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Initial layer problem of the Boussinesq system for Rayleigh-Bénard convection with infinite Prandtl number limit

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Abstract: The main purpose of this paper is to study the initial layer problem and the infinite Prandtl number limit of Rayleigh-Bénard convection with an ill prepared initial data. We use the asymptotic expansion methods of singular perturbation theory and the two-time-scale approach to obtain an exact approximating solution and the convergence rates $O(\varepsilon^{\frac{3}{2}})$ and $O(\varepsilon^2)$.

Keywords: Boussinesq system, Rayleigh-Bénard convection, Infinite Prandtl number limit, Initial layers, Asymptotic expansion, Two-time-scale approach

MSC: 35B25,35B40,35K57

1 Introduction

In atmospheric and oceanographic sciences, fluid phenomena with heat transfer has been extensively studied in a large variety of contexts, see, for instance, [1–4]. The thermal convection of a fluid powered by the difference of temperature between two horizontal parallel plates, known as Rayleigh-Bénard convection see [2, 4–10], obeys the rotating Boussinesq system:

$$\begin{aligned}
&\partial_t u + (u \cdot \nabla)u + \nabla p + 2\Omega e_3 \times u = \nu\Delta u + g\alpha e_3 T,\\
&\nabla \cdot u = 0,\\
&\partial_t T + u \cdot \nabla T = \kappa\Delta T,\\
&u|_{z=0,h} = 0,\\
&T|_{z=0} = T_2, \quad T|_{z=h} = T_1,
\end{aligned}$$

where $T_2 > T_1$, u is the vector velocity field of the fluid, p represents the scalar pressure, Ω is the rotation rate, and e_3 is the unit upward vector. As usual, $e_3 := (0, 0, 1)$, ν is the kinematic viscosity, g is the gravity acceleration constant, α is the thermal expansion coefficient, T is the scalar temperature field of the fluid, and κ is the thermal diffusion coefficient. Here we also impose the periodic boundary conditions in the horizontal directions for simplicity.

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This system with rotation is a dynamic model having 3D incompressible Navier-Stokes equations via a buoyancy force proportional to temperature coupled with the heat advection-diffusion of the temperature [5, 10–13].

We can use the Boussinesq approximation and non-dimensionalization to obtain the simplification of Boussinesq system, namely,

$$\varepsilon[\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon] + \nabla p^\varepsilon + \frac{1}{Ek} e_3 \times u^\varepsilon = \Delta u^\varepsilon + Rae_3 T^\varepsilon, (x, y, z, t) \in Q \times (0, S), \tag{1.1}$$

$$\nabla \cdot u^\varepsilon = 0, (x, y, z, t) \in Q \times (0, S), \tag{1.2}$$

$$\partial_t T^\varepsilon + u^\varepsilon \cdot \nabla T^\varepsilon = \varepsilon \Delta T^\varepsilon, (x, y, z, t) \in Q \times (0, S), \tag{1.3}$$

with the boundary and initial conditions:

$$u^\varepsilon|_{z=0,1} = 0, (x, y, t) \in \mathcal{T} \times (0, S), \tag{1.4}$$

$$T^\varepsilon|_{z=0} = 1, \quad T^\varepsilon|_{z=1} = 0, (x, y, t) \in \mathcal{T} \times (0, S), \tag{1.5}$$

$$u^\varepsilon(t = 0) = u_0^\varepsilon(x, y, z), \ T^\varepsilon(t = 0) = T_0^\varepsilon(x, y, z), (x, y, z) \in Q, \tag{1.6}$$

where $Q = \mathcal{T} \times [0, 1]$, $\mathcal{T} = (R^1/2\pi)^2$ is the torus in R^2, $S > 0$, $\varepsilon = (\frac{1}{Pr})^{\frac{1}{2}}$, $Pr = \frac{\nu}{k}$ is the Prandtl number, $Ek = \frac{\nu}{2\Omega h^2}$ is the Ekman number and $Ra = \frac{g\alpha(T_2 - T_1)h^3}{\nu k}$ $(T_2 > T_1)$ is the Rayleigh number.

This system is different from the nondimensional form in [12, 13]. Encouraged by the results on the global existence and the regularities of the suitable weak solution in [12, 13], and the related models, see [2, 5, 7, 10, 14–18], this system has also suitable weak solution by adopting Galerkin approximation method.

By using the asymptotic expansion methods of the singular perturbation theory and the Stokes operator [9, 19–22], we construct an exact approximating solution and study the infinite Prandtl number limit $Pr \to \infty$ (i.e., $\varepsilon \to 0$), of Rayleigh-Bénard convection (1.1)-(1.6).

The main purpose of this paper is to show that the solutions of Boussinesq system for Rayleigh-Bénard convection converge to those of the infinite Prandtl number limit $Pr \to \infty$ (i.e., $\varepsilon \to 0$) model. It is a singular perturbation problem.

The rest of this paper is outlined as follows. The derivation of initial layer is stated in Section 2. The main convergence results are stated in section 3. The approximating solution is constructed and the properties of approximating solution are showed in section 4. The proofs of main convergence results are showed in Section 5. The conclusion is stated in Section 6.

2 The derivation of initial layer

In this section, formally, when $\varepsilon = 0$, the 3-D Boussinesq system (1.1)-(1.4) turn into:

$$\nabla p^{O,0} + \frac{1}{Ek} e_3 \times u^{O,0} = \Delta u^{O,0} + Rae_3 T^{O,0}, \tag{2.1}$$

$$\nabla \cdot u^{O,0} = 0, \tag{2.2}$$

$$\partial_t T^{O,0} + (u^{O,0} \cdot \nabla) T^{O,0} = 0, \tag{2.3}$$

$$u^{O,0}|_{z=0,1} = 0, \tag{2.4}$$

for $(x, y, z, t) \in Q \times (0, S)$, $S > 0$.

Then we impose the initial condition of $T^{O,0}$ as follws:

$$T^{O,0}(t = 0) = T_0^0(x, y, z), (x, y, z) \in Q, \tag{2.5}$$

where $T_0^0(x, y, z)$ is the limit of $T_0^\varepsilon(x, y, z)$ as $\varepsilon \to 0$.

We now turn to derive the boundary conditions of $T^{0,0}$.

Restricting (2.3) to $z = 0, 1$, one gets

$$\partial_t T^{0,0}|_{z=0,1} + (u^{0,0} \cdot \nabla) T^{0,0}|_{z=0,1} = 0. \tag{2.6}$$

Thus, plugging (2.4) into (2.6), we have

$$\partial_t T^{0,0}|_{z=0,1} = 0. \tag{2.7}$$

Due to the compatibility conditions, we deduce from (1.5), (1.6) and (3.1) (see below section 3) that,

$$T_0^0(x, y, z)|_{z=0} = 1,\ T_0^0(x, y, z)|_{z=1} = 0. \tag{2.8}$$

In view of (2.5), (2.7) and (2.8), one gets

$$T^{0,0}|_{z=0} = 1,\ T^{0,0}|_{z=1} = 0. \tag{2.9}$$

By comparing (1.5) with (2.9), a boundary layer of the the scalar temperature does not occur.

On the other hand, restricting (2.1), (2.2) and (2.4) to $t = 0$, one gets

$$\nabla p^{0,0}(t=0) + \frac{1}{Ek} e_3 \times u^{0,0}(t=0) = \Delta u^{0,0}(t=0) + Ra e_3 T^{0,0}(t=0), \tag{2.10}$$

$$\nabla \cdot u^{0,0}(t=0) = 0, \tag{2.11}$$

$$u^{0,0}|_{z=0,1}(t=0) = 0. \tag{2.12}$$

The equation (2.10) is a stationary Stokes equation with rotation, we solve (2.10), (2.11) and (2.12) and know that the value $u^{0,0}(t = 0)$ is determined by the initial data $T^{0,0}(t = 0)$ of the temperature. But $\lim_{\varepsilon\to 0} u_0^\varepsilon (\neq u^{0,0}(t = 0))$ can be given arbitrarily and independently of $T^{0,0}(t = 0)$.

Thus, an initial layer occurs. We observe that the infinite Prandtl number limit of the Boussinesq system only has an initial layer, which is a singular perturbation problem.

Meanwhile, we get the infinite Prandtl number limit of the Boussinesq system as (2.1)-(2.5), namely,

$$\begin{cases} \nabla p^{0,0} + \frac{1}{Ek} e_3 \times u^{0,0} = \Delta u^{0,0} + Ra e_3 T^{0,0}, \\ \nabla \cdot u^{0,0} = 0, \\ \partial_t T^{0,0} + (u^{0,0} \cdot \nabla) T^{0,0} = 0, \\ u^{0,0}|_{z=0,1} = 0, \\ T^{0,0}(t=0) = T_0^0(x, y, z). \end{cases}$$

3 Main convergence results

Assume that the initial data have an expansion up to the 1st order as follows

$$(u^\varepsilon, T^\varepsilon)(t=0) = (u_0^0 + \varepsilon u_0^1 + u_{0E}^\varepsilon, T_0^0 + \varepsilon T_0^1 + T_{0E}^\varepsilon)(x, y, z), \tag{3.1}$$

where $u_0^0, u_0^1, T_0^0, T_0^1$ are all $C^\infty(Q)$ functions, and $u_{0E}^\varepsilon(x, y, z), T_{0E}^\varepsilon(x, y, z) \in C^\infty(Q)$ satisfy

$$\| (u_{0E}^\varepsilon, T_{0E}^\varepsilon)(x, y, z) \|_{L^2(Q)} \leq C\varepsilon^2, \tag{3.2}$$

for some positive constant C independent of ε.

Theorem 3.1. *Assume that (3.1) holds. Also, assume that u_0^0, u_0^1, T_0^0, $T_0^1 \in C^\infty(Q)$ satisfy the suitable compatibility conditions like $u_0^0|_{z=0,1} = 0$, $\nabla \cdot u_0^0 = 0$, $T_0^0|_{z=0} = 1$, $T_0^0|_{z=1} = 0$, $u_0^1|_{z=0,1} = 0$, $\nabla \cdot u_0^1 = 0$, $T_0^1|_{z=0,1} = 0$ etc. Then, as $\varepsilon \to 0$, for any $0 < S < \infty$, we have the following convergence:*

$$\| (u^\varepsilon - u_{app}^\varepsilon, T^\varepsilon - T_{app}^\varepsilon) \|_{L^\infty(0,S;L^2(Q))} \leq C\varepsilon^{\frac{3}{2}}, \tag{3.3}$$

$$\| (u^\varepsilon - u_{app}^\varepsilon, T^\varepsilon - T_{app}^\varepsilon) \|_{L^2(0,S;H^1(Q))} \leq C\varepsilon^{\frac{3}{2}}, \tag{3.4}$$

where $H^1(Q)=W^{1,2}(Q)$, for some positive constant C independent of ε.

Remark 3.2. *The functions u^ε_{app}, T^ε_{app}, p^ε_{app} are given in Section 3.*

Remark 3.3. *By standard method [20, 23, 24], we also formulate any m^{th}, $m = 0, 1, 2, \ldots$, order compatibility conditions.*

Remark 3.4. *Due to the assumption (3.2), we can get the optimal convergence rate by adding assumption*

$$T^0_0(x, y, z) = 1 \ near\ z = 0, \ and\ T^0_0(x, y, z) = 0\ near\ z = 1. \tag{3.5}$$

Then we have the following theorem.

Theorem 3.5. *Let the assumptions of Theorem 3.1 hold. Furthermore, assume that (3.5) hold. Then, as $\varepsilon \to +\infty$, for any $0 < S < \infty$, we arrive the following convergence:*

$$\| (u^\varepsilon - u^\varepsilon_{app}, T^\varepsilon - T^\varepsilon_{app} - \varepsilon^2 T^{I,2}) \|_{L^\infty(0,S;L^2(Q))} \le C\varepsilon^2, \tag{3.6}$$

for some positive constant C independent of ε, where $T^{I,2} = T^{I,2}(x, y, z, \tau)$, $\tau = \frac{t}{\varepsilon}$, is the solution of the following linear problem

$$\begin{aligned} &\partial_\tau T^{I,2} + (u^{O,0}(t = 0)\nabla)T^{I,1} + (u^{I,0}\nabla)T^{O,1}(t = 0) + (u^{I,0}\nabla)T^{I,1} \\ &\quad +(u^{I,0}\nabla)\partial_t T^{O,0}(t = 0)\tau + (u^{I,1}\nabla)T^{O,0}(t = 0) = 0, (x, y, z) \in Q, \tau > 0, \end{aligned} \tag{3.7}$$

$$T^{I,2}(x, y, z, \tau \to \infty) = 0, (x, y, z) \in Q. \tag{3.8}$$

Remark 3.6. *The functions $u^{O,0}$, $T^{I,1}$, $u^{I,0}$, $T^{O,1}$, $T^{O,0}$, $u^{I,1}$ are given in Section 4.*

4 Approximating solutions and the properties

In this section, we carry out the method of matched asymptotic expansions [25, 26] and the two-time-scale approach [2, 26]. We construct the approximating solution including the outer one away from $t = 0$ and the initial layer expansion near $t = 0$. We also derive the corresponding properties of this approximating solution.

Let u^ε, T^ε, p^ε be the global weak solution to (1.1)-(1.6) in the Leray's sense. It is easy to see that

$$\begin{aligned} (u^\varepsilon, p^\varepsilon, T^\varepsilon)(x, y, z, t) \sim \sum_{i=0}^{\infty} \varepsilon^i (u^{O,i}(x, y, z, t) + u^{I,i}(x, y, z, \tau), \\ p^{O,i}(x, y, z, t) + p^{I,i}(x, y, z, \tau), T^{O,i}(x, y, z, t) + T^{I,i}(x, y, z, \tau)), \end{aligned}$$

where ε is the length of the initial layers, $\tau = \frac{t}{\varepsilon}$ is the fast time variable; $(u^{O,i}, p^{O,i}, T^{O,i})(x, y, z, t)$ are the outer functions for the velocity field, pressure and temperature field, respectively, which are independent of ε; $(u^{I,i}, p^{I,i}, T^{I,i})(x, y, z, \tau)$ are the initial layer functions for the velocity field, pressure and temperature field, respectively. The initial layer functions satisfy: $u^{I,i}$, $p^{I,i}$, $T^{I,i}$ decay to zero exponentially, as $\tau \to \infty$.

We seek for the solutions of the system (1.1)-(1.6) having the approximating expansions as follows:

$$\begin{aligned} &(u^\varepsilon_{app}, p^\varepsilon_{app}, T^\varepsilon_{app})(x, y, z, t) = \sum_{i=0}^{1} \varepsilon^i (u^{O,i}(x, y, z, t) + u^{I,i}(x, y, z, \tau), \\ &p^{O,i}(x, y, z, t) + p^{I,i}(x, y, z, \tau), T^{O,i}(x, y, z, t) + T^{I,i}(x, y, z, \tau)). \end{aligned} \tag{4.1}$$

We discuss in detail the construction of the outer and initial layer functions here as

$$(u^\varepsilon_{app}, p^\varepsilon_{app}, T^\varepsilon_{app}) = (u^\varepsilon_{ou}, p^\varepsilon_{ou}, T^\varepsilon_{ou})(x, y, z, t) + (u^\varepsilon_I, p^\varepsilon_I, T^\varepsilon_I)(x, y, z, \tau), \tau = \frac{t}{\varepsilon}. \tag{4.2}$$

4.1 Outer functions

Away from the initial time $t = 0$, the solution to the system (1.1)-(1.5) are expected to be the following expansions

$$(u_{ou}^{\varepsilon}, p_{ou}^{\varepsilon}, T_{ou}^{\varepsilon}) = \sum_{i=0}^{1} \varepsilon^i (u^{O,i}, p^{O,i}, T^{O,i})(x, y, z, t), \tag{4.3}$$

with $(u^{O,i}, p^{O,i}, T^{O,i})(x, y, z, t)$ to be determined later.

Inserting (4.3) into the system (1.1)-(1.5), then by direct calculation and the matched asymptotic expansions, some equations do not hold and need to be added remainders as

$$\varepsilon[\partial_t u_{ou}^{\varepsilon} + (u_{ou}^{\varepsilon} \cdot \nabla) u_{ou}^{\varepsilon}] + \nabla p_{ou}^{\varepsilon} + \frac{1}{Ek} e_3 \times u_{ou}^{\varepsilon} = \Delta u_{ou}^{\varepsilon} + Rae_3 T_{ou}^{\varepsilon} + R_{ou,u}^{\varepsilon}, \tag{4.4}$$

$$\nabla \cdot u_{ou}^{\varepsilon} = 0, \tag{4.5}$$

$$\partial_t T_{ou}^{\varepsilon} + (u_{ou}^{\varepsilon} \cdot \nabla) T_{ou}^{\varepsilon} = \varepsilon \Delta T_{ou}^{\varepsilon} + R_{ou,T}^{\varepsilon}, \tag{4.6}$$

$$u_{ou}^{\varepsilon}|_{z=0,1} = 0, \tag{4.7}$$

$$T_{ou}^{\varepsilon}|_{z=0,1} = 0, \tag{4.8}$$

where the remainders $R_{ou,u}^{\varepsilon}$ and $R_{ou,T}^{\varepsilon}$ satisfy the estimates

$$\| (R_{ou,u}^{\varepsilon}, R_{ou,T}^{\varepsilon}) \|_{L^\infty(0,S;H^s(Q))} \leq C\varepsilon^2, \tag{4.9}$$

for any fixed $S > 0$ and any $s \geq 1$.

Now, we first consider the coefficient of leading order $O(\varepsilon^0)$ in the outer equations. We set the coefficient of $O(\varepsilon^0)$ in the system (4.4)-(4.6) as zero and use the boundary conditions (4.7)-(4.8) and the initial data (2.5).

At leading order, $(u^{O,0}, p^{O,0}, T^{O,0})$ satisfy infinite Prandtl number system (2.1)-(2.5), namely,

$$\begin{cases} \nabla p^{O,0} + \frac{1}{Ek} e_3 \times u^{O,0} = \Delta u^{O,0} + Rae_3 T^{O,0}, \\ \nabla \cdot u^{O,0} = 0, \\ \partial_t T^{O,0} + (u^{O,0} \cdot \nabla) T^{O,0} = 0, \\ u^{O,0}|_{z=0,1} = 0, \\ T^{O,0}(t = 0) = T_0^0(x, y, z). \end{cases}$$

Similarly, we consider the coefficient of the first order $O(\varepsilon^1)$ in the outer equations. We set the coefficient of $O(\varepsilon^1)$ in the system (4.4)-(4.6) as zero and use the boundary conditions (4.7)-(4.8).

At first order, $(u^{O,1}, p^{O,1}, T^{O,1})$ satisfy the following system:

$$\nabla p^{O,1} + \frac{1}{Ek} e_3 \times u^{O,1} = \Delta u^{O,1} + Rae_3 T^{O,1} - \partial_t u^{O,0} - (u^{O,0} \cdot \nabla) u^{O,0}, \tag{4.10}$$

$$\nabla \cdot u^{O,1} = 0, \tag{4.11}$$

$$\partial_t T^{O,1} + u^{O,0} \cdot \nabla T^{O,1} + u^{O,1} \cdot \nabla T^{O,0} = \Delta T^{O,0}, \tag{4.12}$$

$$u^{O,1}|_{z=0,1} = 0, \tag{4.13}$$

$$T^{O,1}(t = 0) = T_0^1(x, y, z) - T^{I,1}(\tau = 0), \tag{4.14}$$

where $T^{I,1}(\tau = 0)$ will be determined later (see below (4.31)). The proof of (4.14) is complete by the initial data (3.1) (see below (4.33)).

The infinite Prandtl number rotating system (2.1)-(2.5) has stationary Stokes equations via a buoyancy force proportional to temperature coupled with heat advection of the temperature. The linearized infinite Prandtl number type rotating system (4.10)-(4.14) has Stokes equations via a buoyancy force proportional to temperature coupled with linearized heat advection of the temperature. Therefore, the existence of the smooth solutions is the same as the incompressible Stokes equations. We find that:

Proposition 4.1. *Assume that T_0^0, $T^{O,1}(t=0) \in C^\infty(Q)$ satisfy the suitable compatibility conditions like $T_0^0|_{z=0} = 1$, $T_0^0|_{z=1} = 0$, $T^{O,1}(t=0)|_{z=0,1} = 0$ etc. Then there exists a unique and global $C^\infty(Q \times [0,+\infty))$ smooth solution to the system* (2.1)-(2.5) *and* (4.10)-(4.14)*, respectively.*

Proof. The proof of Proposition 4.1 is elementary and we omit it. □

Now we turn to the construction of the initial layer functions.

4.2 Initial layer functions

Near $t = 0$, we will approximate the solution uniformly up to $t = 0$ by the two-scale expansions (4.2)

$$(u^\varepsilon_{app}, p^\varepsilon_{app}, T^\varepsilon_{app}) = (u^\varepsilon_{ou}, p^\varepsilon_{ou}, T^\varepsilon_{ou})(x,y,z,t) + (u^\varepsilon_I, p^\varepsilon_I, T^\varepsilon_I)(x,y,z,\tau), \tau = \frac{t}{\varepsilon},$$

where $(u^\varepsilon_{ou}, p^\varepsilon_{ou}, T^\varepsilon_{ou})$ is given by (4.4)-(4.8) and

$$(u^\varepsilon_I, p^\varepsilon_I, T^\varepsilon_I) = \sum_{i=0}^{1} \varepsilon^i (u^{I,i}, p^{I,i}, T^{I,i})(x,y,z,\tau), (u^{I,i}, p^{I,i}, T^{I,i})(\tau \to +\infty) = 0. \tag{4.15}$$

Inserting (4.2) into the system (1.1)-(1.6), due to the matched asymptotic expansions, some equations do not hold by direct calculation and need to be added remainders as

$$\begin{aligned}
&\varepsilon[\partial_t u^\varepsilon_{app} + (u^\varepsilon_{app}\cdot\nabla)u^\varepsilon_{app}] + \nabla p^\varepsilon_{app} + \frac{1}{Ek} e_3 \times u^\varepsilon_{app} - \Delta u^\varepsilon_{app} - Rae_3 T^\varepsilon_{app} \\
&= \varepsilon[\partial_t(u^\varepsilon_{ou} + u^\varepsilon_I) + ((u^\varepsilon_{ou} + u^\varepsilon_I)\cdot\nabla)(u^\varepsilon_{ou} + u^\varepsilon_I)] + \nabla(p^\varepsilon_{ou} + P^\varepsilon_I) + \frac{1}{Ek} e_3 \times (u^\varepsilon_{ou} + u^\varepsilon_I) \\
&\quad -\Delta(u^\varepsilon_{ou} + u^\varepsilon_I) - Rae_3(T^\varepsilon_{ou} + T^\varepsilon_I) \\
&= R^\varepsilon_{ou,u} + \varepsilon[\partial_t u^\varepsilon_I + (u^\varepsilon_{ou}\cdot\nabla)u^\varepsilon_I + u^\varepsilon_I\cdot\nabla(u^\varepsilon_{ou} + u^\varepsilon_I)] + \nabla p^\varepsilon_I + \frac{1}{Ek} e_3 \times u^\varepsilon_I - \Delta u^\varepsilon_I - Rae_3 T^\varepsilon_I,
\end{aligned} \tag{4.16}$$

$$\nabla\cdot u^\varepsilon_{app} = \nabla\cdot(u^\varepsilon_{ou} + u^\varepsilon_I) = \nabla\cdot u^\varepsilon_I = \sum_{i=0}^{1}\varepsilon^i \nabla\cdot u^{I,i}, \tag{4.17}$$

$$\begin{aligned}
&\partial_t T^\varepsilon_{app} + (u^\varepsilon_{app}\cdot\nabla)T^\varepsilon_{app} - \varepsilon\Delta T^\varepsilon_{app} \\
&= \partial_t(T^\varepsilon_{ou} + T^\varepsilon_I) + ((u^\varepsilon_{ou} + u^\varepsilon_I)\cdot\nabla)(T^\varepsilon_{ou} + T^\varepsilon_I) - \varepsilon\Delta(T^\varepsilon_{ou} + T^\varepsilon_I) \\
&= R^\varepsilon_{ou,T} + \partial_t T^\varepsilon_I + u^\varepsilon_{ou}\cdot\nabla T^\varepsilon_I + (u^\varepsilon_I\cdot\nabla)(T^\varepsilon_{ou} + T^\varepsilon_I) - \varepsilon\Delta T^\varepsilon_I,
\end{aligned} \tag{4.18}$$

$$u^\varepsilon_{app}|_{z=0,1} = (u^\varepsilon_{ou} + u^\varepsilon_I)|_{z=0,1} = u^\varepsilon_I|_{z=0,1} = \sum_{i=0}^{1}\varepsilon^i u^{I,i}|_{z=0,1}, \tag{4.19}$$

$$T^\varepsilon_{app}|_{z=0} = (T^\varepsilon_{ou} + T^\varepsilon_I)|_{z=0} = 1 + \sum_{i=0}^{1}\varepsilon^i T^{I,i}|_{z=0}, \tag{4.20}$$

$$T^\varepsilon_{app}|_{z=1} = (T^\varepsilon_{ou} + T^\varepsilon_I)|_{z=1} = \sum_{i=0}^{1}\varepsilon^i T^{I,i}|_{z=1}, \tag{4.21}$$

$$u^\varepsilon_{app}(t=0) = \sum_{i=0}^{1}\varepsilon^i (u^{O,i}(t=0) + u^{I,i}(\tau=0)), \tag{4.22}$$

$$T^\varepsilon_{app}(t=0) = \sum_{i=0}^{1}\varepsilon^i (T^{O,i}(t=0) + T^{I,i}(\tau=0)). \tag{4.23}$$

We use the Taylor series expansion

$$u^{O,i}(x,y,z,t) = u^{O,i}(x,y,z,\varepsilon\tau) = u^{O,i}(x,y,z,0) + \varepsilon\partial_t u^{O,i}(t=0)\tau + \cdots.$$

$$T^{O,i}(x,y,z,t) = T^{O,i}(x,y,z,\varepsilon\tau) = T^{O,i}(x,y,z,0) + \varepsilon\partial_t T^{O,i}(t=0)\tau + \cdots.$$

Now we compare the coefficients of $O(\varepsilon^i)$, $i \geq 0$ in the resulting system and derive the systems satisfying the initial layer functions.

First taking the coefficient of $O(\varepsilon^{-1})$ in (4.18) as zero, we find $\partial_\tau T^{I,0} = 0$, which, together with $T^{I,0}(\tau \to +\infty) = 0$ in (4.15), one yields

$$T^{I,0}(x,y,z,\tau) = 0, \tag{4.24}$$

this show that the temperature has no zero order initial layer.

Then, setting the coefficients of $O(\varepsilon^0)$ in (4.16)-(4.18) as zero, using (4.24) and requiring that the approximating solution satisfies the boundary and initial conditions (4.19) and (4.22), the initial layer functions $(u^{I,0}, p^{I,0}, T^{I,0})$ satisfy the system as

$$\partial_\tau u^{I,0} + \frac{1}{Ek} e_3 \times u^{I,0} + \nabla p^{I,0} = \Delta u^{I,0}, \tag{4.25}$$

$$\nabla \cdot u^{I,0} = 0, \tag{4.26}$$

$$\partial_\tau T^{I,1} + u^{I,0} \cdot \nabla(T^{O,0}(t=0)) = 0, \tag{4.27}$$

$$u^{I,0}|_{z=0,1} = 0, \tag{4.28}$$

$$u^{I,0}(\tau = 0) = u_0^0 - u^{O,0}(t=0), (u^{I,0}, T^{I,1})(\tau \to +\infty) = 0. \tag{4.29}$$

Now, we turn to derive the initial and boundary conditions of $T^{I,1}$.

Using (4.27) and the decay condition $T^{I,1}(\tau \to +\infty) = 0$ in (4.15), one gets

$$T^{I,1} = -\int_\tau^\infty [u^{I,0} \cdot \nabla(T^{O,0}(t=0))](s)ds. \tag{4.30}$$

In fact, we restrict (4.30) to $\tau = 0$, and replace the right term of result by $\overline{T}^{I,1}$, that is,

$$T^{I,1}(\tau = 0) = \overline{T}^{I,1}. \tag{4.31}$$

We restrict (4.30) to $z = 0, 1$ and by using the boundary condition (4.28) we get

$$T^{I,1}|_{z=0,1} = 0. \tag{4.32}$$

Moreover, we deduce the initial condition (3.15) from (3.1) and (4.23) that

$$T_0^1 = T^{O,1}(t=0) + T^{I,1}(\tau = 0). \tag{4.33}$$

So, $(u^{I,0}, p^{I,0}, T^{I,0})$ satisfy the system (4.24)-(4.26), (4.28) and (4.29) as

$$\begin{cases} T^{I,0}(x,y,z,\tau) = 0, \\ \partial_\tau u^{I,0} + \frac{1}{Ek} e_3 \times u^{I,0} + \nabla p^{I,0} = \Delta u^{I,0}, \\ \nabla \cdot u^{I,0} = 0, \\ u^{I,0}|_{z=0,1} = 0, \\ u^{I,0}(\tau = 0) = u_0^0 - u^{O,0}(t=0), u^{I,0}(\tau \to +\infty) = 0. \end{cases}$$

Similarly, setting the coefficients of $O(\varepsilon^1)$ in (4.16), (4.17), (4.19) and (4.22) as zero, and using the above method, we have

$$\partial_\tau u^{I,1} + \frac{1}{Ek} e_3 \times u^{I,1} + \nabla p^{I,1}$$
$$= \Delta u^{I,1} + Rae_3 T^{I,1} - (u^{I,0} \cdot \nabla)u^{O,0}(t=0) - (u^{O,0}(t=0) \cdot \nabla)u^{I,0} - (u^{I,0} \cdot \nabla)u^{I,0}, \tag{4.34}$$

$$\nabla \cdot u^{I,1} = 0, \tag{4.35}$$

$$u^{I,1}|_{z=0,1} = 0, \tag{4.36}$$

$$u^{I,1}(\tau=0)=u_0^1-u^{O,1}(t=0),\, u^{I,1}(\tau\to+\infty)=0. \tag{4.37}$$

So, $(u^{I,1}, p^{I,1}, T^{I,1})$ satisfy the system (4.27) and (4.34)-(4.37) as

$$\begin{cases}\partial_\tau T^{I,1}+u^{I,0}\cdot\nabla(T^{O,0}(t=0))=0,\\ \partial_\tau u^{I,1}+\frac{1}{Ek}e_3\times u^{I,1}+\nabla p^{I,1}\\ =\Delta u^{I,1}+Rae_3T^{I,1}-(u^{I,0}\cdot\nabla)u^{O,0}(t=0)-(u^{O,0}(t=0)\cdot\nabla)u^{I,0}-(u^{I,0}\cdot\nabla)u^{I,0},\\ \nabla\cdot u^{I,1}=0,\\ u^{I,1}\,|_{z=0,1}=0,\\ u^{I,1}(\tau=0)=u_0^1-u^{O,1}(t=0),\,(u^{I,1},T^{I,1})(\tau\to+\infty)=0.\end{cases}$$

Now we turn to state the exponentially decay properties of the initial layer functions.

Proposition 4.2. *Let the assumptions of Theorem 3.1 hold. Then there exist a unique and smooth solution* $(u^{I,0}, p^{I,0})$ *to the system* (4.24)-(4.26), (4.28) *and* (4.29) *and a unique and smooth solution* $(u^{I,1}, p^{I,1}, T^{I,1})$ *to the system* (4.27) *and* (4.34)-(4.37) *satisfying the exponential decay to zero as* $\tau\to\infty$ *, namely,*

$$\|\,(u^{I,0}, T^{I,1})(\cdot,\tau)\,\|_{H^s(Q)}\le Ce^{-\beta\tau}, \tag{4.38}$$

for some positive constants C, β *and any* $s\ge 1$.

Proof. The proof of Proposition 4.2 is elementary, see [18]. □

We summarize the approximating solution in the next subsection.

4.3 Approximating solution

With outer functions and initial layer functions defined in section 4.1 and 4.2, one gets

$$\begin{aligned}&\varepsilon[\partial_t u^\varepsilon_{app}+(u^\varepsilon_{app}\cdot\nabla)u^\varepsilon_{app}]+\nabla p^\varepsilon_{app}+\frac{1}{Ek}e_3\times u^\varepsilon_{app}-\Delta u^\varepsilon_{app}-Rae_3T^\varepsilon_{app}\\ &=R^\varepsilon_{ou,u}+(\partial_\tau u^{I,0}+\nabla p^{I,0}+\frac{1}{Ek}e_3\times u^{I,0}-\Delta u^{I,0}-Rae_3T^{I,0})\\ &\quad+\varepsilon(\partial_\tau u^{I,1}+\nabla p^{I,1}+\frac{1}{Ek}e_3\times u^{I,1}-\Delta u^{I,1}-Rae_3T^{I,1}\\ &\quad+(u^{I,0}\cdot\nabla)u^{O,0}(t=0)+(u^{O,0}(t=0)\cdot\nabla)u^{I,0}+(u^{I,0}\cdot\nabla)u^{I,0})+R^\varepsilon_{I,u},\end{aligned} \tag{4.39}$$

$$\begin{aligned}&\partial_t T^\varepsilon_{app}+(u^\varepsilon_{app}\cdot\nabla)T^\varepsilon_{app}-\varepsilon\Delta T^\varepsilon_{app}\\ &=R^\varepsilon_{ou,T}+\varepsilon^{-1}\partial_\tau T^{I,0}+(\partial_\tau T^{I,1}+u^{I,0}\cdot\nabla(T^{O,0}(t=0))\\ &\quad+(u^{O,0}(t=0)+u^{I,0})\cdot\nabla T^{I,0}+R^\varepsilon_{I,T},\end{aligned} \tag{4.40}$$

where the remainders $R^\varepsilon_{I,u}$ and $R^\varepsilon_{I,T}$, caused by the initial layer, are given exactly by

$$\begin{aligned}R^\varepsilon_{I,u}&=\varepsilon^2[\tau(u^{I,0}\cdot\nabla)\partial_t u^{O,0}(\theta_1 t)+\tau\partial_t u^{O,0}(\theta_2 t)\cdot\nabla u^{I,0}+(u^{O,0}(t=0)\cdot\nabla)u^{I,1}\\ &\quad+(u^{O,1}\cdot\nabla)u^{I,0}+u^{I,0}\cdot\nabla(u^{O,1}+u^{I,1})+u^{I,1}\cdot\nabla(u^{O,0}(t=0)+u^{I,0})]\\ &\quad+\varepsilon^3(\tau(u^{I,1}\cdot\nabla)\partial_t u^{O,0}(\theta_3 t)+\tau\partial_t u^{O,0}(\theta_4 t)\cdot\nabla u^{I,1}\\ &\quad+(u^{O,1}\cdot\nabla)u^{I,1}+u^{I,1}\cdot\nabla(u^{O,1}+u^{I,1}),\, 0<\theta_i<1,\, i=1,2,3,4,\end{aligned} \tag{4.41}$$

$$\begin{aligned}R^\varepsilon_{I,T}&=\varepsilon[\tau(u^{I,0}\cdot\nabla)\partial_t T^{O,0}(t=0)+u^{O,0}(t=0)\cdot\nabla T^{I,1}\\ &\quad+u^{I,0}\cdot\nabla(T^{O,1}(t=0)+T^{I,1})+u^{I,1}\cdot\nabla T^{O,0}(t=0)]\\ &\quad+\varepsilon^2[\frac{1}{2}\tau^2(u^{I,0}\cdot\nabla)\partial_{tt}T^{O,0}(\theta_5 t)+\tau(u^{I,1}\cdot\nabla)\partial_t T^{O,0}(\theta_6 t)\end{aligned}$$

$$+\tau(u^{I,0}\cdot\nabla)\partial_t T^{O,1}(\theta_7 t)+\tau\partial_t u^{O,0}(\theta_8 t)\cdot\nabla T^{I,1}$$
$$+(u^{O,1}\cdot\nabla)T^{I,1}+u^{I,1}\cdot\nabla(T^{O,1}+T^{I,1})-\Delta T^{I,1}],\ 0<\theta_i<1,\ i=5,6,7,8. \tag{4.42}$$

Hence, the previous computations show that $(u^\varepsilon_{app}, p^\varepsilon_{app}, T^\varepsilon_{app})$ solves the following initial-boundary problem:

$$\varepsilon[\partial_t u^\varepsilon_{app}+(u^\varepsilon_{app}\cdot\nabla)u^\varepsilon_{app}]+\nabla p^\varepsilon_{app}+\frac{1}{Ek}e_3\times u^\varepsilon_{app}=\Delta u^\varepsilon_{app}+Rae_3T^\varepsilon_{app}+R^\varepsilon_{ou,u}+R^\varepsilon_{I,u}, \tag{4.43}$$

$$\nabla\cdot u^\varepsilon_{app}=0, \tag{4.44}$$

$$\partial_t T^\varepsilon_{app}+(u^\varepsilon_{app}\cdot\nabla)T^\varepsilon_{app}=\Delta T^\varepsilon_{app}+R^\varepsilon_{ou,T}+R^\varepsilon_{I,T}, \tag{4.45}$$

$$u^\varepsilon_{app}|_{z=0,1}=0, \tag{4.46}$$

$$T^\varepsilon_{app}|_{z=0}=1, \tag{4.47}$$

$$T^\varepsilon_{app}|_{z=1}=0, \tag{4.48}$$

$$(u^\varepsilon_{app}, T^\varepsilon_{app})(t=0)=(u^0_0+\varepsilon u^1_0, T^0_0+\varepsilon T^1_0), \tag{4.49}$$

where the remainders $R^\varepsilon_{ou,u}$, $R^\varepsilon_{ou,T}$ satisfy the estimate (4.9) and $R^\varepsilon_{I,u}$, $R^\varepsilon_{I,T}$ defined by (4.41) and (4.42) respectively satisfy the the following estimate

$$\| R^\varepsilon_{I,u}(t)\|_{L^\infty(Q)}\leq C\varepsilon^2(\tau+1)e^{-\beta\tau},\ \| R^\varepsilon_{I,T}(t)\|_{L^\infty(Q)}\leq C\varepsilon(\tau+1)e^{-\beta\tau}, \tag{4.50}$$

for some positive constant C and β and for any $t\in[0,S]$ and any fixed $S>0$. The estimate (4.50) can easily be obtained by the definitions of $R^\varepsilon_{I,u}$, $R^\varepsilon_{I,T}$ and the decay estimate (4.38).

We now turn to the proofs of convergence results.

5 The proofs of main convergence results

Without loss of generality, we denote C by a positive generic constant independent of ε. Noting that C may depend upon S for any fixed $S>0$. Let $t\in[0,S]$. We use the standard L^2-energy method to prove Theorems 3.1 and 3.5.

5.1 The proof of Theorem 3.1

In this subsection we assume that (3.1) holds and define error functions

$$(u^\varepsilon_E, p^\varepsilon_E, T^\varepsilon_E)=(u^\varepsilon-u^\varepsilon_{app}, p^\varepsilon-p^\varepsilon_{app}, T^\varepsilon-T^\varepsilon_{app}).$$

Step 1. Combining (1.1)-(1.6) and (4.43)-(4.49), $(u^\varepsilon_E, p^\varepsilon_E, T^\varepsilon_E)$ satisfy the following equations

$$\varepsilon[\partial_t u^\varepsilon_E+(u^\varepsilon_{app}\cdot\nabla)u^\varepsilon_E+(u^\varepsilon_E\cdot\nabla)(u^\varepsilon_{app}+u^\varepsilon_E)]+\nabla p^\varepsilon_E+\frac{1}{Ek}e_3\times u^\varepsilon_E$$
$$=\Delta u^\varepsilon_E+Rae_3T^\varepsilon_E-R^\varepsilon_{ou,u}-R^\varepsilon_{I,u}, \tag{5.1}$$

$$\nabla\cdot u^\varepsilon_E=0, \tag{5.2}$$

$$\partial_t T^\varepsilon_E+(u^\varepsilon_{app}\cdot\nabla)T^\varepsilon_E+(u^\varepsilon_E\cdot\nabla)(T^\varepsilon_{app}+T^\varepsilon_E)=\varepsilon\Delta T^\varepsilon_E-R^\varepsilon_{ou,T}-R^\varepsilon_{I,T}, \tag{5.3}$$

$$u^\varepsilon_E|_{z=0,1}=0, \tag{5.4}$$

$$T^\varepsilon_E|_{z=0,1}=0, \tag{5.5}$$

$$u^\varepsilon_E(t=0)=u^\varepsilon_{0E}(x,y,z),\ T^\varepsilon_E(t=0)=T^\varepsilon_{0E}(x,y,z). \tag{5.6}$$

Step 2. Taking the L^2-inner product of temperature error equation (5.3) with T_E^ε and integrating over Q with respect to (x, y, z) yield

$$\frac{1}{2}\frac{d}{dt} \| T_E^\varepsilon \|_{L^2(Q)}^2 = \int_Q \varepsilon \Delta T_E^\varepsilon T_E^\varepsilon dxdydz - \int_Q (R_{ou,T}^\varepsilon + R_{I,T}^\varepsilon) T_E^\varepsilon dxdydz$$
$$- \int_Q \left(u_{app}^\varepsilon \cdot \nabla\right) T_E^\varepsilon T_E^\varepsilon dxdydz - \int_Q \left(u_E^\varepsilon \cdot \nabla\right) (T_{app}^\varepsilon + T_E^\varepsilon) T_E^\varepsilon dxdydz$$
$$=: I_1 + I_2 + I_3 + I_4. \tag{5.7}$$

We first estimate I_1 by Green's first formula and the boundary condition (5.5). We have that

$$I_1 = \oint_\Gamma \oint \varepsilon T_E^\varepsilon \frac{\partial T_E^\varepsilon}{\partial n} dS - \varepsilon \int_Q |\nabla T_E^\varepsilon|^2 dxdydz$$
$$= -\varepsilon \int_Q |\nabla T_E^\varepsilon|^2 dxdydz, \tag{5.8}$$

where Γ is the boundary surface.

Next, we estimate the integral term I_2 by virtue of Hölder inequality, Young inequality and the estimates (4.9), (4.50). We get that

$$|I_2| \le \eta_1 \| T_E^\varepsilon \|_{L^2(Q)}^2 + C(\eta_1) \| R_{ou,T}^\varepsilon + R_{I,T}^\varepsilon \|_{L^2(Q)}^2$$
$$\le \eta_1 \| T_E^\varepsilon \|_{L^2(Q)}^2 + C(\eta_1)(C\varepsilon^4 + C\varepsilon^2(\tau + 1)^2 e^{-2\beta\tau}). \tag{5.9}$$

Here η_1 is a small constant, $C(\eta_1) > 0$ is a constant, independent of ε. We have used the estimate

$$\| R_{ou,T}^\varepsilon + R_{I,T}^\varepsilon \|_{L^2(Q)} \le C\varepsilon^2 + C\varepsilon(\tau + 1)e^{-\beta\tau}.$$

Then, we estimate the integral term I_3 by divergence formula, divergence theorem, (4.44) and the boundary condition (4.46), (5.5) as follows

$$I_3 = -\int_Q u_{app}^\varepsilon \cdot \nabla \left(\frac{(T_E^\varepsilon)^2}{2} \right) dxdydz$$
$$= -\int_Q \nabla \cdot \left(u_{app}^\varepsilon \frac{(T_E^\varepsilon)^2}{2} \right) dxdydz + \int_Q \nabla \cdot u_{app}^\varepsilon \frac{(T_E^\varepsilon)^2}{2} dxdydz = 0. \tag{5.10}$$

Similarly, we estimate the last integral term I_4 by using same method in estimating I_3. We obtain that

$$I_4 = -\int_Q \left(u_E^\varepsilon \cdot \nabla\right) T_{app}^\varepsilon T_E^\varepsilon dxdydz - \int_Q (u_E^\varepsilon \cdot \nabla) T_E^\varepsilon T_E^\varepsilon dxdydz$$
$$= -\int_Q (u_E^\varepsilon \cdot \nabla) T_{app}^\varepsilon T_E^\varepsilon dxdydz$$
$$\le |-\int_Q (u_E^\varepsilon \cdot \nabla) T_{app}^\varepsilon T_E^\varepsilon dxdydz|$$
$$\le C(\eta_2) \| \nabla T_{app}^\varepsilon \|_{L^\infty(Q)}^2 \| T_E^\varepsilon \|_{L^2(Q)}^2 + \eta_2 \| u_E^\varepsilon \|_{L^2(Q)}^2, \tag{5.11}$$

where we have used Hölder inequality, Young inequality, the properties of the approximating solution, (5.2) and (5.5). η_2 is a small constant, $C(\eta_2) > 0$ is a constant, independent of ε.

Finally, inserting the estimates derived in (5.8)-(5.11) into (5.7) leads to the inequality

$$\frac{1}{2}\frac{d}{dt} \| T_E^\varepsilon \|_{L^2(Q)}^2 + \varepsilon \| \nabla T_E^\varepsilon \|_{L^2(Q)}^2$$
$$\le C(\eta_1)(C\varepsilon^4 + C\varepsilon^2(\tau + 1)^2 e^{-2\beta\tau}) + \eta_1 \| T_E^\varepsilon \|_{L^2(Q)}^2$$

$$+C(\eta_2) \parallel \nabla T^{\varepsilon}_{app} \parallel^2_{L^\infty(Q)} \parallel T^{\varepsilon}_E \parallel^2_{L^2(Q)} + \eta_2 \parallel u^{\varepsilon}_E \parallel^2_{L^2(Q)} .$$

With the help of Poincaré inequality and taking η_1 to be sufficiently small but independent of ε, one gets

$$\begin{aligned}&\frac{d}{dt} \parallel T^{\varepsilon}_E \parallel^2_{L^2(Q)} + 2\varepsilon \parallel \nabla T^{\varepsilon}_E \parallel^2_{L^2(Q)}\\ &\leq 2 \parallel \nabla T^{\varepsilon}_{app} \parallel^2_{L^\infty(Q)} C(\eta_2) \parallel T^{\varepsilon}_E \parallel^2_{L^2(Q)} + 2\eta_2 \parallel u^{\varepsilon}_E \parallel^2_{L^2(Q)}\\ &+2C(\eta_1)(C\varepsilon^4 + C\varepsilon^2(\tau+1)^2 e^{-2\beta\tau}). \end{aligned} \tag{5.12}$$

Step 3. Similarly, testing the velocity equation (5.1) by u^{ε}_E and integrating over Q with respect to (x, y, z).

$$\begin{aligned}&\int_Q (\varepsilon[\partial_t u^{\varepsilon}_E + (u^{\varepsilon}_{app} \cdot \nabla) u^{\varepsilon}_E + (u^{\varepsilon}_E \cdot \nabla)(u^{\varepsilon}_{app} + u^{\varepsilon}_E)] + \nabla p^{\varepsilon}_E + \frac{1}{Ek} e_3 \times u^{\varepsilon}_E) u^{\varepsilon}_E dxdydz\\ &= \int_Q (\Delta u^{\varepsilon}_E + Ra e_3 T^{\varepsilon}_E - R^{\varepsilon}_{ou,u} - R^{\varepsilon}_{I,u}) u^{\varepsilon}_E dxdydz. \end{aligned} \tag{5.13}$$

First, we deal with the left-hand side terms of (5.13) by divergence formula, divergence theorem, the approximating solution's property (4.38), (4.44), (5.2) and the boundary condition (4.46), (5.4).

$$\int_Q \varepsilon \partial_t u^{\varepsilon}_E u^{\varepsilon}_E dxdydz = \frac{\varepsilon}{2}\frac{d}{dt} \parallel u^{\varepsilon}_E \parallel^2_{L^2(Q)},$$

$$\begin{aligned}&\int_Q \varepsilon (u^{\varepsilon}_{app} \cdot \nabla) u^{\varepsilon}_E u^{\varepsilon}_E dxdydz\\ &= \int_Q \varepsilon \nabla \cdot \left(u^{\varepsilon}_{app} \frac{(u^{\varepsilon}_E)^2}{2} \right) dxdydz - \int_Q \varepsilon \nabla \cdot u^{\varepsilon}_{app} \frac{(u^{\varepsilon}_E)^2}{2} dxdydz = 0,\end{aligned}$$

$$\begin{aligned}&\int_Q \varepsilon (u^{\varepsilon}_E \cdot \nabla)(u^{\varepsilon}_{app} + u^{\varepsilon}_E) u^{\varepsilon}_E dxdydz\\ &= \int_Q \varepsilon (u^{\varepsilon}_E \cdot \nabla) u^{\varepsilon}_{app} u^{\varepsilon}_E dxdydz + \int_Q \varepsilon \nabla \cdot \left(u^{\varepsilon}_E \frac{(u^{\varepsilon}_E)^2}{2} \right) dxdydz - \int_Q \varepsilon \nabla \cdot u^{\varepsilon}_E \frac{(u^{\varepsilon}_E)^2}{2} dxdydz\\ &= \int_Q \varepsilon \left(u^{\varepsilon}_E \cdot \nabla \right) u^{\varepsilon}_{app} u^{\varepsilon}_E dxdydz\\ &\leq \left| \int_Q \varepsilon \left(u^{\varepsilon}_E \cdot \nabla \right) u^{\varepsilon}_{app} u^{\varepsilon}_E dxdydz \right|\\ &\leq \varepsilon \parallel \nabla u^{\varepsilon}_{app} \parallel_{L^\infty(Q)} \parallel u^{\varepsilon}_E \parallel^2_{L^2(Q)},\\ &\int_Q \nabla p^{\varepsilon}_E u^{\varepsilon}_E dxdydz\\ &= \int_Q \nabla \cdot (p^{\varepsilon}_E u^{\varepsilon}_E) dxdydz - \int_Q \nabla \cdot u^{\varepsilon}_E p^{\varepsilon}_E dxdydz = 0,\\ &\int_Q \frac{1}{Ek} e_3 \times u^{\varepsilon}_E u^{\varepsilon}_E dxdydz\\ &= \int_Q \frac{1}{Ek} (-u^{\varepsilon}_{2E}, u^{\varepsilon}_{1E}, 0)(u^{\varepsilon}_{1E}, u^{\varepsilon}_{2E}, u^{\varepsilon}_{3E})^T dxdydz = 0.\end{aligned}$$

Next, we deal with the right-hand side terms of (5.13) as follows:

$$\int_Q \Delta u^{\varepsilon}_E u^{\varepsilon}_E dxdydz$$

$$
\begin{aligned}
&= \int_Q \sum_{i=1}^{3} (\partial_{xx} + \partial_{yy} + \partial_{zz}) u_{iE}^{\varepsilon} u_{iE}^{\varepsilon} dxdydz \\
&= \int_0^1 \int_0^{2\pi} \sum_{i=1}^{3} \left(\partial_x u_{iE}^{\varepsilon} u_{iE}^{\varepsilon} |_{x=0}^{x=2\pi} - \int_0^{2\pi} (\partial_x u_{iE}^{\varepsilon})^2 dx \right) dydz \\
&+ \int_0^1 \int_0^{2\pi} \sum_{i=1}^{3} \left(\partial_y u_{iE}^{\varepsilon} u_{iE}^{\varepsilon} |_{y=0}^{y=2\pi} - \int_0^{2\pi} (\partial_y u_{iE}^{\varepsilon})^2 dy \right) dxdz \\
&+ \int_0^{2\pi} \int_0^{2\pi} \sum_{i=1}^{3} \left(\partial_z u_{iE}^{\varepsilon} u_{iE}^{\varepsilon} |_{z=0}^{z=1} - \int_0^{1} (\partial_z u_{iE}^{\varepsilon})^2 dz \right) dxdy \\
&= -\int_Q (\nabla u_E^{\varepsilon})^2 dxdydz,
\end{aligned}
$$

where we use $u_E^{\varepsilon} = (u_{1E}^{\varepsilon}, u_{2E}^{\varepsilon}, u_{3E}^{\varepsilon})$ and the boundary condition (5.4).

$$
\begin{aligned}
&\int_Q Rae_3 T_E^{\varepsilon} u_E^{\varepsilon} dxdydz \\
&\leq |\int_Q Rae_3 T_E^{\varepsilon} u_E^{\varepsilon} dxdydz| \\
&\leq \eta_3 \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + C(\eta_3) Ra^2 \parallel T_E^{\varepsilon} \parallel_{L^2(Q)}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_Q (R_{ou,u}^{\varepsilon} + R_{I,u}^{\varepsilon}) u_E^{\varepsilon} dxdydz \\
&\leq |\int_Q (R_{ou,u}^{\varepsilon} + R_{I,u}^{\varepsilon}) u_E^{\varepsilon} dxdydz| \\
&\leq \eta_4 \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + C(\eta_4) \parallel R_{ou,u}^{\varepsilon} + R_{I,u}^{\varepsilon} \parallel_{L^2(Q)}^2 \\
&\leq \eta_4 \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + C(\eta_4)(C\varepsilon^4 + C\varepsilon^4 (\tau+1)^2 e^{-2\beta\tau}),
\end{aligned}
$$

where we have used Hölder inequality, Young inequality and the estimates (4.9), (4.50). Here $\eta_i > 0$, $i = 3, 4$ is a small constant, $C(\eta_i) > 0$ is a constant, independent of ε.

Then, putting the above derivation equations into (5.13), we obtain that

$$
\begin{aligned}
&\frac{\varepsilon}{2} \frac{d}{dt} \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + \parallel \nabla u_E^{\varepsilon} \parallel_{L^2(Q)}^2 \\
&\leq \varepsilon \parallel \nabla u_{app}^{\varepsilon} \parallel_{L^\infty(Q)} \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + \eta_3 \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + C(\eta_3) Ra^2 \parallel T_E^{\varepsilon} \parallel_{L^2(Q)}^2 \\
&+ \eta_4 \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + C(\eta_4)(C\varepsilon^4 + C\varepsilon^4 (\tau+1)^2 e^{-2\beta\tau}).
\end{aligned}
$$

With the help of the Poincaré inequality, restricting ε to be sufficiently small such that $\varepsilon \parallel \nabla u_{app}^{\varepsilon} \parallel_{L^\infty(Q)} \leq C\varepsilon \leq \frac{1}{4}$ and taking η_3, η_4 to be sufficiently small ($\eta_3 + \eta_4 = \frac{1}{4}$) but independent of ε, one gets

$$
\begin{aligned}
&\varepsilon \frac{d}{dt} \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + \parallel \nabla u_E^{\varepsilon} \parallel_{L^2(Q)}^2 \\
&\leq 2C(\eta_3) Ra^2 \parallel T_E^{\varepsilon} \parallel_{L^2(Q)}^2 + 2C(\eta_4)(C\varepsilon^4 + C\varepsilon^4 (\tau+1)^2 e^{-2\beta\tau}),
\end{aligned}
\tag{5.14}
$$

that is,

$$
\begin{aligned}
&\varepsilon \frac{d}{dt} \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 + \parallel u_E^{\varepsilon} \parallel_{L^2(Q)}^2 \\
&\leq 2C(\eta_3) Ra^2 \parallel T_E^{\varepsilon} \parallel_{L^2(Q)}^2 + 2C(\eta_4)(C\varepsilon^4 + C\varepsilon^4 (\tau+1)^2 e^{-2\beta\tau}),
\end{aligned}
$$

i.e.,

$$\frac{d}{dt}(e^{\frac{t}{\varepsilon}} \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2)$$
$$\leq [2C(\eta_3)Ra^2 \parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + 2C(\eta_4)(C\varepsilon^4 + C\varepsilon^4(\tau+1)^2 e^{-2\beta\tau})]\varepsilon^{-1} e^{\frac{t}{\varepsilon}}. \tag{5.15}$$

Integrating (5.15) with respect to t over $[0, t]$ for any $t \in [0, S]$ and any fixed $S > 0$, one gets

$$\parallel u_E^\varepsilon(t) \parallel_{L^2(Q)}^2 \leq \parallel u_E^\varepsilon(t=0) \parallel_{L^2(Q)}^2 + 2C(\eta_3)Ra^2 \parallel T_E^\varepsilon(t) \parallel_{L^\infty(0,t;L^2(Q))}^2 + 2C(\eta_4)C\varepsilon^4. \tag{5.16}$$

Step 4. Then, combing (5.12) and (5.14), using the Poincaré inequality and restricting η_2 to be sufficiently small independent of ε yield

$$\frac{d}{dt} \parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + \varepsilon \frac{d}{dt} \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2 + 2\varepsilon \parallel \nabla T_E^\varepsilon \parallel_{L^2(Q)}^2 + \parallel \nabla u_E^\varepsilon \parallel_{L^2(Q)}^2$$
$$\leq 2 \parallel \nabla T_{app}^\varepsilon \parallel_{L^\infty(Q)}^2 C(\eta_2) \parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + 2\eta_2 \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2 + 2C(\eta_1)(C\varepsilon^4 + C\varepsilon^2(\tau+1)^2 e^{-2\beta\tau})$$
$$+ 2C(\eta_3)Ra^2 \parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + 2C(\eta_4)(C\varepsilon^4 + C\varepsilon^4(\tau+1)^2 e^{-2\beta\tau})$$
$$\leq C_1 \parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + C_2(\varepsilon^4 + \varepsilon^2(\tau+1)^2 e^{-2\beta\tau})$$
$$\leq C_1 \parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + C_1\varepsilon \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2 + C_2(\varepsilon^4 + \varepsilon^2(\tau+1)^2 e^{-2\beta\tau}), \tag{5.17}$$

where $C_1 = 2 \parallel \nabla T_{app}^\varepsilon \parallel_{L^\infty(Q)}^2 C(\eta_2) + 2C(\eta_3)Ra^2$, $C_2 = 2C(\eta_1)C + 2C(\eta_4)C$.

So,

$$\frac{d}{dt}(\parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + \varepsilon \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2)$$
$$\leq C_1(\parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + \varepsilon \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2) + C_2(\varepsilon^4 + \varepsilon^2(\tau+1)^2 e^{-2\beta\tau}).$$

Using Gronwall's lemma and the assumption (3.2) yield

$$\parallel T_E^\varepsilon \parallel_{L^2(Q)}^2 + \varepsilon \parallel u_E^\varepsilon \parallel_{L^2(Q)}^2$$
$$\leq e^{\int_0^t C_1 d\xi}[\parallel T_E^\varepsilon(t=0) \parallel_{L^2(Q)}^2 + \varepsilon \parallel u_E^\varepsilon(t=0) \parallel_{L^2(Q)}^2 + \int_0^t C_2(\varepsilon^4 + \varepsilon^2(\tau+1)^2 e^{-2\beta\tau}) d\xi]$$
$$\leq C\varepsilon^3, \tag{5.18}$$

where we have used the estimate

$$\int_0^t (\tau+1)^2 e^{-2\beta\tau} d\xi \leq C\varepsilon.$$

We deduce from (5.18) that

$$\parallel T_E^\varepsilon(t) \parallel_{L^\infty(0,S;L^2(Q))}^2 \leq C\varepsilon^3, \tag{5.19}$$

and

$$\parallel u_E^\varepsilon(t) \parallel_{L^\infty(0,S;L^2(Q))}^2 \leq C\varepsilon^2.$$

Inserting (5.19) into (5.16) yields

$$\parallel u_E^\varepsilon(t) \parallel_{L^\infty(0,S;L^2(Q))}^2 \leq C\varepsilon^3. \tag{5.20}$$

Inserting (5.18) into (5.17) and integrating (5.17) with respect to t over $[0, t]$ for any $t \in [0, S]$ any fixed $S > 0$ yield

$$2\varepsilon \int_0^t \parallel \nabla T_E^\varepsilon \parallel_{L^2(Q)}^2 d\xi + \int_0^t \parallel \nabla u_E^\varepsilon \parallel_{L^2(Q)}^2 d\xi \leq C\varepsilon^3.$$

So that,

$$\parallel T_E^\varepsilon(t) \parallel_{L^2(0,S;H^1(Q))}^2 \leq C\varepsilon^2, \tag{5.21}$$

and

$$\| u_E^\varepsilon(t) \|_{L^2(0,S;H^1(Q))}^2 \le C\varepsilon^3. \tag{5.22}$$

Here $H^1(Q) = W^{1,2}(Q)$, the estimates (5.19)-(5.22) yield to (3.3)-(3.5) in Theorem 3.1.

The proof of Theorem 3.1 is complete.

Obviously, the convergence rate $O(\varepsilon^{\frac{3}{2}})$ is not optimal one, so we derive the optimal convergence rate in the next subsection.

5.2 The proof of Theorem 3.5

First we assume that (3.1) and (3.6) hold.

Step 1. We cancel the order $O(\varepsilon)$ term of (4.42) to get the optimal convergence rate, and regard new result as $R_{I,T1}^\varepsilon$ in the remainder. Moreover, by virtue of another initial layer function $T^{I,2}$, we define it as $R_{I,T2}^\varepsilon$.

We define $T^{I,2}$ to be the solution of the system (3.8)-(3.9), which can be solved by

$$T^{I,2} = \int_\tau^\infty [(u^{O,0}(t=0)\nabla)T^{I,1} + (u^{I,0}\nabla)T^{O,1}(t=0) + (u^{I,0}\nabla)T^{I,1}$$
$$+(u^{I,0}\nabla)\partial_t T^{O,0}(t=0)\tau + (u^{I,1}\nabla)T^{O,0}(t=0)](s)ds. \tag{5.23}$$

Thus,

$$T^{I,2}|_{z=0,1} = 0. \tag{5.24}$$

In fact, the assumption (3.6) and the definition (4.30) of $T^{I,1}$ give

$$T^{I,1} = 0, \ near\, z = 0, 1,$$

which, together with (5.23) and the boundary condition $(u^{I,0}, u^{I,1})|_{z=0,1} = 0$, yields to the boundary condition (5.24).

By the exponential decay of the initial layer functions $(u^{I,0}, u^{I,1}, T^{I,1})$, it follows that

$$\| (T^{I,2})(\cdot, \tau) \|_{H^s(Q)} \le Ce^{-\gamma\tau}, \tag{5.25}$$

for some positive constants C, γ and any $s \ge 1$.

Step 2. Now set $\widetilde{T}_E^\varepsilon = T_E^\varepsilon - \varepsilon^2 T^{I,2} = T^\varepsilon - T_{app}^\varepsilon - \varepsilon^2 T^{I,2}$.

Then $(u_E^\varepsilon, p_E^\varepsilon, \widetilde{T}_E^\varepsilon)$ satisfies the following error equations

$$\varepsilon[\partial_t u_E^\varepsilon + (u_{app}^\varepsilon \cdot \nabla)u_E^\varepsilon + (u_E^\varepsilon \cdot \nabla)(u_{app}^\varepsilon + u_E^\varepsilon)] + \nabla p_E^\varepsilon + \frac{1}{Ek} e_3 \times u_E^\varepsilon$$
$$= \Delta u_E^\varepsilon + Rae_3 \widetilde{T}_E^\varepsilon + Rae_3\varepsilon^2 T^{I,2} - R_{ou,u}^\varepsilon - R_{I,u}^\varepsilon, \tag{5.26}$$
$$\nabla \cdot u_E^\varepsilon = 0, \tag{5.27}$$
$$\partial_t \widetilde{T}_E^\varepsilon + (u_{app}^\varepsilon \cdot \nabla)\widetilde{T}_E^\varepsilon + (u_E^\varepsilon \cdot \nabla)(T_{app}^\varepsilon + \varepsilon^2 T^{I,2} + \widetilde{T}_E^\varepsilon)$$
$$= \varepsilon\Delta \widetilde{T}_E^\varepsilon - R_{ou,T}^\varepsilon - R_{I,T2}^\varepsilon - \varepsilon^2 (u_{app}^\varepsilon \cdot \nabla)T^{I,2} + \varepsilon^2 \Delta T^{I,2}, \tag{5.28}$$
$$u_E^\varepsilon|_{z=0.1} = 0, \tag{5.29}$$
$$\widetilde{T}_E^\varepsilon|_{z=0,1} = 0, \tag{5.30}$$
$$u_E^\varepsilon(t=0) = u_{0E}^\varepsilon(x,y,z), \widetilde{T}_E^\varepsilon(t=0) = T_{0E}^\varepsilon(x,y,z) - \varepsilon^2 T^{I,2}(\tau = 0). \tag{5.31}$$

Step 3. Using the decay property (5.25) of $T^{I,2}$ and the definition of $R_{I,T2}^\varepsilon$, one has the following estimates on the remainder $-R_{ou,T}^\varepsilon - R_{I,T2}^\varepsilon - \varepsilon^2(u_{app}^\varepsilon \cdot \nabla)T^{I,2} + \varepsilon^2 \Delta T^{I,2}$ appearing in (5.28):

$$\| -R_{ou,T}^\varepsilon - R_{I,T2}^\varepsilon - \varepsilon^2 (u_{app}^\varepsilon \cdot \nabla)T^{I,2} + \varepsilon^2 \Delta T^{I,2} \|_{L^2(Q)} \le C\varepsilon^2, \tag{5.32}$$

where the estimate (5.32) is much better than the estimate in (5.9).

Now we use the estimate (5.32) in subsection 5.1, and can derive the optimal convergence rate $O(\varepsilon^2)$ in Theorem 3.5 by using the method in the proof of Theorem 3.1.

The proof of Theorem 3.5 is complete.

6 Conclusion

In this paper, we have used matched asymptotic expansion analysis to study the Boussinesq system for Rayleigh-Bénard convection with infinite Prandtl number limit, which involves initial layers. It is a singular perturbation problem. We have derived the convergence of the solution of the Boussinesq system for Rayleigh-Bénard convection to that of the infinite Prandtl number limit system by adopting the effective approximating expansion.

The boundary value of the limit $\lim_{\varepsilon\to 0}(u^{\varepsilon}, T^{\varepsilon})$ are not equal to $(u^{O,0}, T^{O,0})$, due to the initial and boundary conditions effect, the boundary layer occurs. This need an extra correction term of boundary layer with two fast variables. We will discuss it in the future.

The initial data satisfies a higher order correction in powers of ε, then the similar higher-order correction result in powers of ε can be obtained in the same way. We leave it for further investigation.

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