

## Open Mathematics

## Research Article

Varadharajan Radhika, Jay M. Jahangiri, Srikandan Sivasubramanian, and Gangadharan Murugusundaramoorthy\*

# Toeplitz matrices whose elements are coefficients of Bazilevič functions

<https://doi.org/10.1515/math-2018-0093>

Received November 15, 2016; accepted February 7, 2018.

**Abstract:** We consider the Toeplitz matrices whose elements are the coefficients of Bazilevič functions and obtain upper bounds for the first four determinants of these Toeplitz matrices. The results presented here are new and noble and the only prior compatible results are the recent publications by Thomas and Halim [1] for the classes of starlike and close-to-convex functions and Radhika et al. [2] for the class of functions with bounded boundary rotation.

**Keywords:** Toeplitz Determinants, Analytic, Univalent and Bazilevič Functions

**MSC:** 30C45, 33C50, 30C80

## 1 Introduction

Let  $\mathcal{A}$  denote the class of all functions  $f$  of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk  $\mathbb{U} = \{z : |z| < 1\}$  and let  $\mathcal{S}$  denote the subclass of  $\mathcal{A}$  consisting of univalent functions. Obviously, for functions  $f \in \mathcal{S}$  we must have  $f' \neq 0$  in  $\mathbb{U}$ . For  $f \in \mathcal{S}$ , we consider the family  $\mathcal{B}(\beta)$  of Bazilevič functions of type  $\beta$ ;  $0 \leq \beta \leq 1$  so that

$$\Re \left( \frac{z^{1-\beta} f'(z)}{[f(z)]^{1-\beta}} \right) > 0.$$

The family  $\mathcal{B}(\beta)$  of Bazilevič functions of type  $\beta$ ;  $0 \leq \beta \leq 1$  provides a transition from the class of starlike functions to the class of functions of bounded boundary rotation. To see this, we note that for the choice of  $\beta = 0$ , we have  $\mathcal{B}(\beta) \equiv \mathcal{S}^*(0) \equiv \mathcal{S}^*$ , the class of starlike functions  $f \in \mathcal{S}$  so that  $\Re(zf'/f) > 0$  in  $\mathbb{U}$  and for the choice of  $\beta = 1$ , we get the family  $\mathcal{R}$  of functions  $f \in \mathcal{S}$  of bounded boundary rotation so that  $\Re(f') > 0$  in  $\mathbb{U}$ . (For further details see [3].)

Several authors (e.g. see [4–8]) have discussed various subfamilies of the well-known Bazilevič functions of type  $\beta$  from various viewpoints including their coefficient estimates. It is interesting to note in this

**Varadharajan Radhika:** Department of Mathematics, Easwari Engineering College, Ramapuram, Chennai 600089, India, E-mail: radhimukund2006@gmail.com

**Jay M. Jahangiri:** Department of Mathematical Sciences, Kent State University, Burton, OH 44021-9500, USA, E-mail: jjahangi@kent.edu

**Srikandan Sivasubramanian:** Department of Mathematics, University College of Engineering Tindivanam, Anna University, Tindivanam 604001, Tamil Nadu, India, E-mail: sivasaisastha@rediffmail.com

**\*Corresponding Author: Gangadharan Murugusundaramoorthy:** School of Advanced Sciences, VIT University, Vellore Campus, Vellore 632 014, Tamil Nadu, India, E-mail: gmsmoorthy@yahoo.com

connection that the earlier investigations on the subject do not seem to have made use of Toeplitz matrices and determinants. Toeplitz matrices are one of the well-studied classes of structured matrices. They arise in all branches of pure and applied mathematics, statistics and probability, image processing, quantum mechanics, queueing networks, signal processing and time series analysis, to name a few (e.g. see Ye and Lim [9]). Toeplitz matrices have some of the most attractive computational properties and are amenable to a wide range of disparate algorithms and determinant computations.

Here we consider the symmetric Toeplitz determinant

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & & \vdots \\ \vdots & & & \\ a_{n+q-1} & \cdots & & a_n \end{vmatrix}$$

and obtain upper bounds for the coefficient body  $T_q(n)$ ;  $q = 2, 3$ ;  $n = 1, 2, 3$  where the entries of  $T_q(n)$  are the coefficients of functions  $f$  of the form (1) that are in the family of Bazilevič functions  $\mathcal{B}(\beta)$ . As far as we are concerned, the results presented here are new and noble and the only prior compatible results are the recent publications by Thomas and Halim [1] for the classes of starlike and close-to-convex functions and Radhika et al. [2] for the class of functions with bounded boundary rotation. We shall need the following result [10] in order to prove our main theorems.

**Lemma 1.1.** *Let  $h(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in \mathcal{P}$ . Then for some complex valued  $x$  with  $|x| \leq 1$  and some complex valued  $\zeta$  with  $|\zeta| \leq 1$ .*

$$\begin{aligned} 2p_2 &= p_1^2 + x(4 - p_1^2) \\ 4p_3 &= p_1^3 + 2(4 - p_1^2)p_1x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)\zeta. \end{aligned}$$

## 2 Coefficient estimates for Toeplitz determinant

In our first theorem we determine a sharp upper bound for the coefficient body  $T_2(2)$ .

**Theorem 2.1.** *Let  $f$  given by (1) be in the class  $\mathcal{B}(\beta)$ ;  $0 \leq \beta \leq 1$ . Then we have the sharp bound*

$$|T_2(2)| = |a_3^2 - a_2^2| \leq \frac{4}{(\beta + 2)^2} \max \left\{ 1, \left| \frac{(-\beta^4 - 6\beta^3 - 12\beta^2 - 6\beta + 5)}{(\beta + 1)^4} \right| \right\}.$$

*Proof.* First note that by equating the corresponding coefficients in the equation

$$\frac{z^{1-\beta} f'(z)}{[f(z)]^{1-\beta}} = h(z)$$

we obtain

$$a_2 = \frac{p_1}{(\beta + 1)}, \quad (2)$$

$$a_3 = \frac{p_2}{\beta + 2} - \frac{(\beta - 1)p_1^2}{2(\beta + 1)^2}. \quad (3)$$

In view of (2) and (3), a simple computation leads to

$$a_3^2 - a_2^2 = \frac{p_2^2}{(\beta + 2)^2} + \frac{(\beta - 1)^2 p_1^4}{4(\beta + 1)^4} - \frac{p_1^2 p_2 (\beta - 1)}{(\beta + 2)(\beta + 1)^2} - \frac{p_1^2}{(\beta + 1)^2}. \quad (4)$$

Note that, by Lemma 1.1, we may write  $2p_2 = p^2 + x(4 - p^2)$  where without loss of generality we let  $0 \leq p_1 = p \leq 2$ . Substituting this into the above equation we obtain the following quadratic equation in terms of  $x$ .

$$a_3^2 - a_2^2 = \frac{(4 - p^2)^2}{4(\beta + 2)^2} x^2 + \frac{(\beta + 3)(4 - p^2)p^2}{2(\beta + 1)^2(\beta + 2)^2} x + \frac{[(\beta^2 + 6\beta + 9)p^2 - 4(\beta + 1)^2(2 + \beta)^2]p^2}{4(\beta + 1)^4(\beta + 2)^2}.$$

Using the triangle inequality we obtain

$$|a_3^2 - a_2^2| \leq \frac{(4 - p^2)^2}{4(\beta + 2)^2} + \frac{(\beta + 3)(4 - p^2)p^2}{2(\beta + 1)^2(\beta + 2)^2} + \frac{[(\beta^2 + 6\beta + 9)p^2 + 4(\beta + 1)^2(2 + \beta)^2]p^2}{4(\beta + 1)^4(\beta + 2)^2} = \Phi(p, \beta).$$

Differentiating  $\Phi(p, \beta)$  with respect to  $p$  we obtain

$$\frac{\partial(\Phi(p, \beta))}{\partial p} = \frac{p[p^2(\beta^3 - 3\beta + 2) - 2\beta^3 + 4\beta^2 + 14\beta + 8]}{(\beta + 1)^4(\beta + 2)}.$$

Setting  $\frac{\partial(\Phi(p, \beta))}{\partial p} = 0$  yields either  $p = 0$  or

$$p^2 = \frac{2\beta^3 - 4\beta^2 - 14\beta - 8}{\beta^3 - 3\beta + 2}.$$

But  $2\beta^3 - 4\beta^2 - 14\beta - 8 < 0$  for  $0 \leq \beta \leq 1$ . Therefore, the maximum of  $|a_3^2 - a_2^2|$  is attained at the end points  $p_1 = p \in [0, 2]$ .

For  $p_1 = 0$ , we have  $p_2 = 2x$ . Therefore, from (4),

$$|a_3^2 - a_2^2| = \frac{4|x|^2}{(\beta + 2)^2} \leq \frac{4}{(\beta + 2)^2}.$$

For  $p_1 = 2$  we have  $a_2 = \frac{2}{\beta+1}$  and  $a_3 = \frac{2}{\beta+2} - \frac{2(\beta-1)}{(\beta+1)^2}$  which yields

$$|a_3^2 - a_2^2| \leq \left| \frac{4(-\beta^4 - 6\beta^3 - 12\beta^2 - 6\beta + 5)}{(\beta + 1)^4(\beta + 2)^2} \right|.$$

The result is sharp for the functions given by

$$\frac{z^{1-\beta}f'(z)}{[f(z)]^{1-\beta}} = \frac{1+z}{1-z}. \quad \square$$

**Remark 2.2.** Theorem 2.1 for  $\beta = 0$  yields the bound  $|a_3^2 - a_2^2| \leq 5$  for the class of starlike functions  $\mathcal{S}^*$  confirming the bound obtained by Thomas and Halim [1] and for  $\beta = 1$  yields the bound  $|a_3^2 - a_2^2| \leq 5/9$  for the class of functions with bounded boundary rotation  $\mathcal{R}$  confirming the bound obtained by Radhika et al. [2].

In our next theorem, we determine an upper bound for the coefficient body  $T_2(3)$ .

**Theorem 2.3.** Let  $f$  given by (1), be in the class  $\mathcal{B}(\beta)$ ,  $0 \leq \beta \leq 1$ . Then

$$|T_2(3)| = |a_4^2 - a_3^2| \leq \max \left\{ \frac{|N(\beta)|}{9(\beta + 3)^2(\beta + 1)^6(\beta + 2)^2}, \frac{4}{(2 + \beta)^2} \right\}.$$

where

$$N(\beta) = 4\beta^8 + 40\beta^7 + 152\beta^6 + 88\beta^5 - 1312\beta^4 - 5096\beta^3 - 8024\beta^2 - 4248\beta + 2268.$$

*Proof.* By equating the corresponding coefficients in the equation,

$$\frac{z^{1-\beta}f'(z)}{[f(z)]^{1-\beta}} = h(z)$$

we obtain

$$a_2 = \frac{p_1}{(\beta + 1)}, \quad (5)$$

$$a_3 = \frac{p_2}{\beta + 2} - \frac{(\beta - 1)p_1^2}{2(\beta + 1)^2}, \text{ and} \quad (6)$$

$$a_4 = \frac{p_3}{\beta+3} - \frac{(\beta-1)p_1p_2}{(\beta+1)(\beta+2)} + \frac{(\beta-1)(2\beta-1)p_1^3}{6(\beta+1)^3}. \quad (7)$$

In view of (6) and (7) and applying Lemma 1, denoting  $X = 4 - p^2$  and  $Y = (1 - |x|^2)\zeta$ , where  $0 \leq p \leq 2$  and  $|\zeta| < 1$  we get,

$$\begin{aligned} a_4^2 - a_3^2 = & \frac{\beta^8 + 10\beta^7 + 47\beta^6 + 148\beta^5 + 383\beta^4 + 778\beta^3 + 1153\beta^2 + 1368\beta + 1296}{144(\beta+3)^2(\beta+1)^6(\beta+2)^2} p_1^6 \\ & - \frac{\beta^2 + 6\beta + 9}{4(\beta+1)^4(\beta+2)^2} p_1^4 + \frac{X^2 Y^2}{4(\beta+3)^2} - \frac{p_1 x^2 X^2 Y}{4(\beta+3)^2} \\ & + \frac{(\beta+5)xX^2 p_1 Y}{2(\beta+3)^2(\beta+2)(\beta+1)} + \frac{\beta^4 + 5\beta^3 + 11\beta^2 + 19\beta + 36}{12(\beta+3)^2(\beta+2)(\beta+1)^3} p_1^3 X Y \\ & + \frac{p_1^2 X^2 x^4}{16(\beta+3)^2} - \frac{\beta+5}{4(\beta+3)^2(\beta+2)(\beta+1)} p_1^2 x^3 X^2 \\ & + \frac{\beta^2 + 10\beta + 25}{4(\beta+3)^2(\beta+2)^2(\beta+1)^2} p_1^2 X^2 x^2 - \frac{x^2 X^2}{4(\beta+2)^2} \\ & - \frac{\beta^4 + 5\beta^3 + 11\beta^2 + 19\beta + 36}{24(\beta+3)^2(\beta+2)(\beta+1)^3} X x^2 p_1^4 \\ & + \frac{\beta^5 + 10\beta^4 + 36\beta^3 + 74\beta^2 + 131\beta + 180}{12(\beta+3)^2(\beta+2)^2(\beta+1)^4} X x p_1^4 \\ & - \frac{(\beta+3)}{2(\beta+2)^2(\beta+1)^2} X x p_1^2. \end{aligned}$$

As in the proof of Theorem 2.1, without loss of generality, we can write  $p_1 = p$ , where  $0 \leq p \leq 2$ . Then an application of triangle inequality gives,

$$\begin{aligned} |a_4^2 - a_3^2| \leq & \frac{(2-p)^2}{16(\beta+3)^2} |x|^4 + \frac{(\beta+5)(p^2-2p)(4-p^2)^2}{4(\beta+3)^2(\beta+2)(\beta+1)} |x|^3 \\ & + \left[ \frac{\beta^2 + 10\beta + 25}{4(\beta+3)^2(\beta+2)(\beta+1)^2} p^2 (4-p^2) \right. \\ & + \frac{p^3(p-2)(4-p^2)(\beta^4 + 5\beta^3 + 11\beta^2 + 19\beta + 36)}{24(\beta+3)^2(\beta+2)(\beta+1)^3} \\ & - \frac{(\beta^2 + 2\beta - 1)(4-p^2)^2}{4(\beta+3)^2(\beta+2)^2} + \left. \frac{p(4-p^2)^2}{4(\beta+3)^2} \right] |x|^2 \\ & + \left[ \frac{(\beta+3)}{2(\beta+2)^2(\beta+1)^2} (4-p^2)p^2 + \frac{(\beta+5)(4-p^2)^2 p}{2(\beta+3)^2(\beta+2)(\beta+1)} \right. \\ & + \left. \frac{(\beta^5 + 10\beta^4 + 36\beta^3 + 74\beta^2 + 131\beta + 180)p^4(4-p^2)}{12(\beta+3)^2(\beta+1)^4(\beta+2)^2} \right] |x| \\ & + |N_1(\beta)p^6 - N_2(\beta)p^4| + \frac{(4-p^2)^2}{4(\beta+3)^2} \\ & + \frac{(\beta^4 + 5\beta^3 + 11\beta^2 + 19\beta + 36)p^3(4-p^2)}{12(\beta+3)^2(\beta+2)(\beta+1)^3} \\ = & \Psi(p, |x|) \end{aligned}$$

where

$$\begin{aligned} N_1(\beta) &= \frac{\beta^8 + 10\beta^7 + 47\beta^6 + 148\beta^5 + 383\beta^4 + 778\beta^3 + 1153\beta^2 + 1368\beta + 1296}{144(\beta+3)^2(\beta+2)^2(\beta+1)^6}, \\ N_2(\beta) &= \frac{\beta^2 + 6\beta + 9}{4(\beta+2)^2(\beta+1)^4}. \end{aligned}$$

We need to find the maximum value of  $\psi(p, |x|)$  on  $[0, 2] \times [0, 1]$ . First, assume that there is a maximum at an interior point  $\psi(p_0, |x_0|)$  of  $[0, 2] \times [0, 1]$ . Differentiating  $\psi(p, |x|)$  with respect to  $|x|$  and equating it to 0 implies that  $p = p_0 = 2$ , which is a contradiction. Thus for the maximum of  $\psi(p, |x|)$ , we need only to consider the end points of  $[0, 2] \times [0, 1]$ .

For  $p = 0$  we obtain

$$\psi(0, |x|) = \frac{4}{(\beta+3)^2} |x|^4 - \frac{4(\beta^2+2\beta-1)}{(\beta+3)^2(\beta+2)^2} |x|^2 + \frac{4}{(\beta+3)^2} \leq \frac{4}{(\beta+2)^2}.$$

For  $p = 2$  we obtain

$$\Phi(2, |x|) = |64N_1(\beta) - 16N_2(\beta)|.$$

For  $|x| = 0$  we obtain

$$\begin{aligned} \psi(p, 0) &= |N_1(\beta)p^6 - N_2(\beta)p^4| \\ &\quad + \frac{(\beta^4 + 5\beta^3 + 11\beta^2 + 19\beta + 36)p^3(4-p^2)}{12(\beta+3)^2(\beta+2)(\beta+1)^3} + \frac{p(4-p^2)^2}{4(\beta+3)^2}, \end{aligned}$$

which has the maximum value  $|N_1(\beta)p^6 - N_2(\beta)p^4|$  on  $[0, 2]$ .

For  $|x| = 1$  we obtain

$$\begin{aligned} \psi(p, 1) &= \frac{(2-p)^2(4-p^2)^2}{16(\beta+3)^2} + \frac{(\beta+5)(p^2-2p)(4-p^2)^2}{4(\beta+3)^2(\beta+2)(\beta+1)} \\ &\quad + \left[ \frac{(\beta^2+10\beta+25)p^2(4-p^2)^2}{4(\beta+3)^2(\beta+2)(\beta+1)^2} + \frac{p(4-p^2)^2}{4(\beta+3)^2} \right. \\ &\quad + \frac{(\beta^4+5\beta^3+11\beta^2+19\beta+36)p^3(p-2)(4-p^2)}{24(\beta+3)^2(\beta+2)(\beta+1)^3} \\ &\quad \left. - \frac{(\beta^2+2\beta-1)(4-p^2)^2}{4(\beta+3)^2(\beta+2)^2} \right] \\ &\quad + \left[ \frac{(\beta+3)}{2(\beta+2)^2(\beta+1)^2} (4-p^2)p^2 + \frac{(\beta+5)(4-p^2)^2p}{2(\beta+3)^2(\beta+2)(\beta+1)} \right. \\ &\quad \left. + \frac{(\beta^5+10\beta^4+36\beta^3+74\beta^2+131\beta+180)p^4(4-p^2)}{12(\beta+3)^2(\beta+2)^2(\beta+1)^4} \right] \\ &\quad + |N_1(\beta)p^6 - N_2(\beta)p^4| + \frac{(4-p^2)^2}{4(\beta+3)^2} \\ &\quad + \frac{(\beta^4+5\beta^3+11\beta^2+19\beta+36)p^3(4-p^2)}{12(\beta+3)^2(\beta+2)(\beta+1)^3} \end{aligned}$$

which has the maximum values  $|64N_1(\beta) - 16N_2(\beta)|$  for  $p = 2$  and  $\frac{4}{(2+\beta)^2}$  for  $p = 0$ .  $\square$

**Remark 2.4.** Theorem 2.3 for  $\beta = 0$  yields the bound  $|T_2(3)| \leq 7$  for the class of starlike functions  $S^*$  confirming the bound obtained by Thomas and Halim [1] and for  $\beta = 1$  yields the bound  $|T_2(3)| \leq 4/9$  for the class of functions with bounded boundary rotation  $\mathcal{R}$  confirming the bound obtained by Radhika et al. [2].

**Theorem 2.5.** Let  $f$  given by (1) be in the class  $\mathcal{B}(\beta)$ ,  $(0 \leq \beta \leq 1; \beta \neq \beta_0)$ , then

$$|T_3(2)| = \left| \begin{pmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{pmatrix} \right| \leq \begin{cases} \max \left\{ |M_1(\beta)M_2(\beta)|, \frac{8|M_1(\beta)|}{(\beta+2)^2} \right\}; & \text{if } \beta \neq \beta_0, \\ \max \left\{ |M_2(\beta)M_3(\beta)|, \frac{8|M_3(\beta)|}{(\beta+2)^2} \right\}; & \text{if } \beta = \beta_0, \end{cases}$$

where  $\beta_0 \approx 0.3676$  is the positive root of the polynomial

$$4\beta^4 + 32\beta^3 + 80\beta^2 + 64\beta - 36 = 0,$$

$$M_1(\beta) = \frac{4\beta^4 + 32\beta^3 + 80\beta^2 + 64\beta - 36}{3(\beta + 1)^3(\beta + 2)(\beta + 3)},$$

$$M_2(\beta) = \frac{4(4\beta^5 + 34\beta^4 + 108\beta^3 + 140\beta^2 + 32\beta - 54)}{3(\beta + 1)^4(\beta + 2)^2(\beta + 3)}$$

and

$$M_3(\beta) = \frac{8\beta^4 + 52\beta^3 + 124\beta^2 + 140\beta + 108}{3(\beta + 1)^3(\beta + 2)(\beta + 3)}.$$

*Proof.* Write

$$\begin{aligned} |T_3(2)| &= |a_2^3 - 2a_2a_3^2 + 2a_3^2a_4 - a_2a_4| \\ &= |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)|. \end{aligned}$$

Using the same techniques as in Theorem 2.3, one can obtain with simple computations that

$$|a_2 - a_4| \leq |M_1(\beta)| \quad \text{for } \beta \neq \beta_0.$$

We need to show that

$$|a_2^2 - 2a_3^2 + a_2a_4| \leq |M_2(\beta)|.$$

In view of (2), (3) and (7) and Lemma 1, where we denote  $X = 4 - p^2$  and  $Y = (1 - |x|^2)\zeta$ , where  $0 \leq p \leq 2$  and  $|\zeta| < 1$ , one may easily get,

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| &= \left| \frac{p_1^2}{(\beta + 1)^2} - 2 \left( \frac{p_2^2}{(\beta + 2)^2} + \frac{(\beta - 1)^2 p_1^4}{4(\beta + 1)^4} - \frac{(\beta - 1)p_1^2 p_2}{(\beta + 1)^2(\beta + 2)} \right) \right. \\ &\quad \left. + \frac{p_1}{(\beta + 1)} \left( \frac{p_3}{\beta + 3} - \frac{(\beta - 1)p_1 p_2}{(\beta + 1)(\beta + 2)} + \frac{(\beta - 1)(2\beta - 1)p_1^3}{6(\beta + 1)^3} \right) \right| \\ &= \left| \frac{p_1^2}{(\beta + 1)^2} - 2 \frac{p_2^2}{(\beta + 2)^2} - \frac{(\beta - 1)^2 p_1^4}{2(\beta + 1)^4} + 2 \frac{(\beta - 1)p_1^2 p_2}{(\beta + 1)^2(\beta + 2)} \right. \\ &\quad \left. + \frac{p_1 p_3}{(\beta + 1)(\beta + 3)} - \frac{(\beta - 1)p_1^2 p_2}{(\beta + 1)^2(\beta + 2)} + \frac{(\beta - 1)(2\beta - 1)p_1^4}{6(\beta + 1)^4} \right| \\ &= \left| \frac{p_1^2}{(\beta + 1)^2} - \frac{1}{2(\beta + 2)^2} [p_1^4 + X^2 x^2 + 2p_1^2 Xx] \right. \\ &\quad \left. - \frac{(\beta - 1)^2 p_1^4}{2(\beta + 1)^4} + \frac{(\beta - 1)p_1^2}{2(\beta + 1)^2(\beta + 2)} [p_1^2 + Xx] \right. \\ &\quad \left. + \frac{p_1}{4(\beta + 1)(\beta + 3)} [p_1^3 + 2p_1 Xx - p_1 Xx^2 + 2XY] \right. \\ &\quad \left. + \frac{(\beta - 1)(2\beta - 1)p_1^4}{6(\beta + 1)^4} \right|. \end{aligned}$$

Applying the triangle inequality and assuming that  $p_1 = p$ , where  $0 \leq p \leq 2$  we obtain

$$\begin{aligned} |a_2^2 - 2a_3^2 + a_2a_4| &\leq \left[ \frac{1}{4(\beta + 1)(\beta + 3)} (p^2 - 2p) + \frac{1}{2(2 + \beta)^2} (4 - p^2) \right] (4 - p^2)|x|^2 \\ &\quad + \frac{(\beta^2 + 5\beta + 8)}{2(\beta + 1)^2(\beta + 2)^2(\beta + 3)} p^2 (4 - p^2)|x| \\ &\quad + \left| \frac{p^2}{(\beta + 1)^2} - \frac{(90 - \beta^5 - 7\beta^4 - 15\beta^3 + 13\beta^2 + 88\beta)p^4}{12(\beta + 1)^4(\beta + 2)^2(\beta + 3)} \right| \\ &\quad + \frac{p}{2(\beta + 1)(\beta + 3)} (4 - p^2) \\ &= \Omega(p, |x|) \end{aligned}$$

We need to find the maximum value of  $\Omega(p, |x|)$  on  $[0, 2] \times [0, 1]$ . First, assume that there is a maximum at an interior point  $\Omega(p_0, |x_0|)$  of  $[0, 2] \times [0, 1]$ . Differentiating  $\Omega(p, |x|)$  with respect to  $|x|$  and equating it to zero implies that  $p = p_0 = 2$ , which is a contradiction. Thus for the maximum of  $\Omega(p, |x|)$ , we need only to consider the end points of  $[0, 2] \times [0, 1]$ .

For  $p = 0$  we obtain

$$\Omega(0, |x|) = \frac{8}{(\beta + 2)^2} |x|^2 \leq \frac{8}{(\beta + 2)^2}.$$

For  $p = 2$  we obtain

$$\Omega(2, |x|) = \frac{4(4\beta^5 + 34\beta^4 + 108\beta^3 + 140\beta^2 + 32\beta - 54)}{3(\beta + 1)^4(\beta + 2)^2(\beta + 3)} = M_2(\beta).$$

For  $|x| = 0$  we obtain

$$\begin{aligned} \Omega(p, 0) = & \left| \frac{p^2}{(\beta + 1)^2} - \frac{(90 - \beta^5 - 7\beta^4 - 15\beta^3 + 13\beta^2 + 88\beta)}{12(\beta + 1)^4(\beta + 2)^2(\beta + 3)} p^4 \right| \\ & + \frac{p}{2(\beta + 1)(\beta + 3)} (4 - p^2) \end{aligned}$$

which has maximum value  $\Omega(p, 0) = M_2(\beta)$  attained at the end point  $p = 2$ .

For  $|x| = 1$  we obtain

$$\begin{aligned} \Omega(p, 1) = & \left| \frac{p^2}{(\beta + 1)^2} - \frac{(90 - \beta^5 - 7\beta^4 - 15\beta^3 + 13\beta^2 + 88\beta)}{12(\beta + 1)^4(\beta + 2)^2(\beta + 3)} p^4 \right| \\ & + \frac{1}{2(\beta + 2)^2} (4 - p^2)^2 + \frac{1}{4(\beta + 1)(\beta + 3)} p^2 (4 - p^2) \\ & + \frac{(\beta^2 + 5\beta + 8)}{2(\beta + 1)^2(\beta + 2)^2(\beta + 3)} p^2 (4 - p^2), \end{aligned}$$

which has maximum value  $\Omega(p, 1) = \frac{8}{(\beta + 2)^2}$  at  $p = 0$  and  $\Omega(p, 1) = M_2(\beta)$  at  $p = 2$ . Hence

$$|a_2^2 - 2a_3^2 + a_2a_4| \leq \max \left\{ |M_2(\beta)|, \frac{8}{(\beta + 2)^2} \right\}.$$

Thus

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)| \leq \max \left\{ |M_1(\beta)M_2(\beta)|, \frac{8|M_1(\beta)|}{(\beta + 2)^2} \right\}.$$

For the case  $\beta = \beta_0$ , we compute  $|a_2 - a_4|$  as follows

$$|a_2 - a_4| = \left| \frac{p_1}{\beta + 1} - \frac{p_3}{\beta + 3} + \frac{(\beta - 1)p_1p_2}{(\beta + 1)(\beta + 2)} - \frac{(\beta - 1)(2\beta - 1)p_1^3}{6(\beta + 1)^3} \right|.$$

Since, each  $|p_i| \leq 2$ , an application of triangle inequality shows that

$$|a_2 - a_4| \leq |M_3(\beta)| = \frac{8\beta^4 + 52\beta^3 + 124\beta^2 + 140\beta + 108}{3(\beta + 1)^3(\beta + 2)(\beta + 3)}.$$

Therefore

$$|T_3(2)| = |(a_2 - a_4)(a_2^2 - 2a_3^2 + a_2a_4)| \leq \max \left\{ |M_2(\beta)M_3(\beta)|, \frac{8|M_3(\beta)|}{(\beta + 2)^2} \right\}.$$

This completes the proof of Theorem 2.5.  $\square$

**Remark 2.6.** Theorem 2.5 for  $\beta = 0$  yields the bound  $|T_3(2)| \leq 8$  for the class of starlike functions  $S^*$  confirming the bound obtained by Thomas and Halim [1] and for  $\beta = 1$  yields the bound  $|T_3(2)| \leq 4/9$  for the class of functions with bounded boundary rotation  $\mathcal{R}$  confirming the bound obtained by Radhika et al. [2].

**Theorem 2.7.** Let  $f$  given by (1), be in the class  $\mathcal{B}(\beta)$ ,  $0 \leq \beta \leq 1$ . Then

$$|T_3(1)| = \left| \begin{pmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{pmatrix} \right| \leq \max \left\{ 1 + \frac{1}{4(\beta+2)^2}, |M_4(\beta)| \right\}$$

where

$$M_4(\beta) = \frac{\beta^6 + 8\beta^5 + 18\beta^4 - 4\beta^3 - 51\beta^2 - 20\beta + 32}{(\beta+1)^4(\beta+2)^2}.$$

*Proof.* Expanding the determinant by using equations (2) and (3) and applying Lemma 1.1, we have

$$\begin{aligned} T_3(1) &= 1 + 2a_2^2(a_3 - 1) - a_3^2 \\ &= 1 + \frac{2p_1^2}{(\beta+1)^2} \left( \frac{p_2}{(\beta+2)} - \frac{(\beta-1)p_1^2}{2(\beta+1)^2} - 1 \right) \\ &\quad - \frac{p_2^2}{(\beta+2)^2} - \frac{(\beta-1)^2 p_1^4}{4(\beta+1)^4} + \frac{p_2 p_1^2 (\beta-1)}{(\beta+1)^2 (\beta+2)} \\ &= 1 + \frac{2p_1^2}{(\beta+1)^2 (2+\beta)} \left( \frac{p_1^2 + Xx}{2} \right) - \frac{(\beta-1)p_1^4}{(\beta+1)^4} - \frac{2p_1^2}{(\beta+1)^2} - \frac{(\beta-1)^2 p_1^4}{4(\beta+1)^4} \\ &\quad - \frac{1}{4(\beta+2)^2} [p_1^4 + X^2 x^2 + 2p_1^2 Xx] + \frac{p_1^2 (\beta-1)}{(\beta+1)^2 (\beta+2)} \left( \frac{p_1^2 + Xx}{2} \right) \\ &= 1 + \frac{p_1^4}{(\beta+1)^2 (\beta+2)} + \frac{p_1^2 Xx}{(\beta+1)^2 (\beta+2)} - \frac{(\beta-1)p_1^4}{(\beta+1)^4} - \frac{2p_1^2}{(\beta+1)^2} \\ &\quad - \frac{p_1^4}{4(\beta+2)^2} - \frac{X^2 x^2}{4(\beta+2)^2} - \frac{p_1^2 Xx}{2(\beta+2)^2} - \frac{(1-\beta)^2 p_1^4}{4(\beta+1)^4} \\ &\quad - \frac{p_1^4 (1-\beta)}{2(\beta+1)^2 (\beta+2)} - \frac{p_1^2 Xx (1-\beta)}{2(\beta+1)^2 (\beta+2)} \\ &= 1 + p_1^4 \left[ \frac{1}{(\beta+1)^2 (2+\beta)} + \frac{(1-\beta)}{(\beta+1)^4} - \frac{1}{4(\beta+2)^2} - \frac{(1-\beta)^2}{4(\beta+1)^4} \right. \\ &\quad \left. - \frac{(1-\beta)}{2(\beta+1)^4 (\beta+2)} \right] - \frac{2p_1^2}{(\beta+1)^2} + p_1^2 Xx \left[ \frac{(1+\beta)(\beta+2) - (\beta+1)^2}{2(\beta+1)^2 (\beta+2)^2} \right] \\ &\quad - \frac{X^2 x^2}{4(\beta+2)^2} - \frac{2p_1^2}{(\beta+1)^2} \\ &= 1 + \frac{p_1^4}{4(\beta+1)^4 (\beta+2)^2} [4(\beta+1)^2 (\beta+2) + 4(1-\beta)(\beta+2)^2 - (\beta+1)^4 \\ &\quad - (1-\beta)^2 (\beta+2)^2 - 2(1-\beta)(\beta+1)^2 (\beta+2)] \\ &\quad - \frac{2p_1^2}{(\beta+1)^2} + p_1^2 Xx \left[ \frac{(1+\beta)(\beta+2) - (\beta+1)^2}{2(\beta+1)^2 (\beta+2)^2} \right] - \frac{X^2 x^2}{4(\beta+2)^2} - \frac{2p_1^2}{(\beta+1)^2} \\ &= 1 + \frac{3\beta^2 + 14\beta + 15}{4(\beta+1)^4 (\beta+2)^2} p_1^4 - \frac{2p_1^2}{(\beta+1)^2} + \frac{1}{2(\beta+1)(\beta+2)^2} p_1^2 xX - \frac{1}{4(\beta+2)^2} x^2 X^2. \end{aligned}$$

Without loss of generality, we let  $0 \leq p_1 = p \leq 2$ . Now substituting this into the above equation and applying the triangle inequality we obtain the following quadratic equation in terms of  $x$ .

$$\begin{aligned} |T_3(1)| &\leq \frac{(4-p^2)^2}{4(\beta+2)^2} |x|^2 + \frac{p^2(4-p^2)}{2(\beta+1)(\beta+2)^2} |x| + \left[ 1 + \frac{|(3\beta^2 + 14\beta + 15)p^2 - 8(\beta+1)^2(\beta+2)^2|}{4(\beta+1)^4(\beta+2)^2} p^2 \right] \\ &\leq \frac{(4-p^2)^2}{4(\beta+2)^2} + \frac{p^2(4-p^2)}{2(\beta+1)(\beta+2)^2} + \left[ 1 + \frac{|(3\beta^2 + 14\beta + 15)p^2 - 8(\beta+1)^2(\beta+2)^2|}{4(\beta+1)^4(\beta+2)^2} p^2 \right] \\ &= \Gamma(p, \beta). \end{aligned}$$



Differentiating  $\Gamma(p, \beta)$  with respect to  $p$  we obtain

$$\frac{\partial (\Gamma(p, \beta))}{\partial p} = \frac{p [p^2 (\beta^4 + 2\beta^3 + 3\beta^2 + 12\beta + 14) + 4\beta^3 + 13\beta^2 + 14\beta + 15]}{(\beta + 1)^4 (\beta + 2)^2}.$$

Setting  $\partial (\Gamma(p, \beta)) / \partial p = 0$  yields either  $p = 0$  or

$$p^2 = \frac{-4\beta^3 - 13\beta^2 - 14\beta - 15}{\beta^4 + 2\beta^3 + 3\beta^2 + 12\beta + 14}.$$

But  $-4\beta^3 - 13\beta^2 - 14\beta - 15 < 0$  for  $0 \leq \beta \leq 1$ . Therefore, the maximum of  $|T_3(1)|$  is attained at the end points  $p_1 = p \in [0, 2]$ .

For  $p_1 = 0$  we have  $a_2 = 0$  and  $a_3 = 1 - \frac{x^2}{4(\beta + 2)^2}$  which yields

$$|T_3(1)| = 1 + \frac{|x|^2}{4(\beta + 2)^2} \leq 1 + \frac{1}{4(\beta + 2)^2}.$$

For  $p_1 = 2$  we obtain

$$|T_3(1)| \leq \left| 1 + \frac{4(3\beta^2 + 14\beta + 15)}{(\beta + 1)^4 (\beta + 2)^2} - \frac{8}{(\beta + 1)^2} \right| \leq |M_4(\beta)|.$$

where

$$M_4(\beta) = \frac{\beta^6 + 8\beta^5 + 18\beta^4 - 4\beta^3 - 51\beta^2 - 20\beta + 32}{(\beta + 1)^4 (\beta + 2)^2}.$$

This completes the proof of Theorem 2.7.  $\square$

**Remark 2.8.** Theorem 2.5 for  $\beta = 0$  yields the bound  $|T_3(1)| \leq 8$  for the class of starlike functions  $S^*$  confirming the bound obtained by Thomas and Halim [1] and for  $\beta = 1$  yields the bound  $|T_3(1)| \leq 13/9$  for the class of functions with bounded boundary rotation  $\mathcal{R}$  confirming the bound obtained by Radhika et al. [2].

### Conflict of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgement:** The authors sincerely thank the referees for their insightful suggestions. The work of the third author is supported by a grant from Department of Science and Technology, Government of India vide ref: SR/FTP/MS-022/2012 under fast track scheme.

## References

- [1] Thomas D.K., Halim S.A., Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions, Bull. Malays. Math. Sci. Soc., 2016, DOI: 10.1007/s40840-016-0385-4 (published online)
- [2] Radhika V., Sivasubramanian S., Murugusundaramoorthy G., Jahangiri J.M., Toeplitz matrices whose elements are the coefficients of functions with bounded boundary rotation, J. Complex Analysis, 2016, Article ID4960704
- [3] Duren P.L., Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983, MR0708494
- [4] Al-Oboudi F.M.,  $n$ -Bazilevič functions, Abstr. Appl. Anal., 2012, Article ID 383592, 1–10
- [5] Amer A.A., Darus M., Distortion theorem for certain class of Bazilevič functions, Int. J. Math. Anal., 2012, **6**(9-12), 591–597, MR2881175
- [6] Kim Y.C., Srivastava H.M., The Hardy space of a certain subclass of Bazilevič functions, Appl. Math. Comput., 2006, **183**(2), 1201–1207, MR2294077
- [7] Kim Y.C., Sugawa T., A note on Bazilevič functions, Taiwanese J. Math., 2009, **13**(5) 1489–1495, MR2554472
- [8] Singh R., On Bazilevič functions, Proc. Amer. Math. Soc., 1973, **28**, 261–271, MR0311887
- [9] Ye K., Lim L-H., Every matrix is a product of Toeplitz matrices, Found. Comput. Math., 2016, **16**, 577–598, MR3494505
- [10] Libera R.J., Złotkiewicz E.J., Coefficient bounds for the inverse of a function with derivative in  $\mathcal{P}$ , Proc. Amer. Math. Soc., 1983, **87** (2), 251–257, MR0681830