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Optimality and duality in set-valued optimization utilizing limit sets

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Abstract: This paper deals with optimality conditions and duality theory for vector optimization involving non-convex set-valued maps. Firstly, under the assumption of nearly cone-subconvexlike property for set-valued maps, the necessary and sufficient optimality conditions in terms of limit sets are derived for local weak minimizers of a set-valued constraint optimization problem. Then, applications to Mond-Weir type and Wolfe type dual problems are presented.

Keywords: Limit set, Optimality conditions, Set-valued optimization, Nearly cone-subconvexlike, Duality

MSC: 90C29, 90C46, 26B25

1 Introduction

In the past decades a great deal of attention was given to establish the optimality conditions and duality theory for set-valued optimization problem by employing various notions of derivatives or directional derivatives for set-valued maps. For more details related to this topic, one can refer to the excellent books [1-4]. Let us underline there is a growing interest on optimality and duality by using directional derivatives for set-valued maps. For example, Corley [5] defined cone-directed contingent derivatives for set-valued maps in terms of tangent cones, and used it to establish Fritz-John necessary optimality conditions for a set-valued optimization. Under the assumption of generalized cone-preinvexity, Qiu [6] presented the optimality conditions for a set-valued optimization problem by utilizing cone-directed contingent derivatives; Penot [7] introduced the lower Dini derivative for set-valued mappings, which is the generalization of lower Dini directional derivative for the real valued functions. Making use of this type of drectional derivatives, Kuan and Raciti [8] derived a necessary optimality condition for proper minimizers in set-valued optimization; Crespi et al. [9] obtained the necessary and sufficient conditions for weak minimizers and proper minimizers in Lipschitz set-valued optimization in terms of the upper Dini directional derivatives. Guerraggio et al. [10] proposed the optimality conditions for locally Lipschitz vector optimization problem by using of the upper Dini directional derivatives; Yang [11] introduced Dini directional derivatives for a set-valued mapping in topological spaces and used it to derive the optimality conditions for cone-convex set-valued optimization problems. It is worth noticing that the Dini directional derivatives given by Yang [11] are different from above mentioned directional derivatives, which are in terms of continuous selection functions for the set-valued mappings, and the necessary and sufficient optimality conditions were derived for the generalized cone-preinvex set-valued optimization problems by using this kind of directional derivatives in [12]. In 2012, Alonso-

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Durán and Rodríguez-Marín [13] presented the concept of limit set based upon Dini directional derivatives given by Yang [11], and optimality conditions are given for several approximate solutions in unconstraint set-valued optimization by utilizing limit set.

On the other hand, convex analysis is a powerful tool for the investigation of optimal solutions of set-valued optimization problems. Various notions of generalized convexity have been introduced to weaken convexity. One of such generalizations in set-valued analysis is called cone-convexity [14], which plays a very important role in set-valued optimization. Based upon this concept, some scholars developed further generalizations of cone-convexity to vector optimization involving set-valued maps. For example, cone-convexlikeness [15], cone-subconvexlikeness [15], nearly cone-convexlikeness [16] and nearly cone-subconvexlikeness [17] etc. Among these notions, the nearly cone-subconvexlikeness is the most general one. Sach [18] introduced another more general weak convexity for set-valued maps, called ic-cone-convexlikeness. However, it has been pointed out in [19] that when the ordering cone has nonempty interior, ic-cone-convexlikeness is equivalent to nearly cone-subconvexlikeness. In this paper, we shall make use of nearly cone-subconvexlikeness as the weaker condition on convexity assumption.

Based upon the above observation, this paper is focused on Dini directional derivatives of set-valued maps and weak minimizer of a set-valued optimization problem under weaker condition on convexity. The purpose of this paper is two aspects: first, to establish the optimality conditions for local weak minimizers in two types: separating of sets and Kuhn-Tucker type; second, to provide an employment of optimality conditions for weak minimizer to obtain some duality results for Mond-Weir type and Wofe type dual problems.

We proceed as follows: Section 2 is devoted to preliminaries, in which some well-known definitions and results used in the sequel are recalled. In Section 3, we prove some optimality conditions for local weak minimizers in nearly cone-subconvexlike set-valued optimization problems. In Section 4, we present applications of results obtained in Section 3 to two types duality.

2 Preliminaries

Throughout the paper, we assume that X, Y and Z are three real normed linear spaces with topological dual X^*, Y^* and Z^*, repectively. For any $x \in X$ and $x^* \in X^*$, the canonical form between X and X^* is denoted by ${x^*}^T x$. Let $\bar{x} \in X$, $U(\bar{x})$ is used for the set of all neighborhoods of $\bar{x}$. Assuming A is a nonempty subset of Y, the closure of A is denoted by cl$A$ and the cone generated by A is denoted by $\text{cone}(A) = \{\lambda a : \ a \in A, \lambda \in \mathbb{R}_+\}$. $D \subset Y$ and $E \subset Z$ are closed pointed convex cones, which we also assume that they are solid, i.e., int$D \neq \varnothing$ and int$E \neq \varnothing$. We denote

$$D^* = \{y^* : \ {y^*}^T d \geq 0, \ \forall \ d \in D\},$$

and similarly for E^*. Let $F : \ X \to 2^Y$ be a set-valued mapping. The set

$$\text{dom}(F) := \{x \in X : \ F(x) \neq \varnothing\},$$

is called the domain of F. The set

$$\text{graph}(F) := \{(x, y) \in X \times Y : \ y \in F(x)\}$$

is called the graph of the map F. The profile map of F is written by $F_+(\cdot) := F(\cdot) + D$. We follow the convention $F(S) = \bigcup_{x \in S} F(x)$.

Let $S \subset X$ and $\bar{x} \in S$. We will consider the contingent cone to S at $\bar{x}$, defined by (see [1]):

$$T(S, \bar{x}) = \{x \in X : \ \exists (t_n) \to 0^+, (x_n) \to x \text{ with } \bar{x} + t_n x_n \in S \text{ for all } n \in \mathbb{N}\}.$$

Let S be nonempty set of X and $F : S \to 2^Y$ be a set-valued map. The limit set of the map F at a given point was given by Definition 2.1 in [11] and Definition 1 in [13]. Now, we rewrite this definition in terms of set-valued mapping.

Definition 2.1. *Let $(\bar{x}, \bar{y}) \in graph(F)$ with $\bar{x} \in S$. The limit set of F at $\bar{x}$ in the direction $x \in T(S, \bar{x})$ with respect to $\bar{y}$ is the set-valued map*

$$YF(\bar{x}; \bar{y}) : T(S, \bar{x}) \to 2^Y$$

defined by

$$YF(\bar{x}; \bar{y})(x) = \Big\{ y \in Y : y = \lim_{(t_n, x_n) \to (0^+, x)} \frac{f(\bar{x} + t_n x_n) - f(\bar{x})}{t_n}; \text{ for some } f \in CS(F),\ \bar{y} = f(\bar{x}) \Big\},$$

where $CS(F)$ denotes the set of continuous selections of F.

Let's see an example of limit set for a set-valued mapping.

Example 2.2. *Let $X = Y = \mathbb{R}$, $S = X$, $D = \mathbb{R}_+$ and $F : S \to 2^Y$ be defined by*

$$F(x) := \{y \in Y : y \geq x^2\}, \quad \text{for all } x \in S.$$

Taking $\bar{x} = 0$, obviously, for every continuous selections f of F with $\bar{y} = f(\bar{x}) = 0$, we can derive that

$$YF(0; 0)(x) = \mathbb{R}_+, \quad \text{for all } x \in T(S, 0).$$

Let $F : S \subset X \to 2^Y$, $G : S \subset X \to 2^Z$ be two set-valued maps, $(\bar{x}, \bar{y}) \in \text{graph}(F)$ and $(\bar{x}, \bar{z}) \in \text{graph}(G)$. The notation $(F \times G)(x)$ is used to denote $\big(F(x), G(x)\big)$. The limit set of $F \times G$ at $\bar{x}$ in the direction $x \in T(S, \bar{x})$ with respect to $(\bar{y}, \bar{z})$ is denoted by the set-valued mapping

$$Y(F \times G)\big(\bar{x}; (\bar{y}, \bar{z})\big) : T(S, \bar{x}) \to 2^{Y \times Z}$$

and given by

$$Y(F \times G)\big(\bar{x}; (\bar{y}, \bar{z})\big)(x)$$
$$= \Big\{ (y, z) \in Y \times Z : (y, z) = \lim_{(t_n, x_n) \to (0^+, x)} \frac{(f \times g)(\bar{x} + t_n x_n) - (f \times g)(\bar{x})}{t_n}, \text{ for some } f \in \text{CS}(F) \text{ and } g \in \text{CS}(G),\ (\bar{y}, \bar{z}) = \big(f(\bar{x}), g(\bar{x})\big) \Big\}.$$

Definition 2.3. *Let $F : S \to 2^Y$, $S \subset X$, and $(\bar{x}, \bar{y}) \in graph(F)$ with $\bar{x} \in S$. It is said that $(\bar{x}, \bar{y})$ is a local weak minimizer of F over S, if there exists $U \in U(\bar{x})$ such that*

$$\big(F(S \cap U) - \bar{y}\big) \cap -intD = \emptyset. \tag{2.1}$$

If $U = X$, then the word "local" is omitted from the terminology in the above definition, and in this case, $(\bar{x}, \bar{y})$ is called a weak minimizer of F over S.

Definition 2.4. *Let $S \subset X$ be a nonempty set and $F : S \to 2^Y$ be a set-valued mapping.*
(i) F is D-convex on convex set S if for all $t \in [0, 1]$, $x_1, x_2 \in S$,

$$tF(x_1) + (1 - t)F(x_2) \subset F\big(tx_1 + (1 - t)x_2\big) + D.$$

(ii) F is D-convexlike on S if and only if $F(S) + D$ is a convex set.
(iii) F is D-subconvexlike on S if and only if $F(S) + intD$ is a convex set.
(iii) F is nearly D-convexlike on S if and only if $cl[F(S) + D]$ is a convex set.
(iv) F is nearly D-subconvexlike on S if $cl[cone(F(S) + D)]$ is convex set.

Definition 2.4 (ii)-(iv) is extended from definitions in [14-17] for a set-valued map. In particular, it has been pointed out in [16] that the following relationships hold:

$$D\text{-convex} \Rightarrow D\text{-convexlike} \Rightarrow D\text{-subconvexlike} \Leftrightarrow \text{nearly } D\text{-convexlike} \Rightarrow \text{nearly } D\text{-subconvexlike}.$$

In the above relationships the converses are not true in general as illustrated in the following two examples.

Example 2.5. *Let $X = Y = \mathbb{R}^2$ and $D = \mathbb{R}^2_+$. Define $F : X \to 2^Y$ by $F(x) = \mathbb{R}^2 \backslash (-\mathbb{R}^2_+)$. Clearly, $F(X) + D = \mathbb{R}^2 \backslash (-\mathbb{R}^2_+)$ is not a convex set. However, $cl[cone(F(X) + D)]$ is convex. Hence, F is nearly D-subconvexlike on X but not is D-convexlike.*

Example 2.6. *Let $X = \mathbb{R}^2_+$, $Y = \mathbb{R}^2$, $D = \mathbb{R}^2_+$, $F : X \to 2^Y$ be a set-valued mapping and defined by*

$$F(x_1, x_2) = \begin{cases} \{x_1\} \times [1, +\infty) & x_1 \in [0, 1) \\ \{x_1\} \times [0, +\infty) & x_1 \in [1, +\infty) \end{cases}$$

Clearly, F is nearly D-subconvexlike on X since $cl[cone(F(X)+D)]$ is a convex set. However, F is neither a nearly D-convexlike map nor a D-subconvexlike map, because $cl[F(X) + D]$ and $F(X) + intD$ are not convex.

Definition 2.7 (see [18]). *Let $S \subset X$ be a nonempty set and $F : S \to 2^Y$ be a set-valued mapping. F is called ic-D-convexlike on S if $int\big(cone((F(S) + D)\big)$ is nonempty convex and*

$$cone\big(F(S) + D\big) \subset cl\big[int(cone(F(S) + D))\big].$$

Remark 2.8. *It has been proven in [19] that if the ordering cone $D \subset Y$ has nonempty interior then ic-D-convexlikeness is equivalent to nearly D-subconvexlikeness.*

Lemma 2.9 (see [20]). *Suppose that the map $F : S \to 2^Y$ is ic-D-convexlike on S. Then F is also ic-D_1-convexlike on S, where D_1 is a convex cone satisfying $D \subset D_1$.*

The following lemma can be derived directly from Remark 2.8 and Lemma 2.9.

Lemma 2.10. *Let $intD \neq \emptyset$ and $D \subset D_1$. Suppose that the map $F : S \to 2^Y$ is nearly D-subconvexlike on S. Then F is also nearly D_1-subconvexlike on S.*

Since this paper deals with local solutions of set-valued optimization problems, we introduce the following definition, which is local nearly cone-subconvexlike property of a set-valued map.

Definition 2.11. *Let $S \subset X$ be a nonempty set and $F : S \to 2^Y$ is called to be nearly D-subconvexlike on S around $\bar{x} \in S$ if for each $U \in U(\bar{x})$, there exists $\bar{U} \in U(\bar{x})$ such that $\bar{U} \subset U$ and F is a nearly D-subconvexlike on $S \cap \bar{U}$.*

The following lemma is the alternative theorem for nearly D-subconvexlike set-valued map, which is necessary for the results in next section.

Lemma 2.12 (see [17]). *Let $S \subset X$ be a nonempty set and $F : S \to 2^Y$. Suppose that F is nearly D-subconvexlike map on S. Then one and only one of the following conclusions holds:*
(i) $\exists\, x \in S$ such that $F(x) \cap -intD \neq \emptyset$,
(ii) $\exists\, y^* \in D^* \backslash \{0_{Y^*}\}$ such that ${y^*}^T y \geq 0,\ \forall\, y \in F(x),\ \forall x \in S$.

3 Optimality conditions

Let $S \subset X$ be a nonempty set, $F: S \to 2^Y$ and $G: S \to 2^Z$ be two set-valued maps. We consider the following set-valued optimization problem:

$$\text{(SOP)} \quad \begin{cases} \text{minimize} & F(x) \\ \text{subject to} & G(x) \cap (-E) \neq \varnothing, \\ & x \in S. \end{cases}$$

Let $\Omega = \{x \in S : G(x) \cap (-E) \neq \varnothing\}$. We begin with giving a necessary optimality condition in type of separating sets for a local minimizer of (SOP).

Lemma 3.1 (see [21]). *If $\bar{z} \in -E$, $z \in -int(cone(E + \bar{z}))$, $\frac{1}{t_n}(z_n - \bar{z}) \to z$ and $t_n \to 0^+$, then $z_n \in -intE$ for large n.*

Theorem 3.2. *Let $(\bar{x}, \bar{y}) \in graph(F)$ and $\bar{z} \in G(\bar{x}) \cap (-E)$. Suppose that $(\bar{x}, \bar{y})$ is local weak minimizer of (SOP). Then for all $x \in T(S, \bar{x})$, it holds that*

$$Y(F \times G)_+(\bar{x}; (\bar{y}, \bar{z}))(x) \cap -(intD \times int(cone(E + \bar{z}))) = \varnothing. \tag{3.1}$$

Proof. Because $(\bar{x}, \bar{y})$ is a local weak minimizer of (SOP), we get there exists $U \in U(\bar{x})$ such that $(F(\Omega \cap U) - \bar{y}) \cap -\mathrm{int}D = \varnothing$. This follows that

$$(F(\Omega \cap U) + D - \bar{y}) \cap -\mathrm{int}D = \varnothing. \tag{3.2}$$

We proceed by contradiction. Suppose that (3.1) does not hold. Then there exist $x \in T(S, \bar{x})$, $y \in Y$ and $z \in Z$ such that

$$(y, z) \in Y(F \times G)_+(\bar{x}; (\bar{y}, \bar{z}))(x) \cap -(\mathrm{int}D \times \mathrm{int}(\mathrm{cone}(E + \bar{z}))).$$

Hence, it yields from Definition 2.1 that there exist sequences $t_n \to 0^+$, $x_n \to x$, $(f, g) \in \mathrm{CF}((F \times G)_+)$ such that

$$(y, z) = \lim_{(t_n, x_n) \to (0^+, x)} \frac{(f \times g)(\bar{x} + t_n x_n) - (\bar{y}, \bar{z})}{t_n}, \text{ with } (f \times g)(\bar{x}) = (\bar{y}, \bar{z}), \tag{3.3}$$

and

$$(y, z) \in -(\mathrm{int}D \times \mathrm{int}(\mathrm{cone}(E + \bar{z}))). \tag{3.4}$$

From (3.3) and (3.4), for large n we can get

$$f(\bar{x} + t_n v_n) - \bar{y} \in -\mathrm{int}D. \tag{3.5}$$

On the other hand, it follows from Lemma 3.1 that

$$g(\bar{x} + t_n v_n) \in -\mathrm{int}E, \quad \text{for large } n.$$

Hence, for large n, we have $f(\bar{x} + t_n v_n) \in F_+(\bar{x} + t_n v_n)$, $g(\bar{x} + t_n v_n) \in G_+(\bar{x} + t_n v_n) \cap -E$ and $\bar{x} + t_n v_n \in \Omega \cap U$. We derive from (3.5) that

$$f(\bar{x} + t_n v_n) - \bar{y} \in (F_+(\Omega \cap U) - \bar{y}) \cap -\mathrm{int}D.$$

This contradicts to (3.2). □

Theorem 3.3 (Fritz-John type). *Let $(\bar{x}, \bar{y}) \in graph(F)$, $\bar{z} \in G(\bar{x}) \cap (-E)$ and $(\bar{x}, \bar{y})$ be a local weak minimizer of (SOP). Suppose that $((F - \bar{y}) \times G)$ is nearly $D \times E$-subconvexlike map on S around $\bar{x}$. Then there exists $(y^*, z^*) \in (D^* \times E^*) \backslash \{(0, 0)\}$ such that for all $(y, z) \in Y(F \times G)_+(\bar{x}; (\bar{y}, \bar{z}))(x)$, $x \in T(S, \bar{x})$,*

$$y^{*T} y + z^{*T} z \geq 0, \tag{3.6}$$

and

$$z^{*T} \bar{z} = 0. \tag{3.7}$$

Proof. Because $(\bar{x},\bar{y})$ is a locally weak minimizer of (SOP), we get that there exists $U_0 \in U(\bar{x})$ such that

$$\big((F\times G)(S\cap U_0)-(\bar{y},0)\big)\cap -\mathrm{int}(D\times E)=\varnothing. \tag{3.8}$$

Since $\big((F-\bar{y})\times G\big)$ is nearly $D\times E$-subconvexlike map on S around $\bar{x}$, it follows from Definition 2.11 that for above U_0 there exists $\bar{U}\in U(\bar{x})$ such that $\bar{U}\subset U_0$ and $\big((F-\bar{y})\times G\big)$ is nearly $D\times E$-subconvexlike map on $S\cap\bar{U}$. Denote

$$H(x):=\big(F(x)-\bar{y}\big)\times G(x),\quad \forall\, x\in S.$$

Thus, it implies that H is nearly $D\times E$-subconvexlike map on $S\cap\bar{U}$, and we get from (3.8) that

$$H(S\cap\bar{U})\cap -\mathrm{int}(D\times E)=\varnothing.$$

Hence, It yields from Lemma 2.12 that there exists $(y^*,z^*)\in(D^*\times E^*)\backslash\{(0,0)\}$ that such

$$y^{*T}(y-\bar{y})+z^{*T}z\ge 0,\quad \forall\,(y,z)\in F\times G(S\cap\bar{U}). \tag{3.9}$$

Taking $y=\bar{y}$ and $z=\bar{z}$, we obtain $z^{*T}\bar{z}\ge 0$. Noticing that $\bar{z}\in -E$, we have $z^{*T}\bar{z}\le 0$. Hence, we obtain (3.7). Furthermore, it yields from (3.7) and (3.9) that

$$y^{*T}(y-\bar{y})+z^{*T}(z-\bar{z})\ge 0,\quad \forall\,(y,z)\in F\times G(S\cap\bar{U}). \tag{3.10}$$

Now, for $x\in T(S,\bar{x})$ and $(y,z)\in Y(F\times G)_+\big(\bar{x};(\bar{y},\bar{z})\big)(x)$, we shall prove that (3.6) holds. In fact, it follows from the definition of limit set of $(F\times G)_+$ at $\bar{x}$ in the direction $x\in T(S,\bar{x})$ with respect to $(\bar{y},\bar{z})$ that there exist $t_n\to 0^+$, $x_n\to x$, $(f,g)\in \mathrm{CF}\big((F\times G)_+\big)$ and $(y_n,z_n)\to(y,z)$ such that for all n

$$y_n\in\frac{f(\bar{x}+t_nx_n)+D-\bar{y}}{t_n},\quad z_n\in\frac{g(\bar{x}+t_nx_n)+E-\bar{z}}{t_n}. \tag{3.11}$$

For large n, we can get $\bar{x}+t_nx_n\in S\cap\bar{U}$. It yields from (3.11) that there is $(d_n,e_n)\in D\times E$ such that

$$y_n=\frac{f(\bar{x}+t_nx_n)+d_n-\bar{y}}{t_n},\quad z_n=\frac{g(\bar{x}+t_nx_n)+e_n-\bar{z}}{t_n}.$$

For large n, we derive from (3.10) and $y^{*T}d_n+z^{*T}e_n\ge 0$ that

$$y^{*T}\Big(\frac{f(\bar{x}+t_nx_n)+d_n-\bar{y}}{t_n}\Big)+z^{*T}\Big(\frac{g(\bar{x}+t_nx_n)+e_n-\bar{z}}{t_n}\Big)\ge 0,$$

which shows that $y^{*T}y_n+z^{*T}z_n\ge 0$. Taking $n\to+\infty$, we obtain that $y^{*T}y+z^{*T}z\ge 0$. This completes the proof. □

Remark 3.4. *In Theorem 3.3, if $(\bar{x},\bar{y})$ is a weak minimizer of (SOP), we use the nearly $D\times E$-subconvexlike property of $\big((F-\bar{y})\times G\big)$ on S, and in this case the terminology "around $\bar{x}$" is omitted.*

As we see in the proof of Theorem 3.3, the nearly cone-subconvexlike property of $\big((F-\bar{y})\times G\big)$ is essential. Now we present an example, in which $\big((F-\bar{y})\times G\big)$ has nearly cone-subconvexlike property.

Example 3.5. *Let $X=Y=\mathbb{R}^2$, $Z=\mathbb{R}$, $D=\mathbb{R}^2_+$, $E=\mathbb{R}_+$ and S be defined by*

$$S:=\big\{(x,y)\in\mathbb{R}^2:\ x\ge -2,y\ge 0\big\}\cup\big\{(x,y)\in\mathbb{R}^2:\ x\ge 0,y\ge -2\big\}.$$

The set-valued mappings $F:S\to 2^Y$ and $G:S\to 2^Z$ are defined by $F(x,y)=S$ and $G(x,y)=0$ for all $(x,y)\in S$. Taking $\bar{y}=(-2,0)$, we derive that $\big((F-\bar{y})\times G\big)$ is a nearly $D\times E$-subconvexlike map on S since cl[cone$(((F-\bar{y})\times G\big)(S)+D\times E)$] is convex set. However, $(F\times G)$ is not a nearly $D\times E$-subconvexlike map on S because cl[cone$((F\times G)(S)+D\times E)$] is not convex.

It is well known that optimality condition of Kuhn-Tucker type can be derived from that of Fritz-John type by adding a suitable constraint qualification. Next, we present a necessary optimality condition in Kuhn-Tucker type, which is implied from Theorem 3.3 by giving the local generalized Slater constraint qualification for the constraint set-valued map.

Theorem 3.6 (Kuhn-Tucker type). *Let $(\bar{x}, \bar{y}) \in graph(F)$ and $\bar{z} \in G(\bar{x}) \cap (-E)$. Suppose that $(\bar{x}, \bar{y})$ is a local weak minimizer of (SOP) and $\big((F-\bar{y}) \times G\big)$ is nearly $D \times E$-subconvexlike map on S around $\bar{x}$. If for all $U \in U(\bar{x})$, there exists $\hat{x} \in S \cap U$ such that $G(\hat{x}) \cap -intE \neq \emptyset$, then there exists $y^* \in D^*\backslash\{0\}$ and $z^* \in E^*$ such that (3.6) and (3.7) hold for all $(y, z) \in Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)(x)$, $x \in T(S, \bar{x})$.*

Proof. Since the conditions of Theorem 3.3 are fulfilled, we get that there exists $(y^*, z^*) \in (D^* \times E^*)\backslash\{(0, 0)\}$ such that (3.6) and (3.7) hold for all $(y, z) \in Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)(x)$, $x \in T(S, \bar{x})$. Hence, it is only necessary to prove $y^* \neq 0$. From the proof of Theorem 3.3, there is $\bar{U} \in U(\bar{x})$ such that

$$y^{*T}y - y^{*T}\bar{y} + z^{*T}z \geq 0, \quad \forall\ (y, z) \in F \times G(S \cap \bar{U}).$$

If $y^* = 0$, then $z^* \neq 0$ and

$$z^{*T}z \geq 0, \quad \forall\ z \in G(S \cap \bar{U}). \tag{3.12}$$

By the assumption, with this $\bar{U}$, there exists $\hat{x} \in S \cap \bar{U}$ such that $G(\hat{x}) \cap -\text{int}E \neq \emptyset$. This illustrates that there exists $\hat{z} \in G(\hat{x}) \cap -\text{int}E$ such that $z^{*T}\hat{z} < 0$. This is a contradiction to (3.12). □

Under the assumption of nearly cone-subconvexlike property of $Y(F \times G)_+\big(\bar{x}, (\bar{y}, \bar{z})\big)$, we can obtain the next result.

Theorem 3.7. *Let $(\bar{x}, \bar{y}) \in graph(F)$ and $\bar{z} \in G(\bar{x}) \cap (-E)$. Suppose that $(\bar{x}, \bar{y})$ is a local weak minimizer of (SOP) and $Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)$ is nearly $D \times E$-subconvexlike map on $T(S, \bar{x})$. Then there exists $(y^*, z^*) \in (D^* \times E^*)\backslash\{(0, 0)\}$ such that (3.6) and (3.7) hold for all $(y, z) \in Y(F \times G)_+\big(\bar{x}, (\bar{y}, \bar{z})\big)(x)$, $x \in T(S, \bar{x})$.*

Proof. Since $(\bar{x}, \bar{y})$ is a local weak minimizer of (SOP), we derive from Theorem 3.2 that

$$Y(F \times G)_+\big(\bar{x}, (\bar{y}, \bar{z})\big)(x) \cap -(\text{int}D \times \text{int}(\text{cone}(E + \bar{z}))) = \emptyset. \text{ for all } x \in T(S, \bar{x})$$

On the other hand, since $Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)$ is nearly $D \times E$-subconvexlike map on $T(S, \bar{x})$, it yields from Lemma 2.10 and $E \subset \text{cone}(E + \bar{z})$ that $Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)$ is nearly $D \times \text{cone}(E + \bar{z})$-subconvexlike map on $T(S, \bar{x})$. Hence, there exists $(y^*, z^*) \in (D^* \times (\text{cone}(E + \bar{z}))^*)\backslash\{(0, 0)\}$ such that

$$y^{*T}y + z^{*T}z \geq 0, \text{ for all } (y, z) \in Y(F \times G)_+\big(\bar{x}, (\bar{y}, \bar{z})\big)(x),\ x \in T(S, \bar{x}).$$

Since $(\text{cone}(E+\bar{z}))^* \subset E^*$, we get that $z^* \in E^*$ and (3.6) holds. For (3.7), since $\bar{z} \in -E$, it is clearly that $z^{*T}\bar{z} \leq 0$. In addition, we derive from $z^* \in (\text{cone}(E+\bar{z}))^*$ that $z^{*T}(e+\bar{z}) \geq 0$ for all $e \in E$. Taking $e = 0$, we have $z^{*T}\bar{z} \geq 0$. Thus, $z^{*T}\bar{z} = 0$. □

At the end of this section, we present a sufficient optimality condition for a local weak minimizer of (SOP). This result and Theorem 3.6 will be applied to duality in next section.

Theorem 3.8. *Let $(\bar{x}, \bar{y}) \in graph(F)$, $\bar{z} \in G(\bar{x}) \cap (-E)$ and for all $x \in S$*

$$\big((F \times G)(x) - (\bar{y}, \bar{z})\big) \subset Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)(x - \bar{x}). \tag{3.13}$$

If there exists $y^* \in D^*\backslash\{0\}$ and $z^* \in E^*$ such that (3.6) and (3.7) hold, then $(\bar{x}, \bar{y})$ is a local weak minimizer of (SOP).

Proof. Suppose that $(\bar{x}, \bar{y})$ is not a local weak minimizer of (SOP), then for all $U \in U(\bar{x})$ there is $x \in \Omega \cap U$ such that

$$\big(F(x) - \bar{y}\big) \cap -\text{int}D \neq \emptyset.$$

Hence, there are $y \in F(x)$ and $z \in G(x) \cap -E$ such that

$$y - \bar{y} \in -\text{int}D. \tag{3.14}$$

It yields from the condition (3.13) that

$$(y - \bar{y}, z - \bar{z}) \in \big((F \times G)(x) - (\bar{y}, \bar{z})\big) \subset Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)(x - \bar{x}).$$

So, we get from (3.6) that

$$y^{*T}(y - \bar{y}) + z^{*T}(z - \bar{z}) \geq 0.$$

Noticing that $z^{*T}z \leq 0$ and (3.7) holds, one has

$$y^{*T}(y - \bar{y}) \geq -z^{*T}z + z^{*T}\bar{z} = -z^{*T}z \geq 0.$$

This contradicts to (3.14). □

4 Duality Theorems

4.1 Mond-Weir Type Duality

In this subsection, for the primal problem (SOP) we will construct a Mond-Weir type dual problem. Let $(x', y') \in \text{graph}(F)$ and $z' \in G(x') \cap -E$. Considering the following Mond-Weir dual problem (MWD):

$$\text{(MWD)} \quad \begin{cases} \max & y' \\ \text{s. t.} & y^{*T}y + z^{*T}z \geq 0, \ \forall\, (y, z) \in Y(F \times G)_+\big(x'; (y', z')\big)(x), \ \forall\, x \in T(S, \bar{x}), \\ & z^{*T}z' \geq 0, \\ & (y^*, z^*) \in (D^* \backslash \{0_{Y^*}\}) \times E^*. \end{cases}$$

Denote by K_1 the set of all feasible points of (MWD), i.e. the set of points (x', y', z', y^*, z^*) satisfying all the constraints of (MWD). Let $W_1 := \{y' \in F(x') : \ (x', y', z', y^*, z^*) \in K_1\}$.

Definition 4.1. *A feasible point (x', y', z', y^*, z^*) of the problem (MWD) is said to be a weak maximizer of (MWD) if*

$$(W_1 - y') \cap int(D) = \varnothing.$$

Theorem 4.2 (Weak Duality). *Let $(x', y') \in graph(F)$, $z' \in G(x') \cap -E$, and*

$$\big((F \times G)(x) - (y', z')\big) \subset Y(F \times G)_+\big(x'; (y', z')\big)(x - x'), \text{ for all } x \in T(S, x'). \tag{4.1}$$

Suppose that $(\bar{x}, \bar{y})$ is a feasible solution of (SOP) and (x', y', z', y^*, z^*) is a feasible solution of (MWD). Then

$$\bar{y} - y' \notin -int(D). \tag{4.2}$$

Proof. We proceed by contradiction. Assuming that

$$\bar{y} - y' \in -\text{int}(D).$$

We derive from $y^* \in D^* \backslash \{0\}$ that

$$y^{*T}(\bar{y} - y') < 0. \tag{4.3}$$

Since $(\bar{x}, \bar{y})$ is a feasible solution of (SOP), we get from (4.1) that

$$\big((F \times G)(\bar{x}) - (y', z')\big) \subset Y(F \times G)_+\big(x'; (y', z')\big)(\bar{x} - x'), \tag{4.4}$$

and $G(\bar{x}) \cap -E \neq \varnothing$. Taking $\bar{z} \in G(\bar{x}) \cap -E$, we obtain from the constraint condition $z^* \in E^*$ that

$$z^{*T}\bar{z} \leq 0.$$

Then, we derive from $z^{*T}z' \geq 0$ that

$$z^{*T}(\bar{z} - z') \leq 0. \tag{4.5}$$

Furthermore, it yields from the first constraint of (MWD) and (4.4) that

$$y^{*T}(\bar{y} - y') + z^{*T}(\bar{z} - z') \geq 0. \tag{4.6}$$

By (4.5) and (4.6), we get

$$y^{*T}(\bar{y} - y') \geq 0.$$

This is a contradiction to (4.3). □

Theorem 4.3 (Strong duality)**.** *Let $(\bar{x}, \bar{y}) \in graph(F)$ and $\bar{z} \in G(\bar{x}) \cap (-E)$. Suppose that $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP) and $\big((F - \bar{y}) \times G\big)$ is nearly $D \times E$-subconvexlike map on S. If there exists $\hat{x} \in S$ such that $G(\hat{x}) \cap -intE \neq \varnothing$ and*

$$\big((F \times G)(x) - (\bar{y}, \bar{z})\big) \subset Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)(x - \bar{x}), \text{ for all } x \in S, \tag{4.7}$$

then there exist $y^* \in D^*\backslash\{0\}$ and $z^* \in E^*$ such that $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a feasible solution for (MWD). Furthermore, if the Weak Duality Theorem 4.2 between (SOP) and (MWD) holds, then $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a weak maximizer of (MWD).

Proof. It yields from Theorem 3.6 that there are $y^* \in D^*\backslash\{0\}$ and $z^* \in E^*$ such that $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a feasible solution of (MWD). We only need to prove that $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a weak maximizer of (MWD). We proceed by contradiction. If there exists a feasible solution $(x_0, y_0, z_0, y_0^*, z_0^*)$ of (MWD) such that

$$y_0 - \bar{y} \in \text{int}D,$$

that is

$$\bar{y} - y_0 \in -\text{int}D,$$

which contradicts the Weak Duality Theorem 4.2 between (SOP) and (MWD). □

Theorem 4.4 (Converse duality)**.** *Let $(x', y') \in graph(F)$, $z' \in G(x') \cap (-E)$ and (4.1) be satisfied for any $x \in S$. If there exist $y^* \in D^*\backslash\{0\}$ and $z^* \in E^*$ such that (x', y', z', y^*, z^*) is a feasible solution of (MWD), then (x', y') is a weak minimizer of (SOP).*

Proof. It results directly from Theorem 3.8. □

4.2 Wolfe Type Duality

Let us fix a point $d_0 \in D\backslash\{0_Y\}$. Suppose that $(x', y') \in \text{graph}(F)$ and $z' \in G(x') \cap -E$. Considering the following problem (WD), called Wolfe type dual problem of (SOP):

$$(\text{WD}) \quad \begin{cases} \max & y' + z^{*T}z' \cdot d_0 \\ \text{s. t.} & y^{*T}y + z^{*T}z \geq 0, \ \forall\, (y, z) \in Y(F \times G)_+\big(x'; (y', z')\big)(x), \forall\, x \in T(S, \bar{x}), \\ & y^{*T}d_0 = 1, \\ & (y^*, z^*) \in (D^*\backslash\{0_{Y^*}\}) \times E^*. \end{cases}$$

Denote by K_2 the set of all feasible points of (WD), i.e. the set of points (x', y', z', y^*, z^*) satisfying all the constraints of Problem (WD). Let $W_2 = \{y' + z^{*T}z' \cdot d_0 :\ (x', y', z', y^*, z^*) \in K_2\}$.

Definition 4.5. *A feasible point (x', y', z', y^*, z^*) of the problem (WD) is said to be a weak maximizer of (WD) if*

$$\big(W_2 - (y' + z^{*T}z' \cdot d_0)\big) \cap int(D) = \varnothing.$$

Theorem 4.6 (Weak Duality)**.** *Let $(x', y') \in graph(F)$, $z' \in G(x') \cap -E$, and*

$$\big((F \times G)(x) - (y', z')\big) \subset Y(F \times G)_+\big(x'; (y', z')\big)(x - x'), \text{ for all } x \in S. \tag{4.8}$$

Suppose that $(\bar{x}, \bar{y})$ and (x', y', z', y^*, z^*) are feasible points for (SOP) and (WD), respectively. Then

$$\bar{y} - y' - z^{*T}z' \cdot d_0 \notin -intD. \tag{4.9}$$

Proof. Firstly, since $\bar{x} \in S$ and $G(\bar{x}) \cap (-E) \neq \varnothing$, taking $\bar{z} \in G(\bar{x}) \cap (-E)$, we get from (4.8) that

$$\big((F \times G)(\bar{x}) - (y', z')\big) \subset Y(F \times G)_+\big(x'; (y', z')\big)(\bar{x} - x'), \text{ for all } x \in S.$$

Then, it yields from the first constraint condition of problem (WD) that

$$y^{*T}(\bar{y} - y') + z^{*T}(\bar{z} - z') \geq 0. \tag{4.10}$$

Assuming that

$$\bar{y} - (y' + z^{*T}z' \cdot d_0) \in -\text{int}D.$$

Because $z^{*T}\bar{z} \leq 0$, we get that $z^{*T}\bar{z} \cdot d_0 \in -D$ and

$$\bar{y} + z^{*T}\bar{z} \cdot d_0 - (y' + z^{*T}z' \cdot d_0) \in -D - \text{int}D \subset -\text{int}D.$$

Noticing that $y^* \in D^* \backslash \{0_{Y^*}\}$ and $y^{*T}d_0 = 1$, we have

$$y^{*T}(\bar{y} - y') + z^{*T}(\bar{z} - z') < 0,$$

which contradicts (4.10). Thus, we obtain $\bar{y} - y' - z^{*T}z' \cdot d_0 \notin -\text{int}(D)$, as desired. □

Theorem 4.7 (Strong duality)**.** *Let $(\bar{x}, \bar{y}) \in graph(F)$ and $\bar{z} \in G(\bar{x}) \cap (-E)$. Suppose that $(\bar{x}, \bar{y})$ is a weak minimizer of (SOP) and for some $(y^*, z^*) \in (D^* \backslash \{0\}) \times E^*$ with $y^{*T}d_0 = 1$ such that (3.6) and (3.7) are satisfied. If*

$$\big((F \times G)(x) - (\bar{y}, \bar{z})\big) \subset Y(F \times G)_+\big(\bar{x}; (\bar{y}, \bar{z})\big)(x - \bar{x}), \text{ for all } x \in S, \tag{4.11}$$

then $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a feasible solution for (WD). Furthermore, if the Weak Duality Theorem 4.6 between (SOP) and (WD) holds, then $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a weak maximizer of (WD).

Proof. By the given conditions, it is obvious that $(\bar{x}, \bar{y}, \bar{z}, y^*, z^*)$ is a feasible solution for (WD) and

$$z^{*T}\bar{z} = 0.$$

Next, we show that

$$(W_2 - \bar{y} - z^{*T}\bar{z} \cdot d_0) \cap \text{int}D = \varnothing.$$

Let $(x', y', z', y_1^*, z_1^*)$ be a feasible solution for (WD) such that

$$y' + z_1^{*T}z' \cdot d_0 \in (W_2 - \bar{y} - z^{*T}\bar{z} \cdot d_0) \cap \text{int}D.$$

It yields from $z^{*T}\bar{z} = 0$ that

$$y' + z_1^{*T}z' \cdot d_0 \in (W_2 - \bar{y}) \cap \text{int}(D).$$

Therefore,

$$y' + z_1^{*T}z' \cdot d_0 - \bar{y} \in \text{int}(D).$$

This contradicts the Weak Duality Theorem 4.6 between (SOP) and (WD). □

Theorem 4.8 (Converse duality)**.** *Let $(x', y') \in graph(F)$, $z' \in G(x') \cap (-E)$ and (4.8) be satisfied for any $x \in S$. If there exists $y^* \in D^* \backslash \{0\}$ and $z^* \in E^*$ such that (x', y', z', y^*, z^*) is a feasible solution of (MWD) and $z^{*T}z' = 0$, then (x', y') is a weak minimizer of (SOP).*

Proof. It implies directly from Theorem 3.8. □

5 Conclusions

We have established the separating of sets type and Kuhn-Tucker type optimality conditions for a constrained set-valued optimization problem in the sense of weak efficiency. We also present the weak, strong and converse duality theorems for Mond-Weir type and Wofe type dual problems. The generalized convexity assumed in current paper is called nearly cone-subconvexlikeness, which is more weaker than several existed generalized convexities. The derivative we adopted is so called the limit set, which has very nice properties. In recent years, there has been a growing interest to investigate set-valued optimization by utilizing different derivatives. In [22], we used higher-order radial derivatives to establish the optimality conditions and duality theorems for set-valued optimization, because different types of derivatives of a set-valued mapping vary with respect to the existence and properties. This leads to different methods and results by using different derivatives.

Competing interests
The authors declare that they have no competing interests.

Authors's contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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