Open Mathematics

Research Article

Peter R. Mercer*

An improved Schwarz Lemma at the boundary

https://doi.org/10.1515/math-2018-0096
Received April 26, 2018; accepted September 5, 2018.

Abstract: We obtain an new boundary Schwarz inequality, for analytic functions mapping the unit disk to itself. The result contains and improves a number of known estimates.

Keywords: Schwarz Lemma, Julia's Lemma

MSC: 30C80

1 Introduction

Denote by $\Delta \subset \mathbb{C}$ the open unit disk, and let $f : \Delta \to \Delta$ be analytic. We assume that there is $x \in \partial\Delta$ and $\beta \in \mathbb{R}$ such that

$$\liminf_{z\to x} \frac{1-|f(z)|}{1-|z|} = \beta . \tag{1}$$

By pre-composing with a rotation we may suppose that $x = 1$, and by post-composing with a rotation we may suppose that $f(1) = 1$. Then **Julia's Lemma** (e.g. [1, 2]) gives

$$\frac{|1-f(z)|^2}{1-|f(z)|^2} \leq \beta\frac{|1-z|^2}{1-|z|^2} \qquad \forall z \in \Delta .$$

This inequality has an appealing geometric interpretation, which we do not use here. But two immediate consequences which we do use, are that $\beta > 0$ and that the *radial derivative* of f exists at $1 \in \partial\Delta$:

$$\lim_{r\nearrow 1} \frac{f(r)-f(1)}{r-1} = f'(1) \quad \text{with} \quad |f'(1)| = \beta . \tag{2}$$

(There are many other consequences of Julia's Lemma, the most important being contained in the Julia-Carathéodory Theorems.)

Assuming the normalization $f(0) = 0$, we evidently have $\beta \geq 1$. But even better, Osserman [3] showed that in this case

$$\beta \geq 1 + \frac{1-|f'(0)|}{1+|f'(0)|} . \tag{3}$$

(A proof of (3) can also be found in [4], which is motivated by the influential paper [5].) Now Osserman's inequality was in fact anticipated by Ünkelbach [6], who had already obtained the better estimate

$$\beta \geq \frac{2(1-\operatorname{Re} f'(0))}{1 - |f'(0)|^2} = 1 + \frac{|1-f'(0)|^2}{1-|f'(0)|^2} . \tag{4}$$

However, [3] also contains a non-normalized version, which reduces to (3) if $f(0) = 0$, viz.

$$\beta \geq \frac{2(1-|f(0)|)^2}{1-|f(0)|^2+|f'(0)|} . \tag{5}$$

***Corresponding Author: Peter R. Mercer:** Dept. Mathematics SUNY College, Buffalo, USA, E-mail: mercerpr@buffalostate.edu

Since the appearance of Osserman's paper, a good number of authors have refined and generalized these estimates – as discussed in the next section. The aim here is to provide a different and very elementary approach, which contains and improves many of these modifications. But first we recall some results which are of use in the sequel.

The well-known **Schwarz's Lemma**, which is a consequence of the Maximum Principle, says that if $f : \Delta \to \Delta$ is analytic with $f(0) = 0$, then

$$|f(z)| \le |z| \quad \forall z \in \Delta, \qquad \text{and consequently} \quad |f'(0)| \le 1 .$$

To remove the normalization $f(0) = 0$, one applies Schwarz's Lemma to $\phi_{f(a)} \circ f \circ \phi_a$ where ϕ_a is the automorphism of Δ which interchanges a and 0:

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z}.$$

This gives the **Schwarz-Pick Lemma** which says that for $f : \Delta \to \Delta$ analytic,

$$\left|\frac{f(w) - f(z)}{1 - \overline{f(w)}f(z)}\right| \le \left|\frac{w - z}{1 - \overline{w}z}\right| \quad \forall z, w \in \Delta .$$

Consequently, the *hyperbolic derivative* satisfies

$$|f^*(z)| \le 1 \quad \forall z \in \Delta, \qquad \text{where} \quad f^*(z) = \frac{1 - |z|^2}{1 - |f(z)|^2} f'(z) .$$

It is the Schwarz-Pick Lemma that does most of the work in proving Julia's Lemma. But another consequence of the Schwarz-Pick Lemma is the following (e.g. [7–9]), which we shall also rely upon.

Lemma 1.1 (Dieudonné's Lemma). *Let $f : \Delta \to \Delta$ be analytic, with $f(z) = w$ and $f(z_1) = w_1$. Then*

$$\left|f'(z) - c\right| \le r ,$$

where

$$c = \frac{\phi_w(w_1)}{\phi_z(z_1)} \frac{1 - |\phi_z(z_1)|^2}{1 - |\phi_w(w_1)|^2} \frac{1 - |w|^2}{1 - |z|^2} , \quad r = \frac{|\phi_z(z_1)|^2 - |\phi_w(w_1)|^2}{|\phi_z(z_1)|^2(1 - |\phi_w(w_1)|^2)} \frac{1 - |w|^2}{1 - |z|^2} .$$

2 Main result

We remove the dependence on $f(0)$, while improving many estimates which do contain $f(0)$. We shall rely on Dieudonné's Lemma, the Schwarz-Pick Lemma, and Julia's Lemma.

Theorem 2.1. *Let $f : \Delta \to \Delta$ be analytic with $f(z) = w$ and $f(1) = 1$ as in (1). Then*

$$\beta \ge 2 \frac{|1 - w|^2}{1 - |w|^2} \frac{1 - |z|^2}{|1 - z|^2} \frac{1 - \operatorname{Re}\left(f^*(z) \frac{1-\overline{w}}{1-w} \frac{1-z}{1-\overline{z}}\right)}{1 - |f^*(z)|^2} . \tag{6}$$

Proof. Using the easily verified identity

$$1 - |\phi_a(\lambda)|^2 = \frac{(1 - |a|^2)(1 - |\lambda|^2)}{|1 - \overline{a}\lambda|^2} , \tag{7}$$

we get, in Dieudonné's Lemma,

$$c = \frac{w_1 - w}{1 - \overline{w}w_1} \frac{1 - \overline{z}z_1}{z_1 - z} \frac{1 - |z_1|^2}{|1 - \overline{z}z_1|^2} \frac{|1 - \overline{w}w_1|^2}{1 - |w_1|^2} = \frac{w_1 - w}{z_1 - z} \frac{1 - w\overline{w_1}}{1 - z\overline{z_1}} \frac{1 - |z_1|^2}{1 - |w_1|^2} ,$$

and

$$r = \frac{\left(1-|\phi_w(w_1)|^2\right)-\left(1-|\phi_z(z_1)|^2\right)}{|\phi_z(z_1)|^2(1-|\phi_w(w_1)|^2)}\frac{1-|w|^2}{1-|z|^2}$$
$$= \frac{1}{|\phi_z(z_1)|^2}\left(1-\frac{1-|z|^2}{1-|w|^2}\frac{1-|z_1|^2}{1-|w_1|^2}\frac{|1-\overline{w}w_1|^2}{|1-\overline{z}z_1|^2}\right)\frac{1-|w|^2}{1-|z|^2}.$$

then having $z_1 \to 1$ along a sequence for which β in (1) is attained, we get

$$c \to \widetilde{c} = \left(\frac{1-w}{1-z}\right)^2\frac{1}{\beta} \qquad \text{and} \qquad r \to \widetilde{r} = \frac{1-|w|^2}{1-|z|^2} - \frac{1}{\beta}\frac{|1-w|^2}{|1-z|^2}.$$

That is,

$$\left|f'(z)-\widetilde{c}\right| \le \widetilde{r}. \tag{8}$$

Now, upon squaring both sides of this inequality, there is some cancellation:

$$|f'(z)|^2 - 2\,\mathrm{Re}\left(\overline{f'(z)}\left(\frac{1-w}{1-z}\right)^2\frac{1}{\beta}\right) \le \left(\tfrac{1-|w|^2}{1-|z|^2}\right)^2 - \frac{2}{\beta}\frac{1-|w|^2}{1-|z|^2}\frac{|1-\overline{w}|^2}{|1-\overline{z}|^2}.$$

That is,

$$\left(\tfrac{1-|w|^2}{1-|z|^2}\right)^2\left(|f^*(z)|^2-1\right) \le \tfrac{2}{\beta}\tfrac{|1-\overline{w}|^2}{|1-\overline{z}|^2}\tfrac{1-|w|^2}{1-|z|^2}\left[\mathrm{Re}\left(f^*(z)\frac{1-\overline{w}}{1-w}\frac{1-z}{1-\overline{z}}\right)-1\right].$$

By the Schwarz-Pick Lemma each side of this last inequality is nonpositive, so isolating β we get (6). □

Remark 2.2. *Having $z \to 1$ radially in line (8), and using (2), we obtain*

$$\lim_{r\nearrow 1} f'(r) = f'(1).$$

From this, and using $|\tau| = 1 \Rightarrow \frac{1-\mathrm{Re}(\sigma\tau)}{1-|\sigma|^2} \ge \frac{1}{1+|\sigma|}$, follows the rather comforting fact that the right-hand side of (6) tends to β as $z \to 1$ radially.

Remark 2.3. *In Lemma 6.1 of [8] is the estimate*

$$\beta \ge \frac{2}{1+|f^*(z)|}\frac{1-|f(z)|}{1+|f(z)|}\frac{1-|z|}{1+|z|}, \tag{9}$$

which contains (5), but is quite mild if $|z|$ or $|f(z)|$ is near 1. Anyway, $|\tau| = 1 \Rightarrow \frac{1-\mathrm{Re}(\sigma\tau)}{1-|\sigma|} \ge 1$ shows that (6) improves (9).

Remark 2.4. *Now take $z = 0$, so that (6) reads*

$$\beta \ge 2\frac{|1-f(0)|^2}{1-|f(0)|^2}\frac{1 - \mathrm{Re}\left(f^*(0)\frac{1-\overline{f(0)}}{1-f(0)}\right)}{1 - |f^*(0)|^2}. \tag{10}$$

This may be regarded as an non-normalized version of (4). Indeed, taking also $f(0) = 0$ recovers (4). This is the same estimate which results from having $z = 0$ in Theorem 5 of [10]. However, that result (which is arrived at by very nonelementary means) contains $f(0)$ even for $z \ne 0$, a deficiency from which Theorem 2.1 does not suffer.

Remark 2.5. *Using again $|\tau| = 1 \Rightarrow \frac{1-\mathrm{Re}(\sigma\tau)}{1-|\sigma|^2} \ge \frac{1}{1+|\sigma|}$ in (10), we get*

$$\beta \ge \frac{2|1-f(0)|^2}{1-|f(0)|^2+|f'(0)|},$$

which improves (5), analogously to how (4) improves (3).

Remark 2.6. *But using just $|\tau| = 1 \Rightarrow \frac{1-\mathrm{Re}(\sigma\tau)}{1-|\sigma|^2} \ge \frac{1}{1+\mathrm{Re}(\sigma\tau)}$ in (10), then $\frac{1-|f(0)|^2}{|1-f(0)|^2} = \mathrm{Re}\,\frac{1+f(0)}{1-f(0)}$, we get*

$$\beta \ge \frac{2}{\mathrm{Re}\,\frac{1-f(0)^2+f'(0)}{(1-f(0))^2}}, \tag{11}$$

which improves (5) more effectively. Estimate (11) was obtained differently in each of [11] and [12].

3 Consequences

Cases for which $z = w = 0$ (i.e. $f(0) = 0$) are obviously contained in the remarks above, but when this holds we can do a little better, as follows.

Corollary 3.1. *Let $f : \Delta \to \Delta$ be analytic with $f(0) = 0$ and $f(1) = 1$ as in (1). Then*

$$\beta \;\geq\; 1 \;+\; \frac{2|1-f'(0)|^2}{1-|f'(0)|^2+|f''(0)|/2}\,\frac{1+\operatorname{Re}\left(\frac{f''(0)}{2(1-|f'(0)|^2)}\right)}{1-\frac{|f''(0)|}{2(1-|f'(0)|^2)}}\,. \tag{12}$$

Proof. We introduce $f''(0)$, in standard fashion: Set

$$g(\lambda) \;=\; \frac{f(\lambda)}{\lambda} \;\;(\text{with } g(0) := f'(0)), \quad \text{and} \quad h(\lambda) \;=\; \phi_{g(0)}(g(\lambda))\,.$$

Then h is analytic on Δ with $h(0) = 0$, and by Schwarz's Lemma $h : \Delta \to \Delta$. Here we have

$$h'(0) \;=\; \frac{-f''(0)}{2(1-|f'(0)|^2)}\,. \tag{13}$$

A calculation using the identity (7) and the assumption (1) gives

$$\liminf_{z\to 1} \frac{1-|h(z)|}{1-|z|} \;=\; (\beta-1)\frac{1-|f'(0)|^2}{|1-f'(0)|^2} \;=\; \widehat{\beta}\,, \quad \text{say.} \tag{14}$$

Then in (6), i.e. (4), replacing f with h and β with $\widehat{\beta}$, we obtain

$$\beta \;\geq\; 1 + \frac{|1-f'(0)|^2}{1-|f'(0)|^2}\,\frac{2(1-\operatorname{Re} h'(0))}{1\;-\;|h'(0)|^2}.$$

Inserting (13) and a little tidying yields (12), as desired. □

Remark 3.2. *Corollary 3.1 improves*

$$\beta \;\geq\; 1 + \frac{2(1-|f'(0)|^2)}{1-|f'(0)|^2+|f''(0)|/2}\,, \tag{15}$$

which was obtained by Dubinin [13] using a proof which relies directly on (3). (Incidentally, Schwarz's Lemma applied to h gives $|f''(0)|/2 \leq 1-|f'(0)|^2$, from which it is readily seen that (15) improves (3).)

Remark 3.3. *We add finally that using (4) in the form*

$$\beta \;\geq\; 1 \;+\; \frac{|1-f'(0)|^2}{1-|f'(0)|^2}\,,$$

then replacing f with h and β with $\widehat{\beta}$ here, and using (13) and (14), we get another way of expressing (12) :

$$\begin{aligned}\beta \;&\geq\; 1 \;+\; \frac{|1-f'(0)|^2}{1-|f'(0)|^2}\left(1 \;+\; \frac{\left|1+\frac{f''(0)}{2(1-|f'(0)|^2)}\right|^2}{1\;-\;\left|\frac{f''(0)}{2(1-|f'(0)|^2)}\right|^2}\right)\\ &= 1 \;+\; \frac{|1-f'(0)|^2}{1-|f'(0)|^2} \;+\; \frac{\left|1+\frac{f''(0)}{2(1-|f'(0)|^2)}\right|^2}{1-\left|\frac{f''(0)}{2(1-|f'(0)|^2)}\right|}\,\frac{|1-f'(0)|^2}{1-|f'(0)|^2+|f''(0)|/2}\,.\end{aligned}$$

References

[1] Carathéodory C., *Theory of Functions, Vol II*, 1960, New York: Chelsea.

[2] Cowen C.C., MacCluer B.D., *Composition Operators on Spaces of Analytic Functions*, 1995, Boca Raton: C.R.C. Press.

[3] Osserman R., A sharp Schwarz inequality on the boundary, *Proc. Amer. Math. Soc.*, 2000, 128, 3513-3517.

[4] Krantz S.G., The Schwarz lemma at the boundary, *Complex Var. & Ellip. Equ.*, 2011, 56, 455-468.

[5] Burns D.M., Krantz S.G., Rigidity of holomorphic mappings and a new Schwarz lemma at the boundary, *J. Amer. Math. Soc.*, 1994, 7, 661-676.

[6] Ünkelbach H., Uber die Randverzerrung bei konformer Abbildung, *Math. Zeit.*, 1938, 43, 739-742.

[7] Duren P.L., *Univalent Functions*, 1983, New York & Berlin: Springer-Verlag.

[8] Kaptanoğlu H.T., Some refined Schwarz-Pick Lemmas, *Mich. Math. J.*, 2002, 50, 649-664.

[9] Mercer P.R., Sharpened versions of the Schwarz lemma, *J. Math. Analysis & Appl.*, 1997, 205, 508-511.

[10] Komatu Y., On angular derivative, *Kodai Math. Sem. Rep.*, 1961, 13, 167-179.

[11] Frovlova A., Levenshtein M., Shoikhet D., Vasil'ev A., Boundary distortion estimates for holomorphic maps, *Complex Anal. Oper. Theory*, 2014, 8, 1129-1149.

[12] Ren G., Wang X., Extremal functions of boundary Schwarz lemma. *arXiv.org*. https://arxiv.org/pdf/1502.02369 .

[13] Dubinin V.I., Schwarz inequality on the boundary for functions regular in the disk, *J. Math. Sci*, 2004, 122, 3623-3629.