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Nonlinear elastic beam problems with the parameter near resonance

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Abstract: In this paper, we consider the nonlinear fourth order boundary value problem of the form

$$\begin{cases} u^{(4)}(x) - \lambda u(x) = f(x, u(x)) - h(x), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

which models a statically elastic beam with both end-points cantilevered or fixed. We show the existence of at least one or two solutions depending on the sign of $\lambda - \lambda_1$, where λ_1 is the first eigenvalue of the corresponding linear eigenvalue problem and λ is a parameter. The proof of the main result is based upon the method of lower and upper solutions and global bifurcation techniques.

Keywords: Elastic beam, At resonance, Multiplicity, Eigenvalue, Lower and upper solutions, Bifurcation

MSC: 34B27, 34B15, 34B05, 34A40

1 Introduction

The existence and multiplicity of solutions to the nonlinear second order ordinary differential equation boundary value problem with the parameter near resonance of the form

$$\begin{cases} -u''(x) - \lambda u(x) = f(x, u(x)) - h(x), & x \in (0, 1), \\ u(0) = u(1) = 0 \end{cases} \quad (1)$$

have been extensively studied by many authors, see Mawhin and Schmitt et al. [1-3], Iannacci and Nkashama [4], Costa and Goncalves [5], Ambrosetti and Mancini [6], Fonda and Habets [7], Cac [8], Ahmad [9] and Ma [10], and the references therein. In particular, Chiappinelli, Mawhin and Nugari [1] proved that there exists $\nu > 0$ such that problem (1), with λ near λ_1 , had at least one solution for $\lambda \leq \lambda_1$ and two solutions for $\lambda_1 < \lambda < \lambda_1 + \nu$ under the assumption $\lim_{s \rightarrow +\infty} \frac{f(x,s)}{s} = 0$ and a Landesman-Lazer type condition.

However, relatively little is known about the related work on the existence of solutions of the fourth order boundary value problems. The likely reason is that fewer techniques are available for the fourth order operators and the results known for the second order case do not necessarily hold for the corresponding fourth order problem. A natural motivation for studying higher order boundary value problems exists in their applications. For example, it is well-known that the deformation of an elastic beam in equilibrium state, whose both end-points are cantilevered or fixed, can be described by the fourth order boundary value problem

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u''(x)), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \quad (2)$$

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where $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, see [11]. There are some papers discussing the existence of solutions of the problem by using various methods, such as the lower and upper solution method, the Leray-Schauder continuation method, fixed-point theory, and the monotone iterative method, see Rynne [12], Korman [13], Gupta and Kwong [14], Jurkiewicz [15], Vrabel [16], Cabada et al. [17], Bai and Wang [18] and Ma et al. [19], and the references therein.

But to the best of our knowledge, the analogue of (1) has not been established for fourth order boundary value problems.

The purpose of this paper is to establish the similar existence result for the corresponding fourth order analogue of (1) of the form

$$\begin{cases} u^{(4)}(x) - \lambda u(x) = f(x, u(x)) - h(x), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{3}$$

where λ is a parameter, $h \in C[0, 1]$ and $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$. The proof of our main result is based upon the method of lower and upper solutions and global bifurcation techniques.

Particular significance in these points lie in the fact that for a second-order differential equation, with Neumann or Dirichlet boundary conditions, the existence of a lower solution α and an upper solution β with $\alpha(x) \leq \beta(x)$ in $[0, 1]$ can ensure the existence of solutions in the order interval $[\alpha(x), \beta(x)]$, see Coster and Habets [20]. However, this result is not true for fourth-order boundary value problems, see the counterexample in Cabada, Cid and Sanchez [17, P. 1607]. Thus, new challenges are faced and innovation is required.

To apply the bifurcation techniques to study the existence of solutions of (3), we state and prove a spectrum result for fourth order linear eigenvalue problem

$$\begin{cases} u^{(4)}(x) = \lambda u(x), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{4}$$

More precisely, we can show that the eigenvalues of (4) form a sequence

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots \rightarrow +\infty. \tag{5}$$

Moreover, for each $j \in \mathbb{N}$, λ_j ($\lambda_j = m_j^4$, m_j is the simple root of the equation $\cos m \cosh m - 1 = 0$) is simple. In particular, $\lambda_1 \approx (4.73004)^4 \approx 500.564$ is simple and the corresponding eigenspace is spanned by $\varphi_1(x) = \sin m_1 x - \sinh m_1 x + \frac{\sin m_1 - \sinh m_1}{\cos m_1 - \cosh m_1} (\cosh m_1 x - \cos m_1 x)$ with $\varphi_1(x) > 0$ in $(0, 1)$, and we normalize by $\int_0^1 (\varphi_1(x))^2 dx = 1$.

We shall make the following assumptions:

(H1) $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and satisfies

$$f(x, u_2) - f(x, u_1) + \frac{3}{4} \lambda_1 (u_2 - u_1) \geq 0 \text{ for } -\infty < u_1 \leq u_2 < +\infty \text{ and } x \in [0, 1];$$

(H2) $h \in C[0, 1]$ and satisfies

$$\int_0^1 C(x)\varphi_1(x)dx < \int_0^1 h(x)\varphi_1(x)dx < \int_0^1 c(x)\varphi_1(x)dx,$$

where $c, C \in L^1(0, 1)$ with $\liminf_{s \rightarrow -\infty} f(x, s) \geq c(x)$, $\limsup_{s \rightarrow +\infty} f(x, s) \leq C(x)$ uniformly for $x \in (0, 1)$;

(H3)

$$\lim_{s \rightarrow +\infty} \frac{f(x, s)}{s} = 0 \text{ uniformly for } x \in [0, 1].$$

The main result of this paper is the following

Theorem 1.1. *Assume (H1)-(H3) hold. Then there exist $\delta_1 > 0$ and $\lambda_0 \in (3\lambda_1/4, \lambda_1)$ such that (3) has at least one or at least two solutions according to $\lambda_0 \leq \lambda \leq \lambda_1$ or $\lambda_1 < \lambda \leq \lambda_1 + \delta_1$. Moreover, one of these two solutions is a positive solution.*

We first prove the existence of a lower solution α and an upper solution β of (3) for $\lambda \leq \lambda_1$, which are well ordered, that is $\alpha \leq \beta$ (in fact $\alpha < 0$ and $\beta > 0$) under condition (H2). But this is not enough to ensure the existence of a solution in the order interval $[\alpha, \beta]$, so we also make the assumption (H1). It is precisely this circumstance which gives a priori bound for $\lambda \leq \lambda_1$. Once this is done, since λ_1 is simple and the assumption (H3) hold, the Rabinowitz global bifurcation techniques [21] can be used to obtain the second solution following very much the same lines as in [10]. More precisely, there exists an unbounded connected component Σ_∞ that is bifurcating from infinity. Since we have established a prior bound for solutions of (3) when $\lambda \leq \lambda_1$, the connected component Σ_∞ , must do so for $\lambda > \lambda_1$.

For other results concerning the existence of solutions of the nonlinear fourth order differential or difference equations via the bifurcation techniques, we refer the reader to [22, 23].

The rest of the paper is arranged as follows. In Section 2, we investigate the spectrum structure of the linear eigenvalue problem (4). In Section 3, we give some preliminary results and develop the method of lower and upper solutions for (3). Finally Section 4 is devoted to proving our main result by the well-known Rabinowitz bifurcation techniques and the lower and upper solutions arguments. We also give some examples to illustrate our main result.

2 Spectrum of the linear eigenvalue problem

In this section we state a spectrum result of the linear eigenvalue problem (4).

Lemma 2.1 ([24, Lemma 1]). *The equation*

$$\cos m \cosh m - 1 = 0, \quad m \in \mathbb{R}^+$$

has infinitely many simple roots

$$0 < m_1 < m_2 < m_3 < \dots \rightarrow +\infty.$$

Moreover,

$$m_{2k-1} \in \left(\left(2k - \frac{1}{2}\right)\pi, 2k\pi \right), \quad m_{2k} \in \left(2k\pi, \left(2k + \frac{1}{2}\right)\pi \right)$$

for $k \in \mathbb{N}$.

It is well known that linear eigenvalue problem (4) is completely regular Sturmian system and therefore, has infinitely many simple and positive eigenvalues $0 < \lambda_1 < \lambda_2 < \dots \rightarrow +\infty$. The eigenfunction φ_j , corresponding to λ_j , has exactly $j - 1$ simple zeros in $(0, 1)$. The eigenvalues $\lambda_k, k \in \mathbb{N}$ are the roots of the transcendental equation $\cos m \cosh m - 1 = 0$. See Rynne [12, P. 308], Janczewsky [25] and Courant and Hilbert [26].

Moreover, we have the following

Lemma 2.2 ([24, Lemma 2]). *The linear eigenvalue problem (4) has infinitely many eigenvalues*

$$\lambda_j = m_j^4, \quad j \in \mathbb{N},$$

and the eigenfunction corresponding to λ_j is given by

$$\varphi_j(x) = \sin m_j x - \sinh m_j x + \frac{\sin m_j - \sinh m_j}{\cos m_j - \cosh m_j} (\cosh m_j x - \cos m_j x).$$

Moreover, $\varphi_j \in S_{j,+}$, where $S_{j,+}$ denote the set of $u \in C^3[0, 1]$ such that:

- (i) u has only simple zeros in $(0, 1)$ and has exactly $j - 1$ such zeros;*
- (ii) $u''(0) > 0$ and $u''(1) \neq 0$.*

3 The existence of lower and upper solutions

Let us start with the problem (3) with $\lambda = \lambda_1$ and $h = 0$, i.e.

$$\begin{cases} u^{(4)}(x) - \lambda_1 u(x) = f(x, u(x)), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \quad (6)$$

Definition 3.1. A function $\alpha \in C^4[0, 1]$ is said to be a lower solution of problem (6) if

$$\alpha^{(4)}(x) - \lambda_1 \alpha(x) \leq f(x, \alpha(x)) \quad \text{for } x \in (0, 1), \quad (7)$$

and

$$\alpha(0) = \alpha(1) = 0, \quad \alpha'(0) = \alpha'(1) = 0.$$

Similarly, an upper solution $\beta \in C^4[0, 1]$ is defined by reversing the inequality in (7). Such a lower or upper solution is called strict if the inequality is strict for $x \in (0, 1)$.

Lemma 3.2. Assume that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and

$$\liminf_{s \rightarrow -\infty} f(x, s) \geq \underline{c}(x) \quad \text{uniformly for } x \in (0, 1), \quad (8)$$

where $\underline{c} \in L^1(0, 1)$ and satisfies $\int_0^1 \underline{c}(x) \varphi_1(x) dx > 0$. Then there exist $R > 0$ and $d \in C[0, 1]$ with $\int_0^1 d(x) \varphi_1(x) dx > 0$ such that

$$f(x, u) \geq d$$

for all $x \in (0, 1)$, whenever $u \leq -R\varphi_1$ in $(0, 1)$.

Proof. Let $c_\varepsilon = \underline{c} - \varepsilon$, where $\varepsilon > 0$ is small enough to keep $\int_0^1 c_\varepsilon(x) \varphi_1(x) dx > 0$. Then we can assume that the strict inequality holds in (8) with $\underline{c}(x) = c_\varepsilon(x)$. And subsequently, there exists $R_0 > 0$ such that $s \leq -R_0$ implies

$$f(x, s) > \underline{c}(x) \quad (9)$$

for $x \in (0, 1)$.

Since f is bounded on $(0, 1) \times I$, where $I \subset \mathbb{R}$ is a bounded interval. Combining this fact with above (9) show that there exists $m \in L^1(0, 1)$ such that

$$f(x, s) \geq m(x)$$

for all $s \leq 0$ and $x \in (0, 1)$.

Let $a, b \in \mathbb{R}$ with $0 < a < b < 1$. We can choose a, b so large such that

$$\left| \int_0^a (m(x) - \underline{c}(x)) \varphi_1(x) dx + \int_b^1 (m(x) - \underline{c}(x)) \varphi_1(x) dx \right| < \int_0^1 \underline{c}(x) \varphi_1(x) dx.$$

Therefore, it follows from the fact

$$\begin{aligned} \int_a^b \underline{c}(x) \varphi_1(x) dx + \int_0^a m(x) \varphi_1(x) dx + \int_b^1 m(x) \varphi_1(x) dx &= \int_0^1 \underline{c}(x) \varphi_1(x) dx \\ &+ \int_0^a (m(x) - \underline{c}(x)) \varphi_1(x) dx + \int_b^1 (m(x) - \underline{c}(x)) \varphi_1(x) dx \end{aligned}$$

that

$$\int_a^b \underline{c}(x) \varphi_1(x) dx + \int_0^a m(x) \varphi_1(x) dx + \int_b^1 m(x) \varphi_1(x) dx > 0.$$

Let us define $d_0 : (0, 1) \rightarrow \mathbb{R}$ by setting

$$d_0(x) = \begin{cases} \underline{c}(x), & x \in (a, b), \\ m(x), & x \in (0, 1) \setminus (a, b). \end{cases}$$

Observe that $d_0 \in L^1(0, 1)$ and $\int_0^1 d_0(x)\varphi_1(x)dx > 0$.

Let $x \in (0, 1)$ and $s \in \mathbb{R}$ such that $s \leq -R\varphi_1(x)$. We claim that $f(x, s) \geq d(x)$.

Indeed, for $x \in (a, b)$, since $\min_{x \in [a, b]} \varphi_1(x) > 0$, then there exists $R > 0$ such that $R\varphi_1 \geq R_0$ for $x \in (a, b)$.

Therefore, we have $s \leq -R_0$, and by (9), $f(x, s) > \underline{c}(x)$. For $x \in (0, 1) \setminus (a, b)$, since $s \leq 0$, we conclude that $f(x, s) \geq m(x)$. The conclusion now follows by taking a $d \in C[0, 1]$ with $d_0(x) \geq d(x)$, $x \in (0, 1)$ but still satisfies $\int_0^1 d(x)\varphi_1(x)dx > 0$. \square

Lemma 3.3. *Let the assumptions of Lemma 3.2 be satisfied. Then (6) has a strict lower solution α with $\alpha < 0$ for all $x \in (0, 1)$ and such that $u \geq \alpha$ for all possible solutions u of (6).*

Proof. We divide the proof into two steps.

(i) Let $R > 0$ and $d = d(x)$ be as in Lemma 3.2, such that $\int_0^1 d(x)\varphi_1(x)dx > 0$ and $f(x, u) \geq d$ whenever $u \leq -R\varphi_1$.

Consider the linear problem

$$\begin{cases} u^{(4)}(x) - \lambda_1 u(x) = d(x) - \left(\int_0^1 d(x)\varphi_1(x)dx \right) \varphi_1(x), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \quad (10)$$

It's worth pointing out that the right-hand member $v := d(x) - \left(\int_0^1 d(x)\varphi_1(x)dx \right) \varphi_1(x)$ satisfies the orthogonality condition $\int_0^1 v(x)\varphi_1(x)dx = 0$. Then any $\alpha = s\varphi_1(x) + \alpha_0$, $s \in \mathbb{R}$ is a solution of (10), where α_0 is the unique solution of the problem

$$\begin{cases} \alpha_0^{(4)}(x) - \lambda_1 \alpha_0(x) = v(x), & x \in (0, 1), \\ \alpha_0(0) = \alpha_0(1) = \alpha_0'(0) = \alpha_0'(1) = 0, \end{cases}$$

and satisfies $\int_0^1 (\alpha_0^{(4)}(x) - \lambda_1 \alpha_0(x))\varphi_1(x)dx = 0$.

Therefore, there exist constants a, A such that $a\varphi_1 \leq \alpha_0 \leq A\varphi_1$ for $x \in (0, 1)$, and accordingly, taking s negative sufficiently large (precisely, $s < -(R + A)$), we can arrange such that $\alpha = s\varphi_1(x) + \alpha_0 < -R\varphi_1$ for $x \in (0, 1)$. But then $f(x, \alpha) \geq d$ for all $x \in (0, 1)$, and since $\int_0^1 d(x)\varphi_1(x)dx > 0$, we conclude that

$$\begin{cases} \alpha^{(4)}(x) - \lambda_1 \alpha(x) < d(x) \leq f(x, \alpha(x)), & x \in (0, 1), \\ \alpha(0) = \alpha(1) = \alpha'(0) = \alpha'(1) = 0. \end{cases} \quad (11)$$

This implies α is a strict lower solution of (6).

(ii) Let u be a solution of (6). To prove that $u \geq \alpha$ we set $w = u - \alpha$ and by (11), we observe that w satisfies

$$\begin{cases} w^{(4)}(x) - \lambda_1 w(x) > f(x, u) - d(x), & x \in (0, 1), \\ w(0) = w(1) = w'(0) = w'(1) = 0. \end{cases} \quad (12)$$

Multiplying both sides of the equation in (12) by $v \in \{v \in C^4[0, 1] : v \geq 0, v(0) = v(1) = v'(0) = v'(1) = 0\}$ and integrating from 0 to 1, we get that for all $x \in (0, 1)$,

$$\int_0^1 w''(x)v''(x)dx - \lambda_1 \int_0^1 w(x)v(x)dx > \int_0^1 [f(x, u) - d(x)]v(x)dx. \quad (13)$$

Let $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$ denote the positive and negative parts of w .

We claim $u \geq \alpha$, i.e. $w^- = 0$. Assume on the contrary that $w^- \neq 0$, then choosing $v = w^-$ in (13) we obtain

$$-\int_0^1 (w^-(x))''^2 dx + \lambda_1 \int_0^1 (w^-(x))^2 dx > \int_0^1 [f(x, u) - d(x)]w^-(x) dx.$$

But $w^-(x) > 0$ means $u(x) < \alpha(x)$, which in turn implies $u(x) < -R\varphi_1$ and thus $f(x, u) \geq d(x)$. Therefore, the last integral is nonnegative and

$$\int_0^1 (w^-(x))''^2 dx < \lambda_1 \int_0^1 (w^-(x))^2 dx.$$

However, this contradicts the one-dimensional Poincaré inequality

$$\int_0^1 (w^-(x))''^2 dx \geq \lambda_1 \int_0^1 (w^-(x))^2 dx. \quad \square$$

Now, we extend the above result to the problem

$$\begin{cases} u^{(4)}(x) - \lambda u(x) = f(x, u(x)), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \quad (14)$$

Lemma 3.4. *Under the same assumptions as in Lemma 3.2, there exists a strict lower solution $\alpha < 0$ of (14) such that $u \geq \alpha$ for all possible solutions u of (14) for $\lambda \leq \lambda_1$.*

Proof. Let α be the lower solution for the (14) with $\lambda = \lambda_1$ determined in Lemma 3.3. Since $\alpha < -R\varphi_1 < 0$ for all $x \in (0, 1)$, we have from (11) if $\lambda \leq \lambda_1$,

$$\alpha^{(4)}(x) - \lambda_1 \alpha(x) < d(x) + (\lambda - \lambda_1)\alpha(x) \leq f(x, \alpha) + (\lambda - \lambda_1)\alpha(x)$$

for all $x \in (0, 1)$, and

$$\alpha(0) = \alpha(1) = \alpha'(0) = \alpha'(1) = 0,$$

therefore, α is a strict lower solution of (14) for all $\lambda \leq \lambda_1$. Moreover, for any solution u of (14), setting $w = u - \alpha$ we have

$$w^{(4)}(x) - \lambda w(x) > f(x, u) - d(x)$$

for all $x \in (0, 1)$, and

$$w(0) = w(1) = w'(0) = w'(1) = 0.$$

Now to prove that $w \geq 0$, one has only to remark that the argument in the proof of Lemma 3.3, part (ii), works equally well for any $\lambda \leq \lambda_1$. □

A result similar to the Lemma 3.4 holds for positive strict upper solution of (14) if we impose a symmetric condition on $f = f(x, s)$ as $s \rightarrow +\infty$.

Lemma 3.5. *Assume that $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and*

$$\limsup_{s \rightarrow +\infty} f(x, s) \leq \bar{c}(x) \quad \text{uniformly for } x \in (0, 1), \quad (15)$$

where $\bar{c} \in L^1(0, 1)$ and satisfies $\int_0^1 \bar{c}(x)\varphi_1(x) dx < 0$. Then there exists a strict upper solution $\beta > 0$ of (14) such that $u \leq \beta$ for all possible solutions u of (14) for $\lambda \leq \lambda_1$.

At this point, although existence of a strict lower solution $\alpha < 0$ and an strict upper solution $\beta > 0$ of (14) have been obtained for all $\lambda \leq \lambda_1$, this is no longer true for existence of a solution of (14) in the sector enclosed by $[\alpha(x), \beta(x)]$. Hence, we also assume that there exists $\mu \in (0, \frac{3\lambda_1}{4}]$, such that

$$f(x, u_2) + \mu u_2 - (f(x, u_1) + \mu u_1) \geq 0 \quad \text{for } \alpha(x) \leq u_1 \leq u_2 \leq \beta(x) \quad \text{and } x \in [0, 1]. \quad (16)$$

Based on Corollary 3.3 of [27], it allows us to present a maximum principle for the operator $L_\lambda : \mathcal{D} \rightarrow C[0, 1]$ defined by

$$(L_\lambda u)(x) = u^{(4)}(x) - \lambda u(x), \quad x \in [0, 1], \tag{17}$$

where $u \in \mathcal{D}$ and $\mathcal{D} = \{u \in C^4[0, 1] : u(0) = u(1) = u'(0) = u'(1) = 0\}$.

Lemma 3.6. *Let $\lambda \in [0, \lambda_1)$. If $u \in \mathcal{D}$ satisfies $L_\lambda u \geq 0$, then $u \geq 0$ in $[0, 1]$.*

Theorem 3.7. *Let (8) and (15) hold. Then there exist α and β , strict lower and upper solutions, respectively, for the problem (14) which satisfy*

$$\alpha(x) < 0 < \beta(x)$$

for all $x \in [0, 1]$ and $\lambda \leq \lambda_1$, and if f satisfies (16), then (14) has a solution u such that

$$\alpha(x) \leq u(x) \leq \beta(x)$$

for all $x \in [0, 1]$ and $\lambda \in [3\lambda_1/4, \lambda_1]$.

Proof. Let $\alpha < 0$ and $\beta > 0$ be as in Lemma 3.4 and Lemma 3.5. Let $\lambda = \lambda_1 + \varepsilon$ and $\varepsilon \in [-\lambda_1/4, 0]$.

Let us consider the auxiliary problem

$$\begin{cases} u^{(4)}(x) - [(\lambda_1 + \varepsilon) - \mu]u(x) = f(x, \eta(x)) + \mu\eta(x), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{18}$$

with $\eta \in C[0, 1]$ and $\mu \in (0, 3\lambda_1/4)$ is a fixed constant. For $\varepsilon = 0$, (18) reduces to (14).

Define $T_\lambda : C[0, 1] \rightarrow C[0, 1]$ by

$$T_\lambda \eta = u,$$

where u is the unique solution of (18). Clearly the operator T_λ is compact.

Step 1. We show $T_\lambda C \subseteq C$.

Here $C = \{\eta \in C[0, 1] : \alpha \leq \eta \leq \beta\}$ is a nonempty bounded closed subset in $C[0, 1]$.

In fact, for $\xi \in C$, set $y = T_\lambda \xi$. From the definition of α and C , and using (16), we have that

$$\begin{aligned} & (y - \alpha)^{(4)}(x) - [(\lambda_1 + \varepsilon) - \mu](y - \alpha)(x) \\ & \geq (f(x, \xi(x)) + \mu\xi(x)) - (f(x, \alpha(x)) + \mu\alpha(x)) \\ & \geq 0, \end{aligned}$$

and

$$(y - \alpha)(0) = (y - \alpha)(1) = (y - \alpha)'(0) = (y - \alpha)'(1) = 0.$$

Since $(\lambda_1 + \varepsilon) - \mu \in [0, \lambda_1)$, therefore, by Lemma 3.6, we have $y \geq \alpha$. Analogously, we can show that $y \leq \beta$.

Step 2. $T_\lambda : C[0, 1] \rightarrow C[0, 1]$ is nondecreasing.

Let $\eta_1, \eta_2 \in C[0, 1]$ with $\eta_1 \leq \eta_2$ and put $u_i = T_\lambda \eta_i, i = 1, 2$. Then from (16), $w = u_2 - u_1$ satisfies

$$\begin{cases} w^{(4)}(x) - [(\lambda_1 + \varepsilon) - \mu]w(x) = (f(x, \eta_2(x)) + \mu\eta_2(x)) - (f(x, \eta_1(x)) + \mu\eta_1(x)) \geq 0, & x \in (0, 1), \\ w(0) = w(1) = w'(0) = w'(1) = 0. \end{cases} \tag{19}$$

From Lemma 3.6, it follows that $w \geq 0$ and hence $u_1 \leq u_2$.

Step 3. $\alpha \leq T_\lambda \alpha$ and $T_\lambda \beta \leq \beta$.

Since α is a lower solution we have that

$$\begin{aligned} (T_\lambda \alpha)^{(4)}(x) - [(\lambda_1 + \varepsilon) - \mu](T_\lambda \alpha)(x) &= f(x, \alpha(x)) + \mu\alpha(x) \\ &\geq \alpha^{(4)}(x) - (\lambda_1 + \varepsilon)\alpha(x) + \mu\alpha(x) \\ &= \alpha^{(4)}(x) - [(\lambda_1 + \varepsilon) - \mu]\alpha(x). \end{aligned} \tag{20}$$

Thus $w = T_\lambda \alpha - \alpha$ satisfies that

$$\begin{cases} w^{(4)}(x) - [(\lambda_1 + \varepsilon) - \mu]w(x) \geq 0, & x \in (0, 1), \\ w(0) = w(1) = w'(0) = w'(1) = 0, \end{cases} \tag{21}$$

and then by Lemma 3.6 we deduce that $w = T_\lambda \alpha - \alpha \geq 0$. Analogously, we can prove that $T_\lambda \beta \leq \beta$.

The interval $[\alpha, \beta]$ is a closed, convex, bounded and nonempty subset of the Banach space $C[0, 1]$. Then by *Step 1* we can apply Schauder’s fixed point theorem to obtain the existence of a fixed point of T_λ , which obviously is a solution of problem (14) in $[\alpha, \beta]$. \square

Lemma 3.8. *Assume that the assumptions of Theorem 3.7 are satisfied. Let u be a solution of (14). Then for any $\lambda_0 \in (3\lambda_1/4, \lambda_1)$, there exists $\rho > 0$ such that $\|u\|_{C^3} < \rho$ for $\lambda \in [\lambda_0, \lambda_1]$.*

Proof. Let

$$M_1 := \sup \{ |\lambda u + f(x, u)| : \lambda_0 \leq \lambda \leq \lambda_1, \alpha(x) \leq u(x) \leq \beta(x), x \in [0, 1] \}. \tag{22}$$

By Theorem 3.7, we conclude M_1 is finite and then by (14) we know $|u^{(4)}| \leq M_1$ for all $x \in (0, 1)$.

Combine the boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

we know that there exists $t_0 \in (0, 1)$ such that $u'''(t_0) = 0$, see [23, P. 1212], and subsequently,

$$|u'''(t)| \leq \int_{t_0}^t |u^{(4)}(s)| ds \leq M_1.$$

Using the similar argument of above, we can prove that there exist constants M_2, M_3 and M_4 such that for all possible solutions u of (14) for $\lambda_0 \leq \lambda \leq \lambda_1$,

$$|u''(t)| \leq M_2, \quad |u'(t)| \leq M_3, \quad |u(t)| \leq M_4$$

for $t \in [0, 1]$. Clearly, $\|u\|_{C^3} < \rho$ for $\lambda_0 \leq \lambda \leq \lambda_1$ as long as $\rho := \max\{M_1, M_2, M_3, M_4\} + 1$. \square

Lemma 3.9. *Assume that the assumptions of Theorem 3.7 are satisfied. Let*

$$\Omega_{\alpha, \beta} = \{u \in C^4[0, 1] : \alpha(x) < u(x) < \beta(x), x \in [0, 1]\},$$

and

$$\Omega = B_\rho \cap \Omega_{\alpha, \beta},$$

where $B_\rho = \{u \in C^3[0, 1] : \|u\|_{C^3} < \rho\}$ and ρ is given in Lemma 3.8. Then there exists $\lambda_0 \in (3\lambda_1/4, \lambda_1)$, such that

$$\text{deg}[I - T_\lambda, \Omega, 0] = 1$$

for $\lambda \in [\lambda_0, \lambda_1]$.

Proof. For any $\mu \in [0, 1]$, consider now the homotopy

$$\begin{cases} u^{(4)}(x) = \mu(\lambda u(x) + f(x, u(x))), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases} \tag{23}$$

Reasoning as in Lemma 3.8, we observe that all possible solutions of the problems (23) satisfy

$$\|u\|_{C^3} < \rho$$

for any $\lambda \in [\lambda_0, \lambda_1]$. Therefore, if we write (23) as $u = \mu \hat{T}(u)$, and set

$$H(\mu, u) = u - \mu \hat{T}(u).$$

Then $H(\mu, u) \neq 0$ for all $\mu \in [0, 1]$ and $\|u\|_{C^3} = \rho$, so that by the homotopy invariance of the Leray-Schauder degree

$$\deg[I - T_\lambda, B_\rho, 0] = \deg[I - \hat{T}, B_\rho, 0] = \deg[I, B_\rho, 0] = 1. \tag{24}$$

On the other hand, by Theorem 3.7, all zeros of $I - T_\lambda$ belong to $\Omega_{\alpha,\beta}$. Therefore, if ρ is large enough, then by the excision property of the Leray-Schauder degree, we have

$$\deg[I - T_\lambda, \Omega, 0] = \deg[I - T_\lambda, \Omega_{\alpha,\beta}, 0] = \deg[I - T_\lambda, B_\rho, 0] = 1. \quad \square$$

Lemma 3.10. *Let $\Omega = \Omega_{\alpha,\beta} \cap B_\rho$. Then there exists $\delta > 0$ such that*

$$\deg[I - T_\lambda, \Omega, 0] = 1$$

for all $\lambda \in [\lambda_1, \lambda_1 + \delta]$.

Proof. The proof is trivial, so we omit it. □

4 Bifurcation from infinity and the multiplicity of solutions

It follows from Lemmas 3.9 and 3.10, and using the similar arguments of [28], we have the following

Lemma 4.1. *Let $\lambda_0 \in (3\lambda_1/4, \lambda_1)$, $\rho > 0$ and $\delta > 0$ are sufficiently small constants. Then the set*

$$\Gamma := \{(\lambda, u) : \lambda \in [\lambda_0, \lambda_1 + \delta], \|u\|_{C^3} < \rho, (\lambda, u) \text{ satisfies (14)}\}$$

contains a connected component $\mathcal{C} = \{(\lambda, u_\lambda)\}$.

To apply the argument of [21], let us extend $f(x, \cdot)$ to all of \mathbb{R} by setting

$$\tilde{f}(x, u) = \begin{cases} f(x, u), & u > 0, \\ f(x, 0), & u \leq 0, \end{cases} \tag{25}$$

and deduce that

Lemma 4.2. *Since $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies (H3), and λ_1 is a simple eigenvalue, then there exists an unbounded connected component $\Sigma_\infty \subset \mathbb{R} \times C^3[0, 1]$ of solutions of (14) such that for all sufficiently small $r > 0$,*

$$\Sigma_\infty \cap U_r \neq \emptyset, \tag{26}$$

where $U_r := \{(\lambda, u) \in \mathbb{R} \times C^3[0, 1] : |\lambda - \lambda_1| < r, \|u\|_{C^3} > \frac{1}{r}\}$.

Proof of Theorem 1.1. For any given $h \in C[0, 1]$, let

$$f_h(x, s) = f(x, s) - h(x).$$

Notice that $f_h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and satisfies (H3), and by (H2) and (H1), we know f_h satisfies (8) and (15) with $\underline{c} = c - h$, $\bar{c} = C - h$, and (16). Therefore, we may apply Theorem 3.7 and Lemma 4.1, to deduce that the problem (3) has at least one solution u_1 in

$$B_\rho = \{u \in C^3[0, 1] : \|u\|_{C^3} < \rho\}$$

for $\lambda \in [\lambda_0, \lambda_1 + \delta]$.

On the other hand, since λ_1 is a simple eigenvalue, by Lemma 4.2, we conclude that there exists an unbounded connected component $\Sigma_\infty \subset \mathbb{R} \times C^3[0, 1]$ of solutions of (3) bifurcating from infinity at $\lambda = \lambda_1$.

In Lemma 3.8, we have established a priori bound for solutions of (3) when $\lambda \in [\lambda_0, \lambda_1]$, hence the connected component Σ_∞ must bifurcate to right. More precisely, we infer Σ_∞ must satisfy

$$\Sigma_\infty \subset \left\{ (\lambda, u) \in \mathbb{R} \times C^3[0, 1] : \lambda_1 < \lambda < \lambda_1 + r, \|u\|_{C^3} > \frac{1}{r} \right\},$$

and hence, if $\frac{1}{r} > \rho$, i.e. $r < \frac{1}{\rho}$, we obtain the second solution u_2 with

$$\|u_2\|_{C^3} > \frac{1}{r} > \rho$$

for $\lambda \in [\lambda_1, \lambda_1 + \delta_1]$, where $\delta_1 = \min\{r, \delta\}$.

Next, we will show that $u_2 > 0$. It suffices to show that if $(\mu_n, u_n) \in \Sigma_\infty$ with $\mu_n \rightarrow \lambda_1$ and $\|u_n\|_{C^3} \rightarrow \infty$ then $u_n > 0$ in $(0, 1)$ for n large. In fact, let $w_n = \frac{u_n}{\|u_n\|_{C^3}}$. Then

$$\begin{cases} w_n^{(4)}(x) - \mu_n w_n(x) = \frac{f_h(x, u_n(x))}{u_n(x)} w_n(x), & x \in (0, 1), \\ w_n(0) = w_n(1) = w_n'(0) = w_n'(1) = 0. \end{cases} \tag{27}$$

Assumption (H3) yields that, up to a subsequence, $w_n \rightarrow w$ in $C^1[0, 1]$, where w is such that $\|w\|_{C^1} = 1$ and satisfies

$$\begin{cases} w^{(4)}(x) - \lambda_1 w(x) = 0, & x \in (0, 1), \\ w(0) = w(1) = w'(0) = w'(1) = 0, \end{cases}$$

it follows that $w \geq 0$, and hence there exists $\beta > 0$ such that $w = \beta\varphi_1$. Then, it follows that $u_n > 0$ in $(0, 1)$, for n large. □

Example 4.3. Let us consider the following nonlinear fourth order boundary value problem

$$\begin{cases} u^{(4)}(x) - \lambda u(x) = f(x, u(x)) - h(x), & x \in (0, 1), \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{cases} \tag{28}$$

with

$$f(x, u) = \begin{cases} -\arctan u, & |u| \geq 1, \\ -\frac{\pi}{4}u, & 0 \leq |u| < 1, \end{cases}$$

and

$$h(x) \equiv \frac{1}{2}.$$

Obviously, the function $f(x, u) + \frac{3\lambda_1}{4}u$ is increasing in u .

Let $\underline{c}(x) \equiv 1$ and $\bar{c}(x) \equiv -1$. Then $c(x) \equiv \frac{3}{2}$ and $C(x) \equiv -\frac{1}{2}$. And it is easy to check that all assumptions of Theorem 1.1 are valid. Therefore, from Theorem 1.1, we know there exist $\delta_1 > 0$ and $\lambda_0 \in (3\lambda_1/4, \lambda_1)$ such that (28) has at least one solution for $\lambda \in [\lambda_0, \lambda_1]$, and two solutions for $\lambda \in (\lambda_1, \lambda_1 + \delta_1]$. Moreover, one of these two solutions is a positive solution.

Example 4.4. The functions f and h can be given respectively by

$$f(x, u) = \begin{cases} -\sqrt{u}, & u \geq 100, \\ -\frac{1}{10}u, & |u| < 100, \\ \sqrt{-u}, & u \leq -100, \end{cases}$$

and

$$h(x) = 50\varphi_1(x), \quad x \in [0, 1],$$

for which if we take $c(x) = 100\varphi_1(x)$ and $C(x) = -200\varphi_1(x)$ for $x \in [0, 1]$, then $\underline{c}(x) = 50\varphi_1(x)$ and $\bar{c}(x) = -250\varphi_1(x)$.

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