Delta_\varphi^\tau K - c)^*$, hence, together with inequality (1.13) and the Blaschke-Santaló inequality (2.3), we obtain

$$V(K)V(\Delta_\varphi^{\tau,c}K) \le V(\Delta_\varphi^\tau K)V(\Delta_\varphi^{\tau,c}K) = V(\Delta_\varphi^\tau K - c)V((\Delta_\varphi^\tau K - c)^*) \le \omega_n^2.$$

This gives inequality (1.15).

The equality conditions of inequalities (2.3) and (1.13) show that equality holds in inequality (1.15) if and only if K is an ellipsoid centered at the origin. □

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