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New topology in residuated lattices

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Abstract: In this paper, by using the notion of upsets in residuated lattices and defining the operator $D_a(X)$, for an upset X of a residuated lattice L we construct a new topology denoted by τ_a and (L, τ_a) becomes a topological space. We obtain some of the topological aspects of these structures such as connectivity and compactness. We study the properties of upsets in residuated lattices and we establish the relationship between them and filters. O. Zahiri and R. A. Borzooei studied upsets in the case of BL-algebras, their results become particular cases of our theory, many of them work in residuated lattices and for that we offer complete proofs. Moreover, we investigate some properties of the quotient topology on residuated lattices and some classes of semitopological residuated lattices. We give the relationship between two types of quotient topologies $\tau_{a/F}$ and $\bar{\tau}_a$. Finally, we study the uniform topology $\tau_{\bar{A}}$ and we obtain some conditions under which $(L/J, \tau_{\bar{A}})$ is a Hausdorff space, a discrete space or a regular space relative to the uniform topology. We discuss briefly the applications of our results on classes of residuated lattices such as *divisible residuated lattices*, *MV-algebras* and *involutive residuated lattices* and we find that any of this subclasses of residuated lattices with respect to these topologies form semitopological algebras.

Keywords: Residuated lattice, divisible residuated lattice, Semitopological algebra

MSC: 03G10, 03G25, 06D05, 06D35, 08A72

1 Introduction

Residuation is a fundamental concept of ordered structures and categories. The origin of residuated lattices is in Mathematical Logic without contraction. The general definition of a residuated lattice was given by Galatos et al. (2007)[1]. They first developed the structural theory of this kind of algebra about residuated lattices.

Hajek (1998)[2] introduced the notion of BL-algebras and the concepts of filters and prime filters in BL-algebras in order to provide an algebraic proof of the completeness theorem of *basic logic* (BL, for short), arising from the continuous triangular norms, familiar in the fuzzy logic frame-work. Using prime filters in BL-algebras, he proved the completeness of Basic Logic. Soon after, Turunen (1999)[3] published a study on BL-algebras and their deductive systems.

A weaker logic than BL called *Monoidal t-norm based logic* (MTL, for short) was defined by Esteva and Godo (2001) [4] and proved by Jenei and Montagna (2002)[5] to be the logic of left continuous t-norms and their residua. The algebraic counterpart of this logic is *MTL-algebra*, also introduced by Esteva and Godo (2001)[4]. In Esteva and Godo (2001)[4] a residuated lattice L is called *MTL-algebra* if the prelinearity property holds in L .

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Recently, Turunen and Mertanen (2008)[6] and D. Buşneag et al. (2013)[7] defined the notion of semidismissible residuated lattice and investigated their properties. Also, D. Buşneag et al. (2015) [8] investigated the notion of Stonean residuated lattices and they discussed it from the view of ideal theory.

Semitopological and topological BL-algebras were defined by R. A. Borzooei et al. (2011)[9], and they establish the relationships between them. A. Borumand Saeid and S. Motamed (2009)[10] introduced the set of double complemented elements for any filter F in BL-algebras and studied their properties. In Borzooei and Zahiri (2014)[11] the definition of double complemented elements for any filter F in BL-algebras was generalized to the concept of $D_\gamma(F)$, for any upset F of the BL-algebra L . In fact, they show that any BL-algebra L with that topology is a semitopological BL-algebra.

In the present paper, we work on a special type of topology induced by a modal and closure operator denoted by $D_a(X)$, for an upset X of a residuated lattice L , where a is an element of L . We show that any divisible residuated lattice L with this topology is a semitopological algebra.

The paper is organized as follows:

In Section 2, we recall the basic definitions and we put in evidence rules of calculus in a residuated lattice which we need in the rest of the paper.

In Section 3, we consider the definition of upsets in the case of residuated lattices and we establish the relationship between upsets and filters. We define the operator $D_a(X)$, for an upset X of a residuated lattice L and we construct a new topology denoted by τ_a , where (L, τ_a) becomes a topological space. We discuss briefly the properties and applications of the operator D_a in residuated lattices. Moreover, we obtain some of the topological aspects of these structures such as connectivity and compactness.

In Section 4, we investigate some properties of quotient topologies on residuated lattices as $\tau_{a/F}$ and $\bar{\tau}_a$. Finally, we study some properties of the direct product of residuated lattices. One of the most important findings is the characterization when $(L/J, \tau_{\bar{A}})$ becomes a Hausdorff space, a discrete space or a regular space relative to the uniform topology $\tau_{\bar{A}}$.

In Section 5, we apply our results on classes of residuated lattices such as *divisible residuated lattices*, *MV-algebras* and *involutive residuated lattices* and we find that any of this subclasses of residuated lattices with respect to these topologies form semitopological algebras.

2 Preliminaries

In this section, we recall some basic notions relevant to residuated lattices which will need in the sequel.

Definition 2.1 ([1]). *A residuated lattice is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that*
 (Lr_1) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice;*
 (Lr_2) $(L, \odot, 1)$ *is a commutative monoid;*
 (Lr_3) $\odot$ *and* $\rightarrow$ *form an adjoint pair, i.e.,* $a \odot x \leq b$ *iff* $x \leq a \rightarrow b$.

We call them simply *residuated lattices*. For examples of residuated lattices see [12–17].

In what follows (unless otherwise specified) we denote by L a residuated lattice. If L is totally ordered, then L is called a *chain*.

For $x \in L$ and $n \geq 1$ we define $x^* = x \rightarrow 0$, $x^{**} = (x^*)^*$, $x^0 = 1$ and $x^n = x^{n-1} \odot x$.

We refer to [3–21] for detailed proofs of these well-known results:

Lemma 2.2. *Let L be a residuated lattice. Then for every $x, y, z \in L$, we have:*

- (r_1) $x \rightarrow x = 1$, $x \rightarrow 1 = 1$, $1 \rightarrow x = x$;
- (r_2) $x \leq y$ *iff* $x \rightarrow y = 1$;
- (r_3) *If* $x \vee y = 1$, *then* $x \odot y = x \wedge y$;
- (r_4) *If* $x \leq y$, *then* $z \odot x \leq z \odot y$, $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$;
- (r_5) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z)$;

- $(r_6) \quad x \odot (y \vee z) = (x \odot y) \vee (x \odot z), \text{ and } x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z);$
 $(r_7) \quad (x \rightarrow z) \wedge (y \rightarrow z) = (x \vee y) \rightarrow z;$
 $(r_8) \quad (x \rightarrow z) \vee (y \rightarrow z) \leq (x \wedge y) \rightarrow z;$
 $(r_9) \quad x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z);$
 $(r_{10}) \quad (x \rightarrow y) \vee (x \rightarrow z) \leq x \rightarrow (y \vee z);$
 $(r_{11}) \quad x \vee y \leq ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x);$
 $(r_{12}) \quad (x \vee y)^* = x^* \wedge y^*, (x \wedge y)^* \geq x^* \vee y^*;$
 $(r_{13}) \quad (x \rightarrow y^{**})^{**} = x \rightarrow y^{**};$
 $(r_{14}) \quad x^{**} \rightarrow y^{**} = y^* \rightarrow x^* = x \rightarrow y^{**} = (x \rightarrow y^{**})^{**};$
 $(r_{15}) \quad x \odot x^* = 0, 1^* = 0, 0^* = 1, x^{***} = x^*;$
 $(r_{16}) \quad x \leq x^{**}, x^{**} \leq x^* \rightarrow x, x \rightarrow y \leq y^* \rightarrow x^*;$
 $(r_{17}) \quad x \rightarrow y \leq (x \rightarrow y)^{**} \leq x^{**} \rightarrow y^{**};$
 $(r_{18}) \quad x^{**} \odot y^{**} \leq (x \odot y)^{**}, \text{ so } (x^{**})^n \leq (x^n)^{**}$
 for every natural number n ;
 $(r_{19}) \quad x^* \odot y^* \leq (x \odot y)^*;$
 $(r_{20}) \quad (x \wedge y)^{**} \leq x^{**} \vee y^{**} \leq (x \vee y)^{**}.$

Following the above mentioned literature, we consider the identities:

- $(i_1) \quad x \wedge y = x \odot (x \rightarrow y) \quad (\text{divisibility});$
 $(i_2) \quad (x^* \wedge y^*)^* = [x^* \odot (x^* \rightarrow y^*)]^* \quad (\text{semi-divisibility});$
 $(i_3) \quad (x \rightarrow y) \vee (y \rightarrow x) = 1 \quad (\text{pre-linearity});$
 $(i_4) \quad x^* \vee x^{**} = 1;$
 $(i_5) \quad x^2 = x;$
 $(i_6) \quad x = x^{**}.$

Then the *residuated lattice* L is called:

- (i) *Divisible* if L verifies (i_1) ;
 (ii) *Semi-divisible* if L verifies (i_2) ;
 (iii) *MTL-algebra* if L verifies (i_3) ;
 (iv) *BL-algebra* if L verifies (i_1) and (i_3) ;
 (v) *Stonean* if L verifies (i_4) ;
 (v) *G-algebra* if L verifies (i_5) ;
 (vi) *Involutive* if L verifies (i_6) .

An *MV-algebra* is an algebra $\mathcal{L} = (L, \oplus, *, 0)$ of type $(2, 1, 0)$ satisfying the following equations:

- $(mv_1) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z;$
 $(mv_2) \quad x \oplus y = y \oplus x;$
 $(mv_3) \quad x \oplus 0 = x;$
 $(mv_4) \quad x^{**} = x;$
 $(mv_5) \quad x \oplus 0^* = 0^*;$
 $(mv_6) \quad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x, \text{ for all } x, y, z \in L.$

Note that axioms $(mv_1) - (mv_3)$ state that $(L, \oplus, *, 0)$ is a commutative monoid.

Every *BL-algebra* L with $x^{**} = x$ for all $x \in L$, is an *MV-algebra*.

By Lemma 2.2, in the case of divisible residuated lattices we have:

Corollary 2.3 ([7]). *If L is a divisible residuated lattice, then for every $x, y \in L$ we have:*

- $(r_{21}) \quad (x^{**} \rightarrow x)^* = 0;$
 $(r_{22}) \quad (x \rightarrow y)^{**} = x^{**} \rightarrow y^{**};$
 $(r_{23}) \quad (x \wedge y)^{**} = x^{**} \wedge y^{**};$
 $(r_{24}) \quad x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z);$
 $(r_{25}) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$

On any residuated lattice L ([6, 8]) we may define an operator $\oplus$ by setting for all $x, y \in L$,

$$x \oplus y = (x^* \odot y^*)^*. \quad (1)$$

By (r_5) , the identity (1) is equivalent with

$$x \oplus y = x^* \rightarrow y^{**} = y^* \rightarrow x^{**}, \text{ for all } x, y \in L. \quad (2)$$

Lemma 2.4 ([8]). *Let L be a residuated lattice and $x, y, z \in L$. Then:*

$$(r_{26}) \quad x \oplus 0 = x^{**}, \quad x \oplus 1 = 1, \quad x \oplus x^* = 1;$$

$$(r_{27}) \quad x \oplus y = y \oplus x, \quad x, y \leq x \oplus y;$$

$$(r_{28}) \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z;$$

$$(r_{29}) \quad \text{If } x \leq y, \text{ then } x \oplus z \leq y \oplus z.$$

In what follows we will establish other necessary properties of operator $\oplus$ in a residuated lattice L .

Proposition 2.5. *Let L be a residuated lattice and $x, y, z \in L$. Then:*

$$(r_{30}) \quad (x \oplus y)^{**} = x \oplus y = x^{**} \oplus y^{**};$$

$$(r_{31}) \quad \text{If } x \vee y = 1, \text{ then } x \oplus y = 1;$$

$$(r_{32}) \quad x \oplus (y \vee z)^* = (x \oplus y^*) \wedge (x \oplus z^*);$$

$$(r_{33}) \quad x \oplus (y^* \vee z^*)^* = (x \oplus y) \wedge (x \oplus z).$$

Proof. (r_{30}) . We obtain successively $(x \oplus y)^{**} = (x^* \odot y^*)^{***} \stackrel{(r_{15})}{=} (x^* \odot y^*)^* = x \oplus y \stackrel{(r_{15})}{=} [(x^{**})^* \odot (y^{**})^*]^* = x^{**} \oplus y^{**}$.

(r_{31}) . Since $x \leq x^{**}, y \leq y^{**}$ and $x^{**}, y^{**} \leq x \oplus y$ implies $x \vee y \leq x \oplus y$. Since $1 = x \vee y \leq x \oplus y$, then $x \oplus y = 1$.

(r_{32}) . By $(r_9), (r_{12}), (r_{15})$ and (2) we obtain successively $x \oplus (y \vee z)^* \stackrel{(2)}{=} x^* \rightarrow (y \vee z)^{***} \stackrel{(r_{15})}{=} x^* \rightarrow (y \vee z)^* \stackrel{(r_{12})}{=} x^* \rightarrow (y^* \wedge z^*) \stackrel{(r_9)}{=} (x^* \rightarrow y^*) \wedge (x^* \rightarrow z^*) \stackrel{(r_{14})}{=} (y^{**} \rightarrow x^{**}) \wedge (z^{**} \rightarrow x^{**}) \stackrel{(2)}{=} (x \oplus y^*) \wedge (x \oplus z^*)$.

(r_{33}) . By $(r_9), (r_{12}), (r_{15})$ and (2) we obtain successively $x \oplus (y^* \vee z^*)^* \stackrel{(2)}{=} x^* \rightarrow (y^* \vee z^*)^{***} \stackrel{(r_{15})}{=} x^* \rightarrow (y^* \vee z^*)^* \stackrel{(r_{12})}{=} x^* \rightarrow (y^{**} \wedge z^{**}) \stackrel{(r_9)}{=} (x^* \rightarrow y^{**}) \wedge (x^* \rightarrow z^{**}) \stackrel{(2)}{=} (x \oplus y) \wedge (x \oplus z)$. $\square$

Corollary 2.6. *Let L be a residuated lattice and $x, y \in L$. Then:*

$$(r_{34}) \quad (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x \text{ iff } (x^{**} \rightarrow y^{**})^{**} \rightarrow y^{**} = (y^{**} \rightarrow x^{**})^{**} \rightarrow x^{**}.$$

Proof. (r_{34}) . Since $(x^* \oplus y)^* \oplus y \stackrel{(2)}{=} (x^{**} \rightarrow y^{**})^* \oplus y \stackrel{(2)}{=} (x^{**} \rightarrow y^{**})^{**} \rightarrow y^{**}$ and $(y^* \oplus x)^* \oplus x \stackrel{(2)}{=} (y^{**} \rightarrow x^{**})^* \oplus x \stackrel{(2)}{=} (y^{**} \rightarrow x^{**})^{**} \rightarrow x^{**}$. Thus, our claim holds. $\square$

Deductive systems correspond to subsets closed with respect to Modus Ponens, their are also called filters, too. Apart from their logical interest, filters have important algebraic properties and they have been intensively studied from an algebraic point of view. Filter theory plays an important role in studying logical algebras.

We consider L as a residuated lattice.

Definition 2.7 ([2]). *An implicative filter (filter, for short) is a nonempty subset F of L such that*

$$(F_1) \quad \text{If } x \leq y \text{ and } x \in F, \text{ then } y \in F;$$

$$(F_2) \quad \text{If } x, y \in F, \text{ then } x \odot y \in F.$$

We notice that:

1. F is an implicative filter ([13]) of L iff $1 \in F$ and $x, x \rightarrow y \in F$, then $y \in F$ (that is, F is a *deductive system* of L).

2. Every filter is a latticeal filter in the lattice $(L, \wedge, \vee)$, but the converse does not hold ([3, 14]).

Hence, if we denote by $\mathcal{F}(L)$ ($\mathcal{F}_i(L)$) the set of all latticeal filters (implicative filters) of L , then $\mathcal{F}_i(L) \subseteq \mathcal{F}(L)$.

We have ([13]) $\mathcal{F}_i(L) = \mathcal{F}(L)$ iff $x \odot y = x \wedge y$ for every $x, y \in L$.

We recall ([3, 14]) that for a nonempty subset D of L we denote by $\langle D \rangle$ the filter generated by D , and $\langle D \rangle = \{x \in L : d_1 \odot \dots \odot d_n \leq x, \text{ for some } d_1, \dots, d_n \in D\}$. If $a \in L$, the filter generated by $\{a\}$ will be denoted by $\langle a \rangle$, and $\langle a \rangle = \{x \in L : a^n \leq x \text{ for some } n \geq 1\}$. If $F \in \mathcal{F}_i(L)$ and $a \in L \setminus F$, then $\langle F \cup \{a\} \rangle$ will be denoted by $F(a)$, and $F(a) = \{x \in L : a^n \rightarrow x \in F \text{ for some } n \geq 1\}$. If $F, G \in \mathcal{F}_i(L)$, then $F \vee G = F \vee_{\mathcal{F}_i(L)} G = \langle F \cup G \rangle = \{x \in L : b \odot c \leq x \text{ for some } b \in F, c \in G\}$.

We say that $P \in \mathcal{F}_i(L)$, $P \neq L$ is a *prime filter* ([14]) if for $x, y \in L$ and $x \vee y \in P$, then $x \in P$ or $y \in P$. We denote by $\text{Spec}_i(L)$ the set of all prime filters of L .

We recall that a filter M of L is called *maximal* if $M \neq L$ and M is not strictly contained in any proper filter of L .

Every *maximal filter* M of L is obvious *prime* because, if there exist two proper filters $N, P \in \mathcal{F}_i(L)$ such that $M = N \cap P$, then $M \subseteq N$ and $M \subseteq P$, by the maximality of M we deduce that $M = N = P$, that is, M is an *inf-irreducible*, so *prime* element in the lattice of filters $(\mathcal{F}_i(L), \subseteq)$ of L (by the distributivity of the lattice of filters $(\mathcal{F}_i(L), \subseteq)$ of L).

So, if we denote by $\text{Max}_i(L)$ the set of all maximal filters of L , then $\text{Max}_i(L) \subseteq \text{Spec}_i(L)$.

Proposition 2.8 ([1, 3, 14]). *For $M \in \mathcal{F}_i(L)$, $M \neq L$, the following are equivalent:*

- (i) M is maximal;
- (ii) If $x \notin M$, then there exists $n \geq 1$ such that $(x^n)^* \in M$.

For $F \in \mathcal{F}_i(L)$ we define a relation $\equiv_F$ on L by $x \equiv_F y$ iff $x \rightarrow y, y \rightarrow x \in F$, for all $x, y \in L$ iff $(x \rightarrow y) \odot (y \rightarrow x) \in F$.

Then ([3, 14]) $\equiv_F$ is a congruence relation on L . For $x \in L$ we denote by $[x] = x/F$ the class of congruence of x modulo $\equiv_F$ and $L/F = \{x/F : x \in L\}$.

Define the binary operations $\vee, \wedge, \odot$ and $\rightarrow$ on L/F by $(x/F) \vee (y/F) = (x \vee y)/F$, $(x/F) \wedge (y/F) = (x \wedge y)/F$, $(x/F) \odot (y/F) = (x \odot y)/F$ and $(x/F) \rightarrow (y/F) = (x \rightarrow y)/F$ for all $x, y \in L$.

Then $(L/F, \vee, \wedge, \odot, \rightarrow, \mathbf{0}, \mathbf{1})$ is a residuated lattice, which is called the *quotient residuated lattice of L with respect to F* , where $\mathbf{0} = 0/F$ and $\mathbf{1} = 1/F$.

The relation of order on L/F is defined by $(x/F) \leq (y/F)$ iff $x \rightarrow y \in F$.

For a nonempty subset S of L we denote by $S/F = \{x/F : x \in S\}$. Clearly, for $x \in L$, $x/F = \mathbf{0}$ iff $x^* \in F$ and $x/F = \mathbf{1}$ iff $x \in F$.

We denote by $\pi : L \rightarrow L/F$ the canonical epimorphism defined by $\pi(x) = [x]$.

Definition 2.9. If L, L' are residuated lattices, a map $f : L \rightarrow L'$ is called *morphism of residuated lattices* if f is a morphism of bounded lattices, $f(x \odot y) = f(x) \odot f(y)$ and $f(x \rightarrow y) = f(x) \rightarrow f(y)$ for every $x, y \in L$.

If f is a bijective map (one-to-one and onto), then we say that L and L' are isomorphic and we write $L \approx L'$.

Remark 2.10. If $F \in \mathcal{F}_i(L')$, then $f^{-1}(F) \in \mathcal{F}_i(L)$. In particular $f^{-1}(\{1\}) = \{x \in L : f(x) = 1\}$ is denoted by $\text{Ker}(f)$ and $\text{Ker}(f) \in \mathcal{F}_i(L)$.

Clearly, f is one-to-one iff $\text{Ker}(f) = \{1\}$. Also, $f(L)$ is a subalgebra of L' denoted by $\text{Im}(f)$.

3 Construction of some topologies on residuated lattices

In general, the concept of topology represents the study of topological spaces. Important topological properties include *connectedness* and *compactness*.

A topology tells how elements of a set relate spatially to each other. The same set can have different topologies. For instance, the real line, the complex plane, and the Cantor set can be thought of as the same set with different topologies.

Let X be a set and let τ be a family of subsets of X . We denote by $\mathcal{P}(X)$ the family of all subsets of X . Then τ is called a topology on X if:

T₁. Both the empty set $\emptyset$ and X are elements of τ ;

- T₂**. Any union of elements of τ is an element of τ ;
T₃. Any intersection of finitely many elements of τ is an element of τ .

If τ is a topology on X , then the pair (X, τ) is called a *topological space*. The notation X_τ is used to denote a set X endowed with the particular topology τ .

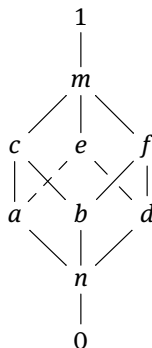
The members of τ are called *open sets* in X . A subset of X is said to be *closed* if its complement is in τ (i.e., its complement is open). A subset of X may be open, closed, both (clopen set), or neither. The empty set $\emptyset$ and X itself are always both closed and open. An open set containing a point x is called a *neighborhood* of x .

A set with a topology is called a *topological space*. A topological space (X, τ) is called *connected* if $\{\emptyset, X\}$ is the set of all closed and open subsets of X .

A *base* (or *basis*) β for a topological space X with topology τ is a collection of open sets in τ such that every open set in τ can be written as a union of elements of β . We say that the base generates the topology τ .

Definition 3.1 ([11]). Let $(X, \leq)$ be an ordered set. Then we define $\uparrow: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, by $\uparrow S = \{x \in X \mid a \leq x, \text{ for some } a \in S\}$, for any subset S of X . A subset F of X is called an *upset* if $\uparrow F = F$. We denote by $U(X)$ the set of all upsets of X . An upset F is called *finitely generated* if there exists $n \in N$ such that $F = \uparrow \{x_1, x_2, \dots, x_n\}$, for some $x_1, x_2, \dots, x_n \in X$.

Remark 3.2. Clearly, if F is a filter of L , then F is an upset of L , but the converse does not hold. Indeed, we consider $L = \{0, n, a, b, c, d, e, f, m, 1\}$ with $0 < n < a < c < m < 1$, $0 < n < a < e < m < 1$, $0 < n < b < c < m < 1$, $0 < n < b < f < m < 1$, $0 < n < d < e < m < 1$ and elements $\{a, b\}$, $\{a, d\}$, $\{b, d\}$, $\{a, f\}$, $\{c, d\}$, $\{b, e\}$, $\{c, e\}$, $\{c, f\}$ and $\{e, f\}$ are pairwise incomparable.



Then ([12]) L becomes a residuated lattice relative to the following operations :

$\rightarrow$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	1	1	e	e	1	1	1	1
c	d	d	e	f	1	d	e	f	1	1
d	c	c	c	c	c	1	1	1	1	1
e	b	b	c	b	c	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	b	b	0	0	b	b	b
c	0	0	a	b	c	0	a	b	c	c
d	0	0	0	0	0	d	d	d	d	d
e	0	0	a	0	a	d	e	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

Then $\uparrow n = \{n, a, b, c, d, e, f, m, 1\}$ is an upset, but $n \odot n = 0$, that is $\uparrow n$ is not a filter.

In [11], O. Zahiri and R. A. Borzooei construct a topology on BL-algebras considering the notion of upsets, and inspired by their work we construct a topology in the case of residuated lattices and some of their results will become particular cases. We offer complete proofs for the results available in residuated lattices.

Definition 3.3 ([11]). Let τ and τ' be two topologies on a given set X . If $\tau' \subseteq \tau$, then we say that τ' is finer than τ . Let (X, τ) and (Y, τ') be two topological spaces. A map $f : X \rightarrow Y$ is called continuous if the inverse image of each open set of Y is open in X . A homeomorphism is a continuous function, bijective and has a continuous inverse.

Lemma 3.4 ([11]). Let β and β' be two bases for topologies τ and τ' , respectively on X . Then the following are equivalent:

- (i) τ' is finer than τ ;
- (ii) For any $x \in X$ and each basis element $B \in \beta$ containing x , there is a basis element $B' \in \beta'$ such that $x \in B' \subseteq B$.

Definition 3.5 ([11]). Let $(A, *)$ be an algebra of type 2 and τ be a topology on A . Then $(A, *, \tau)$ is called:

- (i) Right (left) topological algebra, if for all $a \in A$ the map $*$: $A \rightarrow A$ defined by $x \mapsto a * x$ ($x \mapsto x * a$) is continuous;
- (ii) Semitopological algebra if A is a right and left topological algebra.

If $(A, *)$ is a commutative algebra, then right and left topological algebras are equivalent.

Definition 3.6 ([11]). Let A be a nonempty set, $\{*_i\}_{i \in I}$ be a family of binary operations on A and τ be a topology on A . Then:

- (i) $(A, \{*_i\}_{i \in I}, \tau)$ is a right (left) topological algebra, if for $i \in I$, $(A, *_i, \tau)$ is a right (left) topological algebra;
- (ii) $(A, \{*_i\}_{i \in I}, \tau)$ is a right (left) semitopological algebra, if for $i \in I$, $(A, *_i, \tau)$ is a right (left) semitopological algebra.

On any residuated lattice L we define an operator $\ominus$ by setting for all $x, y \in L$,

$$x \ominus y = x^* \oplus y. \quad (3)$$

By (r_5) , the identity (3) is equivalent with

$$x \ominus y = x^* \oplus y = x^{**} \rightarrow y^{**} = y^* \rightarrow x^*, \text{ for all } x, y \in L. \quad (4)$$

Definition 3.7. Consider L a residuated lattice and $a \in L$. For any nonempty upset X of L we define the set

$$D_a(X) = \{x \in L \mid a^n \ominus x \in X, \text{ for some } n \in \mathbb{N}\},$$

where $a^n \ominus x = a \ominus (a^{n-1} \ominus x)$, for any $n \in \{2, 3, 4, \dots\}$.

Lemma 3.8. Let L be a residuated lattice and $a, x \in L$. Then:

- (r_{35}) $a^n \ominus x = (a^{**})^n \rightarrow x^{**}$;
- (r_{36}) $(a^n \ominus x)^{**} = a^n \ominus x$, for any $n \in \mathbb{N}$.

Proof. (r_{35}) . Mathematical induction relative to n .

If $n = 2$, by (r_5) , (r_{30}) and identity (4) we obtain successively $a^2 \ominus x = a \ominus (a \ominus x) \stackrel{id.(4)}{=} a^{**} \rightarrow (a^* \oplus x)^{**} \stackrel{(r_{30})}{=} a^{**} \rightarrow (a^* \oplus x) \stackrel{id.(4)}{=} a^{**} \rightarrow (a^{**} \rightarrow x^{**}) \stackrel{(r_5)}{=} (a^{**} \odot a^{**}) \rightarrow x^{**} = (a^{**})^2 \rightarrow x^{**}$.

Suppose $a^{n-1} \ominus x = (a^{**})^{n-1} \rightarrow x^{**}$. By (r_5) we obtain successively $a^n \ominus x = a \ominus (a^{n-1} \ominus x) = a^{**} \rightarrow ((a^{**})^{n-1} \rightarrow x^{**}) \stackrel{(r_5)}{=} (a^{**} \odot (a^{**})^{n-1}) \rightarrow x^{**} = (a^{**})^n \rightarrow x^{**}$. Therefore, $a^n \ominus x = (a^{**})^n \rightarrow x^{**}$, for any $n \in \mathbb{N}$.

(r_{36}) . By (r_{30}) we have $(a^n \ominus x)^{**} = [a \ominus (a^{n-1} \ominus x)]^{**} \stackrel{id.(4)}{=} [a^* \oplus (a^{n-1} \ominus x)]^{**} \stackrel{(r_{30})}{=} a^* \oplus (a^{n-1} \ominus x) \stackrel{id.(4)}{=} a \ominus (a^{n-1} \ominus x) = a^n \ominus x$, for any $n \in \mathbb{N}$. $\square$

A directly consequence of (r_{35}) is the following result:

Proposition 3.9. Let L be a residuated lattice and $a \in L$. For any nonempty upset X of L the set $D_a(X) = \{x \in L : (a^{**})^n \rightarrow x^{**} \in X, \text{ for some } n \in \mathbb{N}\}$.

Remark 3.10. If $a = 1$, then $D_a(X) = D_1(X) = \{x \in L : x^{**} \in X\}$ is the set of double complemented elements of X .

Corollary 3.11. If L is a residuated lattice and $a, x \in L$, then:

$$(r_{37}) \quad a \odot x \leq a \wedge x \leq a \vee x \leq a^{**} \vee x^{**} \leq a \oplus x;$$

$$(r_{38}) \quad x^{**} \leq a \odot x \leq a^n \odot x, \text{ for any } n \in \mathbb{N}.$$

Proof. (r_{37}) . Indeed, $a \odot x \leq a \wedge x \leq a \vee x \leq a^{**} \vee x^{**}$. By (r_{27}) and identity (2) we obtain successively $a \oplus x \stackrel{id.(2)}{=} a^* \rightarrow x^{**} \geq x^{**}$ and $a \oplus x \stackrel{(r_{27})}{=} x \oplus a \stackrel{id.(2)}{=} x^* \rightarrow a^{**} \geq a^{**}$, hence $a^{**} \vee x^{**} \leq a \oplus x$.

(r_{38}) . Since $a^n \odot x \stackrel{(r_{35})}{=} (a^{**})^n \rightarrow x^{**} \stackrel{(r_a)}{\geq} a^{**} \rightarrow x^{**} = a \odot x \geq x^{**}$, hence $x^{**} \leq a \odot x \leq a^n \odot x$, for any $n \in \mathbb{N}$. $\square$

The following properties hold for any residuated lattice:

$$(r_{39}) \quad z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y) \text{ and } z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x);$$

$$(r_{40}) \quad (x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z.$$

In the next result, we investigate some properties of the operator $D_a(X)$ for nonempty upsets. Clearly, $D_a(\emptyset) = \emptyset$.

In Proposition 3.3, [11], there are the properties of operator D_a for BL-algebras, we offer complete proofs for them in the case of residuated lattices:

Theorem 3.12. If L is a residuated lattice and $a, x \in L$. Consider X, Y two nonempty upsets of L . Then:

(i) $D_a(X)$ is an upset of L ;

(ii) $1 \in D_a(X)$, $a \in D_a(X)$ and $X \subseteq D_a(X)$;

(iii) $a^m \odot (a^n \odot x) = a^{m+n} \odot x$, for any $m, n \in \mathbb{N}$;

(iv) if $X \subseteq Y$, then $D_a(X) \subseteq D_a(Y)$;

(v) $D_a(D_a(X)) = D_a(X)$;

(vi) if F is a filter of L , then $D_a(F)$ is a filter of L ;

(vii) if $a \leq s$, then $D_s(X) \subseteq D_a(X)$;

(viii) if $\{X_\alpha | \alpha \in I\}$ is a family of upsets of L , then $D_a(\cup\{X_\alpha | \alpha \in I\}) = \cup(D_a(\{X_\alpha | \alpha \in I\}))$;

(ix) if $\{X_1, X_2, \dots, X_n\}$ is a finite set of upsets of L , then $D_a(\cap\{X_i | i = 1, 2, \dots, n\}) = \cap\{D_a(X_i) | i = 1, 2, \dots, n\}$;

(x) $D_a(D_x(X)) = D_x(D_a(X))$;

(xi) if X_1, X_2, X_3 are upsets of L , then $D_a(X_1) \cap [D_a(X_2) \cup D_a(X_3)] = [D_a(X_1) \cap D_a(X_2)] \cup [D_a(X_1) \cap D_a(X_3)]$ and $D_a(X_1) \cup [D_a(X_2) \cap D_a(X_3)] = [D_a(X_1) \cup D_a(X_2)] \cap [D_a(X_1) \cup D_a(X_3)]$.

Proof. (i). Clearly, $D_a(X) \subseteq \uparrow D_a(X)$. We consider $s \in \uparrow D_a(X)$, then there exists $x \in D_a(X)$ such that $x \leq s$. Since $x \in D_a(X)$ we have $a^n \odot x \in X$, by (r_{35}) we deduce that $(a^{**})^n \rightarrow x^{**} \in X$, for some $n \in \mathbb{N}$. By (r_4) and (r_{16}) , since $x \leq s$ we obtain successively $x^{**} \leq s^{**}$, $(a^{**})^n \rightarrow x^{**} \leq (a^{**})^n \rightarrow s^{**}$. Since X is an upset of L and $(a^{**})^n \rightarrow x^{**} \in X$, then $(a^{**})^n \rightarrow s^{**} \in X$, hence $s \in D_a(X)$ and so $\uparrow D_a(X) \subseteq D_a(X)$. We deduce that $D_a(X)$ is an upset of L .

(ii). Since $1 \in X$, by (r_{35}) we have $a^n \odot 1 = (a^{**})^n \rightarrow 1^{**} = (a^{**})^n \rightarrow 1 = 1 \in X$, for any $n \in \mathbb{N}$. Hence $1 \in D_a(X)$.

Since $1 \in X$, by (r_{18}) and (r_{35}) we have $a^n \odot a = (a^{**})^n \rightarrow a^{**} = 1 \in X$, for any $n \in \mathbb{N}$. Hence $a \in D_a(X)$.

Now, we consider $x \in X$. Then by (r_{16}) and (r_{38}) we have $x \leq x^{**} \leq a^n \odot x \in X$, hence $x \in D_a(X)$. We deduce that $X \subseteq D_a(X)$.

(iii). Consider $m, n \in \mathbb{N}$ and $a, x \in L$. By (r_5) and (r_{36}) we obtain successively $a^m \odot (a^n \odot x) \stackrel{id.(4)}{=} (a^{**})^m \rightarrow (a^n \odot x)^{**} \stackrel{(r_{36})}{=} (a^{**})^m \rightarrow (a^n \odot x) \stackrel{id.(4)}{=} (a^{**})^m \rightarrow ((a^{**})^n \rightarrow x^{**}) \stackrel{(r_5)}{=} [(a^{**})^m \odot (a^{**})^n] \rightarrow x^{**} = (a^{**})^{m+n} \rightarrow x^{**} \stackrel{id.(4)}{=} a^{m+n} \odot x$.

(iv). Consider $X \subseteq Y$ and $x \in D_a(X)$. Then there exists $n \in \mathbb{N}$ such that $a^n \odot x \in X \subseteq Y$ and so $x \in D_a(Y)$. Hence $D_a(X) \subseteq D_a(Y)$.

(v). Since $X \subseteq D_a(X)$, by (iv) we have $D_a(X) \subseteq D_a(D_a(X))$. Consider $x \in D_a(D_a(X))$. Then there exists $m \in \mathbb{N}$ such that $a^m \odot x \in D_a(X)$ and so $a^n \odot (a^m \odot x) \in X$, for some $n \in \mathbb{N}$. By (iii) we have $a^{n+m} \odot x \in X$ and so $x \in D_a(X)$. Hence $D_a(X) = D_a(D_a(X))$.

(vi). Consider F a filter of L . Then F is a nonempty upset and by (ii) we have $1 \in D_a(F)$. Let $x, x \rightarrow y \in D_a(F)$, then there are $m, n \in \mathbb{N}$ such that $(a^{**})^n \rightarrow x^{**} \in F$ and $(a^{**})^m \rightarrow (x \rightarrow y)^{**} \in F$. By (r_5) and (r_{17}) we obtain successively $((a^{**})^n \rightarrow x^{**}) \rightarrow ((a^{**})^{n+m} \rightarrow y^{**}) = ((a^{**})^n \rightarrow x^{**}) \rightarrow ((a^{**})^n \odot (a^{**})^m \rightarrow y^{**}) \stackrel{(r_5)}{=} (a^{**})^m \rightarrow [((a^{**})^n \rightarrow x^{**}) \rightarrow ((a^{**})^n \rightarrow y^{**})] \stackrel{(r_{39})}{\geq} (a^{**})^m \rightarrow (x^{**} \rightarrow y^{**}) \stackrel{(r_{17})}{\geq} (a^{**})^m \rightarrow (x \rightarrow y)^{**} \in F$.

Since F is a filter and $(a^{**})^n \rightarrow x^{**} \in F$, then $(a^{**})^{n+m} \rightarrow y^{**} \in F$, hence $y \in D_a(F)$. We deduce that $D_a(F)$ is a filter of L .

(vii). Consider $a \leq s$ and $x \in D_s(X)$, then there exists $n \in \mathbb{N}$ such that $(s^{**})^n \rightarrow x^{**} \in X$. By (r_4) we obtain successively $a \leq s$, $a^{**} \leq s^{**}$, $(a^{**})^n \leq (s^{**})^n$, $(s^{**})^n \rightarrow x^{**} \leq (a^{**})^n \rightarrow x^{**}$. Since $(s^{**})^n \rightarrow x^{**} \in X$, then $(a^{**})^n \rightarrow x^{**} \in X$, hence $x \in D_a(X)$. We deduce that $D_s(X) \subseteq D_a(X)$.

(viii). Consider $x \in L$. Then we have the equivalences:

$$\begin{aligned} x \in D_a(\cup\{X_\alpha | \alpha \in I\}) &\text{ iff } a^n \odot x \in \cup\{X_\alpha | \alpha \in I\} \text{ iff} \\ a^n \odot x \in X_\alpha, &\text{ for some } n \in \mathbb{N}, \text{ and } \alpha \in I \text{ iff} \\ x \in D_a(X_\alpha), &\text{ for some } n \in \mathbb{N} \text{ and } \alpha \in I \text{ iff} \\ x \in \cup(D_a(\{X_\alpha | \alpha \in I\})). &\text{ Hence } D_a(\cup\{X_\alpha | \alpha \in I\}) = \\ \cup(D_a(\{X_\alpha | \alpha \in I\})). \end{aligned}$$

(ix). Following (iv), since $\cap\{X_i | i = 1, 2, \dots, n\} \subseteq X_i$, for any $i \in \{1, 2, \dots, n\}$, then $D_a(\cap\{X_i | i = 1, 2, \dots, n\}) \subseteq D_a(X_i)$, for any $i \in \{1, 2, \dots, n\}$, hence $D_a(\cap\{X_i | i = 1, 2, \dots, n\}) \subseteq \cap\{D_a(X_i) | i = 1, 2, \dots, n\}$. Consider $x \in \cap\{D_a(X_i) | i = 1, 2, \dots, n\}$. Then there exist $m_1, m_2, \dots, m_n \in \mathbb{N}$ such that $(a^{**})^{m_i} \rightarrow x^{**} \in X_i$, for any $i \in \{1, 2, \dots, n\}$.

Consider now $p = \max\{m_1, m_2, \dots, m_n\}$. By (r_4) , since $m_i \leq p$, for any $i \in \{1, 2, \dots, n\}$, then $(a^{**})^{m_i} \rightarrow x^{**} \leq (a^{**})^p \rightarrow x^{**}$. Since $(a^{**})^{m_i} \rightarrow x^{**} \in X_i$, then $(a^{**})^p \rightarrow x^{**} \in X_i$, for any $i \in \{1, 2, \dots, n\}$. It follows that $(a^{**})^p \rightarrow x^{**} \in \cap\{X_i | i = 1, 2, \dots, n\}$, hence $x \in D_a(\cap\{X_i | i = 1, 2, \dots, n\})$. We deduce that $D_a(\cap\{X_i | i = 1, 2, \dots, n\}) = \cap\{D_a(X_i) | i = 1, 2, \dots, n\}$.

(x). We have successively:

$$\begin{aligned} D_a(D_x(X)) &= \{t \in L | a^n \odot t \in D_x(X), \text{ for some } n \in \mathbb{N}\} = \\ &= \{t \in L | x^m \odot (a^n \odot t) \in X, \text{ for some } m, n \in \mathbb{N}\} \\ &\stackrel{id.(4)}{=} \{t \in L | (x^{**})^m \rightarrow (a^n \odot t)^{**} \in X, \text{ for some } m, n \in \mathbb{N}\} \\ &\stackrel{(r_{36})}{=} \{t \in L | (x^{**})^m \rightarrow (a^n \odot t) \in X, \text{ for some } m, n \in \mathbb{N}\} \\ &\stackrel{id.(4)}{=} \{t \in L | (x^{**})^m \rightarrow [(a^{**})^n \rightarrow t^{**}] \in X, \text{ for some } m, n \in \mathbb{N}\} \\ &\stackrel{(r_5)}{=} \{t \in L | (a^{**})^n \rightarrow [(x^{**})^m \rightarrow t^{**}] \in X, \text{ for some } m, n \in \mathbb{N}\} \\ &\stackrel{id.(4)}{=} \{t \in L | a^n \odot (x^m \odot t) \in X, \text{ for some } m, n \in \mathbb{N}\} \\ &= \{t \in L | x^m \odot t \in D_a(X), \text{ for some } m \in \mathbb{N}\} = D_x(D_a(X)). \end{aligned}$$

(xi). It follows from (viii) and (ix). $\square$

In universal algebra, for a nontrivial lattice A , a unary mapping $f : P(A) \rightarrow P(A)$ is called *latticeal modal operator* on A if it satisfies the conditions: For all $A_1, A_2 \subseteq A$,

- (1) $A_1 \subseteq f(A_1)$;
- (2) $f(f(A_1)) = f(A_1)$;
- (3) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$.

A latticeal modal operator is called *monotone* if it satisfies:

- (m) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$.

Following Theorem 3.12 (ii), (iv), (v), and (viii) we deduce:

Corollary 3.13. *Let $a \in L$. Then the map $D_a : U(L) \rightarrow U(L)$ is a latticeal monotone modal operator.*

Proof. Following Theorem 3.12 (ii), (iv), (v), and (viii) we deduce that the map $D_a : U(L) \rightarrow U(L)$ is a lattice monotone modal operator. $\square$

We recall that, for a residuated lattice L , by $U(L)$ we denote the set of all upsets of L .

Corollary 3.14. *For any $a \in L$, the map $D_a : U(L) \rightarrow U(L)$ is a closure operator and $D_a = D_{a^{**}}$.*

Proof. Following Theorem 3.12 (i), (ii), (iv) and (v), we deduce that D_a is a closure operator.

Consider X an upset of L . Following Theorem 3.12 (vii), since by (r_{16}) we have $a \leq a^{**}$, then $D_{a^{**}}(X) \subseteq D_a(X)$. Consider now $x \in D_a(X)$. Then there exists $n \in \mathbb{N}$ such that $(a^{**})^n \rightarrow x^{**} \in X$. By (r_{15}) we have $a^{****} = a^{**}$, then $(a^{****})^n \rightarrow x^{**} \in X$, hence $x \in D_{a^{**}}(X)$. We deduce that $D_a(\cdot) = D_{a^{**}}(\cdot)$. $\square$

Lemma 3.15. *Let $a \in L$ and F be a filter of L . Then $D_a(F) = D_1(F)$ iff $a^{**} \in F$.*

Proof. " $\Rightarrow$ ". Consider $a \in L$ and F a filter of L such that $D_a(F) = D_1(F)$. Following Theorem 3.12 (ii), we have $a \in D_a(F) = D_1(F)$, then $a \in D_1(F)$, and we have successively $a \in D_1(F)$, $(1^{**})^n \rightarrow a^{**} \in F$, $1^n \rightarrow a^{**} \in F$, $1 \rightarrow a^{**} \in F$, $a^{**} \in F$.

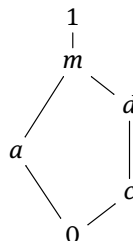
" $\Leftarrow$ ". Suppose $a^{**} \in F$. By Theorem 3.12 (vii), since $a \leq 1$, then $D_1(F) \subseteq D_a(F)$. Consider now $x \in D_a(F)$. Then there exists $n \in \mathbb{N}$ such that $(a^{**})^n \rightarrow x^{**} \in F$. Since $a^{**} \in F$ and F is a filter of L , then $(a^{**})^n \in F$, for any $n \in \mathbb{N}$. By $(a^{**})^n \in F$, $(a^{**})^n \rightarrow x^{**} \in F$, we deduce $x^{**} \in F$. Since $x^{**} = (1^{**})^n \rightarrow x^{**} \in F$, then $x \in D_1(F)$, hence $D_a(F) \subseteq D_1(F)$. We deduce that $D_a(F) = D_1(F)$. $\square$

Corollary 3.16. *Let L be an involutive residuated lattice, $a \in L$ and F be a filter of L . Then $D_a(F) = F$ iff $a \in F$.*

Proof. " $\Rightarrow$ ". Consider $a \in L$ and F a filter of L such that $D_a(F) = F$. Following Theorem 3.12 (ii), we have $a \in D_a(F) = F$, then $a \in F$.

" $\Leftarrow$ ". Suppose $a \in F$. By Theorem 3.12 (ii), $a \in F \subseteq D_a(F)$. Consider now $x \in D_a(F)$. Then there exists $n \in \mathbb{N}$ such that $(a^{**})^n \rightarrow x^{**} \in F$. Since $a \leq a^{**} \in F$ and F is a filter of L , then $(a^{**})^n \in F$, for any $n \in \mathbb{N}$. By $(a^{**})^n \in F$, $(a^{**})^n \rightarrow x^{**} \in F$, we deduce $x^{**} \in F$, hence $x \in F$. Therefore, $D_a(F) \subseteq F$. We deduce that $D_a(F) = F$. $\square$

Remark 3.17. (i). By Theorem 3.12 (vi), if F is a proper filter, then $D_a(F)$ is a proper filter or $D_a(F) = L$. We consider the lattice $L = \{0, a, c, d, m, 1\}$ with $0 < a < m < 1$, $0 < c < d < m < 1$, but a incomparable with c and d .



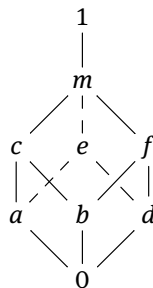
Then ([12], page 233) L becomes a residuated lattice relative to the following operations:

$\rightarrow$	0	a	c	d	m	1	$\odot$	0	a	c	d	m	1
0	1	1	1	1	1	1	0	0	0	0	0	0	0
a	d	1	d	d	1	1	a	0	a	0	0	a	a
c	a	a	1	1	1	1	c	0	0	c	c	c	c
d	a	a	m	1	1	1	d	0	0	c	c	c	d
m	0	a	d	d	1	1	m	0	a	c	c	m	m
1	0	a	c	d	m	1	1	0	a	c	d	m	1

We have $\langle a \rangle = \{a, m, 1\}$ is a proper filter of L , $D_m(\langle a \rangle) = \{x \in L \mid (m^{**})^n \rightarrow x^{**} \in \langle a \rangle, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid 1 \rightarrow x^{**} \in \langle a \rangle\} = \langle a \rangle$ and $D_c(\langle a \rangle) = \{x \in L \mid (c^{**})^n \rightarrow x^{**} \in \langle a \rangle, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid c \rightarrow x^{**}\} = L$.

Moreover, $\langle a \rangle$ is a maximal filter of L and $D_c(\langle a \rangle) = L \neq \langle a \rangle$. So, if M is a maximal filter, then $D_x(M) = M$ or $D_x(M) = L$, for any $x \in L$.

(ii). There are residuated lattices L such that for a proper filter F of L , there is $a \in L$ and $D_a(F) = L$. Indeed, we consider $L = \{0, a, b, c, d, e, f, m, 1\}$ with $0 < a < c < m < 1$, $0 < a < e < m < 1$, $0 < b < c < m < 1$, $0 < b < f < m < 1$, $0 < d < e < m < 1$ and elements $\{a, b\}$, $\{a, f\}$, $\{a, d\}$, $\{b, d\}$, $\{b, e\}$, $\{d, c\}$, $\{c, e\}$, $\{c, f\}$ and $\{e, f\}$ are pairwise incomparable.

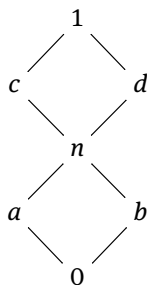


Then ([12]) L is a residuated lattice with the following operations:

$\rightarrow$	0	a	b	c	d	e	f	m	1	$\odot$	0	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
a	m	1	m	1	m	1	m	1	1	a	0	0	0	0	0	0	0	0	a
b	m	m	1	1	m	m	1	1	1	b	0	0	0	0	0	0	0	0	b
c	m	m	m	1	m	m	m	1	1	c	0	0	0	0	0	0	0	0	c
d	m	m	m	m	1	1	1	1	1	d	0	0	0	0	0	0	0	0	d
e	m	m	m	m	m	1	m	1	1	e	0	0	0	0	0	0	0	0	e
f	m	m	m	m	m	m	1	1	1	f	0	0	0	0	0	0	0	0	f
m	m	m	m	m	m	m	m	1	1	m	0	0	0	0	0	0	0	0	m
1	0	a	b	c	d	e	f	m	1	1	0	a	b	c	d	e	f	m	1

Let $F = \langle 1 \rangle = \{1\}$. Since $D_m(\langle 1 \rangle) = \{x \in L \mid (m^{**})^n \rightarrow x^{**} \in \langle 1 \rangle, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid 0 \rightarrow x^{**} \in \langle 1 \rangle\} = \{x \in L \mid 1 \in \langle 1 \rangle\} = L$.

(iii). There are residuated lattices L such that for a prime filter F , there is $a \in L$ such that $D_a(F)$ is not prime. Indeed, we consider $L = \{0, a, b, n, c, d, 1\}$ with $0 < a < n < d < 1$, $0 < b < n < c < 1$, but (a, b) and (c, d) are pairwise incomparable.

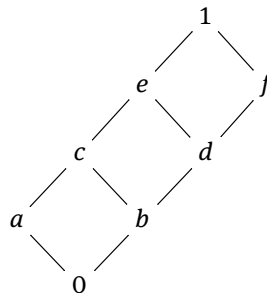


Then ([12], page 191) L becomes a residuated lattice relative to the operations:

$\rightarrow$	0	a	b	n	c	d	1	$\odot$	0	a	b	n	c	d	1
0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
a	b	1	b	1	1	1	1	a	0	a	0	a	a	a	a
b	a	a	1	1	1	1	1	b	0	0	b	b	b	b	b
n	0	a	b	1	1	1	1	n	0	a	b	n	n	n	n
c	0	a	b	d	1	d	1	c	0	a	b	n	c	n	c
d	0	a	b	c	c	1	1	d	0	a	b	n	n	d	d
1	0	a	b	n	c	d	1	1	0	a	b	n	c	d	1

Clearly, $\langle c \rangle = \{c, 1\}$ is a prime filter of L . Since $D_n(\langle c \rangle) = \{x \in L \mid (n^{**})^m \rightarrow x^{**} \in \langle c \rangle, \text{ for some } m \in \mathbb{N}\} = \{x \in L \mid 1 \rightarrow x^{**} \in \langle c \rangle\} = \{x \in L \mid x^{**} \in \langle c \rangle\} = \{n, c, d, 1\}$, hence $D_n(\langle c \rangle) = \{n, c, d, 1\}$. Since $n = a \vee b$, but $a \notin D_n(\langle c \rangle)$ and $b \notin D_n(\langle c \rangle)$, we deduce $D_n(\langle c \rangle)$ is not prime.

(iv). There are residuated lattices L such that for a filter F , there is $a \in L$ such that $D_a(F)$ is prime, but F is not prime. Indeed, we consider $L = \{0, a, b, c, d, e, f, 1\}$ with $0 < a < c < e < 1$, $0 < b < e < f < 1$, $0 < b < c < e < 1$, $0 < b < d < e < 1$, and elements $\{a, b\}$, $\{a, d\}$, $\{a, f\}$, $\{c, d\}$, $\{c, f\}$, and $\{e, f\}$ are pairwise incomparable.



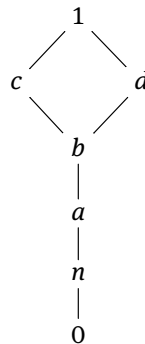
Then [12] L becomes a MTL-algebra with the following operations:

$\rightarrow$	0	a	b	c	d	e	f	1	$\odot$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
a	f	1	f	1	f	1	f	1	a	0	a	0	a	0	a	0	a
b	e	e	1	1	1	1	1	1	b	0	0	0	0	0	b	b	b
c	d	e	f	1	f	1	f	1	c	0	a	0	a	0	a	b	c
d	c	c	c	c	1	1	1	1	d	0	0	0	0	d	d	d	d
e	b	c	b	c	f	1	f	1	e	0	a	0	a	d	e	d	e
f	a	a	c	c	e	e	1	1	f	0	0	b	b	d	d	f	f
1	0	a	b	c	d	e	f	1	1	0	a	b	c	d	e	f	1

Since $\langle e \rangle = \{e, 1\}$, with $e = c \vee d$, but $c \notin \langle e \rangle$ and $d \notin \langle e \rangle$, then $\langle e \rangle$ is not a prime filter. Since $D_d(\langle e \rangle) = \{x \in L \mid (d^{**})^n \rightarrow x^{**} \in \langle e \rangle, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid d \rightarrow x^{**} \in \langle e \rangle\} = \{d, e, f, 1\}$. Hence $D_d(\langle e \rangle) = \{d, e, f, 1\}$, which is a prime filter of L , but $\langle e \rangle$ is not prime.

(v). If $M \in \text{Max}_i(L)$ is a maximal filter and $D_a(M)$ is a proper filter, then $D_a(M) = M$. Indeed, by Theorem 3.12 (ii), $M \subseteq D_a(M)$ and $D_a(M)$ is a proper filter. We get that if M is a maximal filter, then $M = D_a(M)$.

There are residuated lattices L such that $D_a(M)$ is a maximal filter, but $M \notin \text{Max}_i(L)$. We consider $L = \{0, n, a, b, c, d, 1\}$ with $0 < n < a < b < c, d < 1$, but c and d are incomparable.



Then ([12]) L becomes a distributive residuated lattice relative to the following operations:

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	d	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	n	n	a	1	1	1	1
c	0	n	a	d	1	d	1
d	n	n	a	c	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	0	n	0	n
a	0	0	a	a	a	a	a
b	0	0	a	b	b	b	b
c	0	n	a	b	c	b	c
d	0	0	a	b	b	d	d
1	0	n	a	b	c	d	1

Clearly, $\langle a \rangle = \{a, b, c, d, 1\}$ is the unique maximal filter of L and $\langle b \rangle = \{b, c, d, 1\}$ is a filter of L . Since $D_c(\langle b \rangle) = \{x \in L \mid (c^{**})^n \rightarrow x^{**} \in \langle b \rangle, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid 1 \rightarrow x^{**} \in \langle b \rangle\} = \{x \in L \mid x^{**} \in \langle b \rangle\} = \{a, b, c, d, 1\} = \langle a \rangle$. Therefore, $D_c(\langle b \rangle) = \langle a \rangle$ is the unique maximal filter of L , but $\langle b \rangle$ is not maximal.

Proposition 3.18. Let $a \in L$ and F be a filter of L . Then:

- (i) if M is a maximal filter of L and $a^{**} \in M$, then $D_a(M) = M$;
- (ii) $D_1(F)$ is a prime filter iff $D_a(F)$ is prime;
- (iii) $(a^n)^* \in F$, for some $n \in \mathbb{N}$ iff $D_a(F) = L$;
- (iv) if M is a maximal filter of L , then $D_a(M) = L$ iff $a \in L \setminus M$.

Moreover, if M is a maximal filter of L , then $D_a(M) = M$ iff $a \in M$.

Proof. (i). Let M be a maximal filter of L and $a^{**} \in M$. Following Theorem 3.12 (ii) and (vi) we have $D_a(M)$ is a filter and $M \subseteq D_a(M)$. We must to prove that $D_a(M)$ is a proper filter of L . If $0 \in D_a(M)$, then by (r_5) and (r_{15}) we obtain successively $0 \in D_a(M)$, $(a^{**})^n \rightarrow 0^{**} \in M$, $(a^{**})^n \rightarrow 0 \in M$, $(a^{**})^{n-1} \odot a^{**} \rightarrow 0 \in M$, $(a^{**})^{n-1} \rightarrow [a^{**} \rightarrow 0] \in M$, $(a^{**})^{n-1} \rightarrow a^{***} \in M$, $(a^{**})^{n-1} \rightarrow a^* \in M$. Since $a^{**} \in M$ and M is a filter of L , then $(a^{**})^{n-1} \in M$. Since $(a^{**})^{n-1} \in M$, $(a^{**})^{n-1} \rightarrow a^* \in M$, then $a^* \in M$. Since $a^*, a^{**} \in M$ and M is a filter of L , then $0 = a^* \odot a^{**} \in M$, a contradiction. Hence $D_a(M) \neq L$.

By hypothesis, M is a maximal filter and $M \subseteq D_a(M) \neq L$, then $D_a(M) = M$.

(ii). Let F be a filter of L . Following Theorem 3.12 (vi), $D_1(F)$ and $D_a(F)$ are filters of L . Suppose $D_1(F)$ is a prime filter of L and $x \vee y \in D_1(F)$. If $x \vee y \in D_1(F)$, then $(1^{**})^n \rightarrow (x \vee y)^{**} = (x \vee y)^{**} \in F$, for any $n \in \mathbb{N}$. Since F is a filter and $(x \vee y)^{**} \leq (a^{**})^n \rightarrow (x \vee y)^{**}$, then $(a^{**})^n \rightarrow (x \vee y)^{**} \in F$, hence $x \vee y \in D_a(F)$.

Since $x \vee y \in D_1(F)$ and $D_1(F)$ is prime, then $x \in D_1(F)$ or $y \in D_1(F)$. It follows that $x^{**} \in F$ or $y^{**} \in F$. We have successively $x^{**} \in F$, $x^{**} \leq (a^{**})^n \rightarrow x^{**} \in F$ or $y^{**} \in F$, $y^{**} \leq (a^{**})^m \rightarrow y^{**} \in F$, then $x \in D_a(F)$ or $y \in D_a(F)$, for some $m, n \in \mathbb{N}$. Hence $D_a(F)$ is prime.

Now, suppose $D_a(F)$ is prime and for $a = 1$ we deduce $D_1(F)$ is prime, too.

(iii). Suppose $(a^n)^* \in F$, for some $n \in \mathbb{N}$. Following Theorem 3.12 (ii), $F \subseteq D_a(F)$, then $(a^n)^* \in D_a(F)$. By Theorem 3.12 (ii) and (vi), we obtain $a \in D_a(F)$ and $D_a(F)$ is a filter, then $a^n \in D_a(F)$, for any $n \in \mathbb{N}$. Since $a^n \in D_a(F)$, $(a^n)^* \in D_a(F)$ and $D_a(F)$ is a filter, then $0 = a^n \odot (a^n)^* \in D_a(F)$. Hence $D_a(F) = L$.

Now, suppose $D_a(F) = L$. Then $0 \in D_a(F)$, $(a^{**})^n \rightarrow 0^{**} \in F$, $(a^{**})^n \rightarrow 0 \in F$, for some $n \in \mathbb{N}$.

We prove by induction that $(a^{**})^{n+1} \rightarrow 0 = a^{n+1} \rightarrow 0$, for $n \in \mathbb{N}$. Clear, for $n = 0$. Suppose $(a^{**})^n \rightarrow 0 = a^n \rightarrow 0$, for $n \in \mathbb{N}$. By (r_5) , $(a^{**})^{n+1} \rightarrow 0 = ((a^{**})^n \odot a^{**}) \rightarrow 0 = a^{**} \rightarrow ((a^{**})^n \rightarrow 0) = a^{**} \rightarrow (a^n \rightarrow 0) = a^n \rightarrow a^{**} = a^n \rightarrow (a \rightarrow 0) = a^{n+1} \rightarrow 0$.

Since $(a^{**})^n \rightarrow 0 \in F$, then $a^n \rightarrow 0 \in F$, that is $(a^n)^* \in F$.

(iv). Consider M a maximal filter of L .

" $\Rightarrow$ ". Consider $D_a(M) = L$. If $a \in M$, then we get $a^{**} \in M$, and by (i) we obtain $L = D_a(M) = M$, a contradiction. We deduce $a \notin M$, that is, $a \in L \setminus M$.

" $\Leftarrow$ ". Consider $a \in L \setminus M$. Since M is a maximal filter, then following Proposition 2.8 there is $n \in \mathbb{N}$ such that $(a^n)^* \in M$, and by (iii) we obtain $D_a(M) = L$.

Now, the fact that $D_a(M) = M$ iff $a \in M$ is routine. $\square$

Georgescu et al. (2015)[15] called *Gelfand residuated lattices* those residuated lattices in which any prime filter is included in a unique maximal filter. They are also called *normal residuated lattices*. Examples of Gelfand residuated lattices are Boolean algebras, BL-algebras and Stonean residuated lattices (see [8]).

Proposition 3.19. *Let $a \in L$ and $P \in \text{Spec}_i(L)$ be a prime filter. If $P = D_a(P)$, then L is Gelfand (normal) residuated lattice. The converse does not hold.*

Proof. Let P be a prime filter of L such that $P = D_a(P)$. Using Zorn's Lemma we deduce that P is contained in a maximal filter. Suppose that there are two distinct maximal filters M_1 and M_2 such that $P \subseteq M_1$ and $P \subseteq M_2$. Since $M_1 \neq M_2$, there is $a \in M_1$ such that $a \notin M_2$. By Theorem 3.12 (ii) and (vi) we have that $a \in D_a(P)$ and $D_a(P)$ is a filter, then $(a^n)^{**} \in D_a(P) = P$, for any $n \in \mathbb{N}$. Hence $(a^n)^{**} \in P$, for any $n \in \mathbb{N}$.

Following Proposition 2.8, there is $n \geq 1$ such that $(a^n)^* \in M_2$. Then $(a^n)^{**} \notin M_2$, hence $(a^n)^{**} \notin P$, a contradiction.

For the converse we consider the residuated lattice L from Remark 3.17 (v). The prime filter of L are $\langle a \rangle = \{a, b, c, d, 1\}$, $\langle b \rangle = \{b, c, d, 1\}$, $\langle c \rangle = \{c, 1\}$ and $\langle d \rangle = \{d, 1\}$. The maximal filter of L is $\langle a \rangle$, which include the all other prime filters. Hence L is Gelfand, but $D_c(\langle b \rangle) = \langle a \rangle \neq \langle b \rangle$. $\square$

Proposition 3.20. *Let $F \in \mathcal{F}_i(L)$ and $a \in F$. For $x \in D_a(F)$ the following assertions are equivalent:*

- (i) $D_a(F) = F$;
- (ii) $x^{**} \in F$, then $x \in F$.

Proof. (i) $\Rightarrow$ (ii). Consider $D_a(F) = F$. By hypothesis F is a filter and $a \in F$, then $a^{**} \in F$, $(a^{**})^n \in F$, for every $n \in \mathbb{N}$. Since F is a filter and $(a^{**})^n \in F$, $(a^{**})^n \rightarrow x^{**} \in F$, then $x^{**} \in F$. We obtain successively $F = D_a(F) = \{x \in L \mid (a^{**})^n \rightarrow x^{**} \in F, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid x^{**} \in F\}$, that is, if $x^{**} \in F$, then $x \in F$, for all $x \in L$.

(ii) $\Rightarrow$ (i). By hypothesis F is a filter and $a \in F$, then $a^{**} \in F$, $(a^{**})^n \in F$, for every $n \in \mathbb{N}$. Since F is a filter and $(a^{**})^n \in F$, $(a^{**})^n \rightarrow x^{**} \in F$, then $x^{**} \in F$. We obtain successively $D_a(F) = \{x \in L \mid (a^{**})^n \rightarrow x^{**} \in F, \text{ for some } n \in \mathbb{N}\} = \{x \in L \mid x^{**} \in F\} = F$. $\square$

Theorem 3.21. *Let $a \in L$ and F be a filter of L . For $x \in D_a(F)$ the following assertions are equivalent:*

- (i) $D_a(F) = F$;
- (ii) $x^{**} \in F$ iff $x \in F$.

Proof. (i) $\Rightarrow$ (ii). Consider $D_a(F) = F$. By hypothesis F is a filter and $x \in D_a(F) = F$, by (r_{16}) we obtain $x \leq x^{**} \in F$, so, if $x \in F$, then $x^{**} \in F$. Now, we prove that if $x^{**} \in F$, then $x \in F$. Since F is a filter and $x^{**} \in F$, $x^{**} \leq (a^{**})^n \rightarrow x^{**} \in F$, for any $n \in \mathbb{N}$, then $x \in D_a(F) = F$.

(ii) $\Rightarrow$ (i). By Theorem 3.12, (ii) we have $F \subseteq D_a(F)$. Now, we consider $x \in D_a(F)$ such that $x^{**} \in F$ iff $x \in F$. Since $x^{**} \in F$ and $x^{**} \leq (a^{**})^n \rightarrow x^{**} \in F$, for any $n \in \mathbb{N}$, then we obtain successively $D_a(F) = \{x \in L \mid (a^{**})^n \rightarrow x^{**} \in F, \text{ for some } n \in \mathbb{N}\} \subseteq \{x \in L \mid x^{**} \in F\} \subseteq F$. Therefore, $D_a(F) = F$. $\square$

Remark 3.22. Examples of residuated lattices which satisfy the conditions from Proposition 3.20 and Theorem 3.21 are Boolean algebras, MV-algebras and involutive residuated lattices.

Corollary 3.23. The set $\tau_a = \{D_a(X) \mid X \in U(L)\}$ is a topology on L and (L, τ_a) is a topological space.

Proof. Let $a \in L$. Clearly, $D_a(\emptyset) = \emptyset$ and $D_a(L) = L$. Following Theorem 3.12 (viii) and (ix) the set $\tau_a = \{D_a(X) \mid X \in U(L)\}$ is a topology on L and (L, τ_a) is a topological space. $\square$

Proposition 3.24. The set $\beta_a = \{D_a(\uparrow x) \mid x \in L\}$ is a base for the topology τ_a on L .

Proof. Let Z be an open subset of (L, τ_a) . Then there is $X \in U(L)$ such that $Z = D_a(X)$. Since X is an upset, then $X = \cup\{\uparrow x \mid x \in X\}$ and by Theorem 3.12 (viii), we deduce that $D_a(X) = \cup\{D_a(\uparrow x) \mid x \in X\}$. Hence β_a is a base for the topology τ_a on L . $\square$

Corollary 3.25. Every open set X relative to the topology τ_a is an upset. But the converse does not hold.

Proof. Following Corollary 3.23 and Proposition 3.24 we have (L, τ_a) is a topological space with β_a a base for the topology τ_a . Since every union of elements of β_a is an upset and every open set X relative to τ_a can be written as a union of elements of β_a , then X is an upset.

For the converse we consider the residuated lattice L from Remark 3.17 (v), where $\uparrow n = \{n, a, b, c, d, 1\}$ is an upset of L . The upsets of L are $\uparrow 0 = L$, $\uparrow n = \{n, a, b, c, d, 1\}$, $\uparrow a = \{a, b, c, d, 1\}$, $\uparrow b = \{b, c, d, 1\}$, $\uparrow c = \{c, 1\}$, $\uparrow d = \{d, 1\}$ and $\uparrow 1 = \{1\}$, so $U(L) = \{\uparrow 0, \uparrow n, \uparrow a, \uparrow b, \uparrow c, \uparrow d, \uparrow 1\}$. Since the base of topology τ_p on L is the set $\beta_p = \{D_p(\uparrow x) \mid p \in L \text{ and } \uparrow x \in U(L)\} = \{z \in L \mid (p^{**})^n \rightarrow z^{**} \in \uparrow x \text{ for some } n \in \mathbb{N}\}$, then it is easy to verify that $\uparrow n$ can not be written as a union of elements of β_p . Therefore, $\uparrow n$ is not an open set relative to the topology τ_p . $\square$

Proposition 3.26. If $u, v \in L$ such that $u \leq v$, then the topology τ_v is finer than topology τ_u .

Proof. We denote by β_u, β_v basis of the topology τ_u , respectively τ_v .

Consider $t \in L$ and $D_u(\uparrow x) \subseteq \beta_u$ an element of the basis β_u such that $t \in D_u(\uparrow x)$. Then there exists $n \in \mathbb{N}$ such that $(u^{**})^n \rightarrow t^{**} \in \uparrow x$, that is, $x \leq (u^{**})^n \rightarrow t^{**}$. Following Theorem 3.12 (ii) we have $t \in D_v(\uparrow t)$. Following Theorem 3.12 (vii), since $u \leq v$, then $D_v(\uparrow t) \subseteq D_u(\uparrow t)$. We prove that $D_u(\uparrow t) \subseteq D_u(\uparrow x)$. Now, we consider $s \in D_u(\uparrow t)$. Then there exists $m \in \mathbb{N}$ such that $(u^{**})^m \rightarrow s^{**} \in \uparrow t$ and so $t \leq (u^{**})^m \rightarrow s^{**}$. By (r_4) , (r_{35}) and (r_{36}) we obtain successively $t \leq (u^{**})^m \rightarrow s^{**}$, $t^{**} \stackrel{(r_4)}{\leq} [(u^{**})^m \rightarrow s^{**}]^{**} \stackrel{(r_{35}), (r_{36})}{\leq} (u^{**})^m \rightarrow s^{**}$. Since $x \leq (u^{**})^n \rightarrow t^{**} \leq (u^{**})^n \rightarrow [(u^{**})^m \rightarrow s^{**}] \stackrel{(r_5)}{\leq} (u^{**})^{m+n} \rightarrow s^{**}$, then $x \leq (u^{**})^{m+n} \rightarrow s^{**}$, hence $(u^{**})^{m+n} \rightarrow s^{**} \in \uparrow x$ and so $s \in D_u(\uparrow x)$.

Since $D_u(\uparrow t) \subseteq D_u(\uparrow x)$ and $D_v(\uparrow t) \subseteq D_u(\uparrow t)$, it follows that $D_v(\uparrow t) \subseteq D_u(\uparrow t) \subseteq D_u(\uparrow x)$.

Clearly, $D_v(\uparrow t) \subseteq \beta_v$ is an element of the basis β_v . We deduce that for any $t \in D_u(\uparrow x) \subseteq \beta_u$, there is a basis element $D_v(\uparrow t) \subseteq \beta_v$ such that $t \in D_v(\uparrow t) \subseteq D_u(\uparrow t) \subseteq D_u(\uparrow x)$. Following Lemma 3.4 we deduce that the topology τ_v is finer than topology τ_u . $\square$

Lemma 3.27. If X is a nonempty subset of L and $a \in L$, then X is a compact subset of (L, τ_a) iff $X \subseteq D_a(\uparrow \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\})$, for some $x_{i_1}, x_{i_2}, \dots, x_{i_n} \in X$ and $i \in I$.

Proof. " $\Leftarrow$ ". Suppose $X \subseteq D_a(\uparrow \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\})$, for some $x_{i_1}, x_{i_2}, \dots, x_{i_n} \in X$ and $\{D_a(X_i) \mid i \in I\}$ be a family of open subsets of L whose union contains X . For any $j \in \{1, 2, \dots, n\}$, there is $i_j \in I$ such that $x_{i_j} \in D_a(X_{i_j})$. Following Theorem 3.12 (i) and (iv), we have $D_a(x_{i_j}) \subseteq D_a(X_{i_j})$, for any $j \in \{1, 2, \dots, n\}$. By Theorem 3.12 (viii) we obtain $X \subseteq D_a(\uparrow \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\}) = D_a(x_{i_1}) \cup D_a(x_{i_2}) \cup \dots \cup D_a(x_{i_n}) \subseteq D_a(X_{i_1}) \cup D_a(X_{i_2}) \cup \dots \cup D_a(X_{i_n})$. Hence X is compact.

" $\Rightarrow$ ". Now, suppose X be a compact subset of L . Since $X \subseteq \cup\{\uparrow x \mid x \in X\}$, then by Theorem 3.12(ii), $X \subseteq \cup\{D_a(\uparrow x) \mid x \in X\}$. Hence $\{D_a(\uparrow x) \mid x \in X\}$ is a family of open subsets of L whose union contains X .

By hypothesis, there are $x_1, x_2, \dots, x_n \in X$ such that $X \subseteq D_a(\uparrow x_1) \cup D_a(\uparrow x_2) \cup \dots \cup D_a(\uparrow x_n)$. By Theorem 3.12(viii) we obtain $X \subseteq D_a(\uparrow \{x_1, x_2, \dots, x_n\})$. $\square$

Theorem 3.28. *The topological space (L, τ_a) is connected.*

Proof. Consider X a non-empty subset of L such that is both closed and open relative to the topology τ_a . If X is an open set, by Corollary 3.25, then X is an upset. If $0 \in X$ and X is an upset, then $X = L$. If $0 \in L - X$, since X is closed, we get $L - X$ is open, by Corollary 3.25, $L - X$ is an upset of L , since $0 \in L - X$, then $L - X = L$. Hence $X = \emptyset$, a contradiction. We conclude that $\{\emptyset, L\}$ is the set of all subsets of L which are both closed and open. That is, (L, τ_a) is connected. $\square$

4 Some Properties of Quotient Topology on Residuated Lattices

Proposition 4.1. *Let $P, Q \in \mathcal{F}_i(L)$ and $a \in L$ such that $Q \subseteq P$. Then $D_{[a]}(P/Q) = D_a(P)/Q$, where $[a] = \{a/Q \mid a \in L\}$.*

Proof. Consider $x \in L$. Then

$$D_{[a]}(P/Q) = \{[x] \in L/Q \mid [a]^m \ominus [x] \in P/Q, \text{ for some } m \in \mathbb{N}\} = \{[x] \in L/Q \mid [a^m \ominus x] \in P/Q\} = \{[x] \in L/Q \mid a^m \ominus x \in P\} = D_a(P)/Q. \quad \square$$

Following Proposition 3.24 we obtain the following result.

Remark 4.2. (i). *Let $F \in \mathcal{F}_i(L)$ and $a \in L$. Then $(L/F, \tau_{a/F})$ is a topological space, where the set $\tau_{a/F} = \{D_{a/F}(X/F) \mid X \in U(L)\}$ is a topology on L/F and the set $\{D_{a/F}(\uparrow x/F) \mid x/F \in L/F\}$ is a base for the topology $\tau_{a/F}$ on L/F .*

If $\pi : L \rightarrow L/F$ is the canonical morphism defined by $\pi(x) = [x]$, then π is a continuous map. Indeed, if O is an open set with respect the topology $\tau_{a/F}$, then $O = \bigcup \{D_{a/F}(\uparrow x_i/F) \mid \text{for some } x_i/F \in L/F\}$, and $\pi^{-1}(O) = \pi^{-1}(\bigcup D_{a/F}(\uparrow x_i/F)) = \bigcup \pi^{-1}(D_{a/F}(\uparrow x_i/F)) = \bigcup D_a(\pi^{-1}(\uparrow x_i/F))$, which is an open set on (L, τ_a) .

(ii) *Let $F \in \mathcal{F}_i(L)$ and $a \in L$. Then $(L/F, \bar{\tau}_a)$ is a topological space, where the set $\bar{\tau}_a = \{D_a(X)/F \mid X \in U(L)\}$ is a topology on L/F and the set $\{D_a(\uparrow x)/F \mid x/F \in L/F\}$ is a base for the topology $\bar{\tau}_a$.*

If $\pi : L \rightarrow L/F$ is the canonical morphism defined by $\pi(x) = [x]$, then π is a continuous map. Indeed, if O is an open set with respect the topology $\bar{\tau}_a$, then $O = \bigcup \{D_a(\uparrow x_i)/F \mid \text{for some } x_i/F \in L/F\}$, and $\pi^{-1}(O) = \pi^{-1}(\bigcup D_a(\uparrow x_i)/F) = \bigcup \pi^{-1}(D_a(\uparrow x_i)/F) = \bigcup D_a(\pi^{-1}(\uparrow x_i/F))$, which is an open set on (L, τ_a) .

Theorem 4.3. *Let F be a filter of L and $a, x \in L$. Then:*

(i) $D_a(\uparrow x)/F \subseteq D_{a/F}(\uparrow x/F)$.

(ii) *The topology $\tau_{a/F}$ is finer than $\bar{\tau}_a$, where $\bar{\tau}_a$ is the quotient topology of L/F .*

Proof. (i). $D_{a/F}(\uparrow x/F) = \{u/F \in L/F \mid x/F \leq ((a/F)^{**})^n \rightarrow (u/F)^{**}\} = \{u/F \in L/F \mid x \rightarrow [(a^{**})^n \rightarrow u^{**}] \in F\}$ and $D_a(\uparrow x)/F = \{v/F : v \in D_a(\uparrow x)\}$. Let $u/F \in D_{a/F}(\uparrow x/F)$. Then there exists $v \in D_a(\uparrow x)$ such that $u/F = v/F$ and so $x \rightarrow [(a^{**})^n \rightarrow v^{**}] = 1$, for some $n \in \mathbb{N}$. Since $u \equiv_F v$, then $x \rightarrow [(a^{**})^n \rightarrow u^{**}] = x \rightarrow [(a^{**})^n \rightarrow v^{**}] = 1$, that is $x \rightarrow [(a^{**})^n \rightarrow u^{**}] \in F$, hence $u/F \in D_a(\uparrow x)/F$. Therefore, $D_a(\uparrow x)/F \subseteq D_{a/F}(\uparrow x/F)$.

(ii). By Proposition 3.24 the set $\{D_{a/F}(\uparrow x/F) : x/F \in L/F\}$ is a base for the topology $\tau_{a/F}$ on L/F . Moreover, the set $\{D_a(\uparrow x)/F : x \in L\}$ is a base for the topology $\bar{\tau}_a$. Now, the proof of (ii) is straightforward by (i). $\square$

Let L, L' be residuated lattices. On $L \times L'$ we consider the relation of order $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \leq y'$ and the operations

$$\begin{aligned} (x, y) \wedge (x', y') &= (x \wedge x', y \wedge y'), \\ (x, y) \vee (x', y') &= (x \vee x', y \vee y'), \\ (x, y) \odot (x', y') &= (x \odot x', y \odot y'), \\ (x, y) \rightarrow (x', y') &= (x \rightarrow x', y \rightarrow y') \text{ for all } x, y \in L \text{ and } x', y' \in L'. \end{aligned}$$

Then $L \times L'$ with the above operations is a residuated lattice called *direct product* of L and L' .

Lemma 4.4 ([17]). *Let L, L' be residuated lattices. Then K is a filter of $L \times L'$ iff there exist $P \in \mathcal{F}_i(L)$ and $Q \in \mathcal{F}_i(L')$ such that $K = P \times Q$.*

Proof. " $\Rightarrow$ ". If $K \in \mathcal{F}_i(L \times L')$, we consider $P = \{x \in L : (x, x') \in K \text{ for some } x' \in L'\}$ and $Q = \{x' \in L' : (x, x') \in K \text{ for some } x \in L\}$.

Clearly, $K = P \times Q$. Since $(1, 1) \in K$ we get $1 \in P$. Let $x, y \in P$. Then there exist $x', y' \in L'$ such that $(x, x'), (y, y') \in K$. Thus, $(x, x') \odot (y, y') = (x \odot y, x' \odot y') \in K$, so $x \odot y \in P$.

Consider $x \leq y$ and $x \in P$. Then there exists $x' \in L'$ such that $(x, x') \in K$. Since $(x, x') \leq (y, x')$, then $y \in P$, hence $P \in \mathcal{F}_i(L)$. Similarly, $Q \in \mathcal{F}_i(L')$.

" $\Leftarrow$ ". Let $K = P \times Q$ for some $P \in \mathcal{F}_i(L)$ and $Q \in \mathcal{F}_i(L')$. Clearly, $K \in L \times L'$. We consider $(x, y), (p, q) \in K$. Then $x, p \in P$ and $y, q \in Q$, that is $x \odot p \in P$, $y \odot q \in Q$. Therefore, $(x, y) \odot (p, q) = (x \odot p, y \odot q) \in P \times Q = K$.

Now, we consider $(x, y) \in K$ such that $(x, y) \leq (p, q)$. Then $x \leq p$ with $x \in P$ and $y \leq q$ with $y \in Q$. Since $P \in \mathcal{F}_i(L)$ and $Q \in \mathcal{F}_i(L')$, then $p \in P$ and $q \in Q$, that is $(p, q) \in K$. Hence $K \in \mathcal{F}_i(L \times L')$. $\square$

Lemma 4.5. *Let $P, Q \in \mathcal{F}_i(L)$ and $a \in L$. Then $D_a(P) \times D_a(Q) = D_a(P \times Q)$.*

Proof. By Lemma 4.4, $D_a(P) \times D_a(Q) =$

$\{x \in L | a^n \odot x \in P, \text{ for some } n \in \mathbb{N}\} \times \{y \in L | a^m \odot y \in Q, \text{ for some } m \in \mathbb{N}\} = \{x \times y \in L \times L | (a^t \odot x, a^t \odot y) \in P \times Q, \text{ for some } t \geq \max\{m, n\} \in \mathbb{N}\} = D_a(P \times Q)$. $\square$

Consider the topological spaces (L, τ_a) and (L, τ_b) .

Theorem 4.6. *Let $P, Q \in \mathcal{F}_i(L)$ and $a, b \in L$. Then there exists a homeomorphism from $\frac{L \times L}{D_a(P \times Q)}$ to $\frac{L}{D_a(P)} \times \frac{L}{D_a(Q)}$.*

Proof. We define $\phi : L \times L \rightarrow \frac{L}{D_a(P)} \times \frac{L}{D_a(Q)}$, by $\phi(x, y) = (\frac{x}{D_a(P)}, \frac{y}{D_a(Q)})$. Clearly, ϕ is onto. Let $(x, y) \in L \times L$. Then $(x, y) \in \ker(\phi)$ iff $x/D_a(P) = 1/D_a(P)$ and $y/D_a(Q) = 1/D_a(Q)$ iff $x \in D_a(P)$ and $y \in D_a(Q)$. Hence $\ker(\phi) = D_a(P) \times D_a(Q)$. It follows easily by Lemma 4.5 that $D_a(P) \times D_a(Q) = D_a(P \times Q)$. Consider the map $h : \frac{L \times L}{D_a(P \times Q)} \rightarrow \frac{L}{D_a(P)} \times \frac{L}{D_a(Q)}$, defined by $h(\frac{(x, y)}{D_a(P \times Q)}) = (x/D_a(P), y/D_a(Q))$, then by the first isomorphism theorem h becomes an isomorphism. If we suppose that X is an open subset of $\frac{L}{D_a(P)} \times \frac{L}{D_a(Q)}$, then there exist $U, V \in \tau_b$ open subsets such that $X = \frac{U}{D_a(P)} \times \frac{V}{D_a(Q)}$. Clearly, $h^{-1} = \frac{U \times V}{D_a(P \times Q)}$ is an open subset of $\frac{L \times L}{D_a(P \times Q)}$. Hence h is a continuous map. In the same manner we can prove that h^{-1} is a continuous map. Therefore, h is a homeomorphism. $\square$

4.1 Uniform topology on quotient residuated lattice L/J

In this section based on the work of Ghorbani and Hasankhani (2010)[18] we define and study a uniform topology $\tau_{\overline{A}}$ on the quotient residuated lattice L/J , where J is a filter of L .

If X is a non-empty set, U and V are subsets of $X \times X$. Then:

$U \circ V = \{(x, y) \in X \times X : (z, y) \in U \text{ and } (x, z) \in V \text{ for some } z \in X\}$,

$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$,

$\Delta = \{(x, x) \in X \times X : x \in X\}$.

In universal algebra if X is a non-empty set, then a non-empty collection K of subsets of $X \times X$ is called a *uniformity* on X if it satisfies the following conditions:

(U_1) $\Delta \subseteq U$ for any $U \in K$,

(U_2) if $U \in K$, then $U^{-1} \in K$,

(U_3) if $U \in K$, then there exists a set $V \in K$ such that $V \circ V \subseteq U$,

(U_4) if $U, V \in K$, then $U \cap V \subseteq K$,

(U_5) if $U \in K$ and $U \subseteq V \subseteq X \times X$, then $V \in K$.

The pair (X, K) is called a *uniform structure*.

In Ghorbani and Hasankhani (2010)[18], Theorem 2.10, it was proved that if Λ is a family of filters of a residuated lattice L which is closed under intersection. If we consider the sets $U_F = \{(x, y) \in L \times L : x \equiv_F y\}$ for every $F \in \Lambda$ and $K^* = \{U_F : F \in \Lambda\}$, then $K = \{U \subseteq L \times L : U_F \subseteq U \text{ for some } U_F \in K^*\}$ is a uniformity on L . Let $U \in K$, define $U_{[x]} = \{y \in L : (x, y) \in U\}$.

Then $\tau_\Lambda = \{O \subseteq L : \forall x \in O, \exists U \in K \text{ such that } U_{[x]} \subseteq O\}$ is a topology on L and is called the *uniform topology* on L induced by Λ .

In [18], Theorem 3.1, it was proved that if J is a filter of L . For each $F \in \Lambda$, let $\bar{F} = \langle F \vee J \rangle / J$. Then $\Lambda^* = \{\bar{F} : F \in \Lambda\}$ gives a uniform topology on L/J , where $U_{\bar{F}} = \{([x], [y]) \in L/J \times L/J : [x] \equiv_{\bar{F}} [y]\}$.

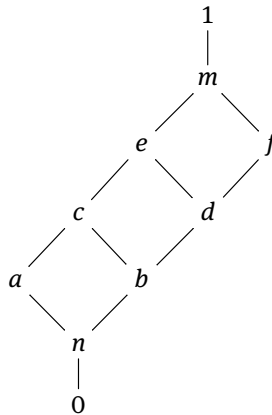
The set $\tau = \{O \subseteq L/J : \pi^{-1}(O) \in \tau_\Lambda\}$, where $\pi : L \rightarrow L/J$ is the canonical epimorphism, τ becomes the quotient topology on L/J , and τ_{Λ^*} becomes the uniform topology induced by Λ^* on L/J .

Lemma 4.7. Let $F, Q \in \mathcal{F}_i(L)$ and $a \in L$. Then:

- (i) $\langle \{a\} \cup F \rangle \subseteq D_a(F)$, but the converse does not hold;
- (ii) $\langle \{a\} \cup Q \cup F \rangle \subseteq D_a(\langle Q \cup F \rangle)$, but the converse does not hold.

Proof. (i). Let $F \in \mathcal{F}_i(L)$ and $a \in L$. If $x \in \langle \{a\} \cup F \rangle$ then $x \geq a \odot f$, for some $f \in F$. We obtain successively $(a^{**})^n \rightarrow x^{**} \stackrel{(r_4)}{\geq} (a^{**})^n \rightarrow (a \odot f)^{**} \stackrel{(r_4)}{\geq} a^{**} \rightarrow (a \odot f)^{**} \stackrel{(r_{14})}{=} a \rightarrow (a \odot f)^{**} = a \rightarrow [(a \odot f)^* \rightarrow 0] \stackrel{(r_5)}{=} [a \odot (a \odot f)^*] \rightarrow 0 \stackrel{(r_5)}{=} [a \odot (a \rightarrow f^*)] \rightarrow 0 \geq f^* \rightarrow 0 = f^{**} \geq f \in F$, that is $x \in D_a(F)$. Thus $\langle \{a\} \cup F \rangle \subseteq D_a(F)$.

For the converse we consider $L = \{0, n, a, b, c, d, e, f, m, 1\}$ with $0 < n < a < c < e < m < 1$, $0 < n < b < d < f < m < 1$ and the elements $\{a, b\}$, $\{c, d\}$, $\{e, f\}$ are pairwise incomparable.



Then([12]) L becomes a distributive residuated lattice relative to the following operations:

$\rightarrow$	0	n	a	b	c	d	e	f	m	1
0	1	1	1	1	1	1	1	1	1	1
n	m	1	1	1	1	1	1	1	1	1
a	f	f	1	f	1	f	1	f	1	1
b	e	e	e	1	1	1	1	1	1	1
c	d	d	e	f	1	f	1	f	1	1
d	c	c	c	e	e	1	1	1	1	1
e	b	b	c	d	e	f	1	f	1	1
f	a	a	a	c	c	e	e	1	1	1
m	n	n	a	b	c	d	e	f	1	1
1	0	n	a	b	c	d	e	f	m	1

$\odot$	0	n	a	b	c	d	e	f	m	1
0	0	0	0	0	0	0	0	0	0	0
n	0	0	0	0	0	0	0	0	0	n
a	0	0	a	0	a	0	a	0	a	a
b	0	0	0	0	0	0	b	b	b	b
c	0	0	a	0	a	0	a	b	c	c
d	0	0	0	0	0	b	b	d	d	d
e	0	0	a	0	a	b	c	d	e	e
f	0	0	0	b	b	d	d	f	f	f
m	0	0	a	b	c	d	e	f	m	m
1	0	n	a	b	c	d	e	f	m	1

Let $F = \{1, m, f\}$, then $\langle \{d\} \cup F \rangle = \{x \in L : x \geq d \odot t \text{ for some } t \in F\} = \{d, e, f, m, 1\}$. But $D_a(F) = \{x \in L : (d^{**})^n \rightarrow x^{**} \in F\} = \{L\}$, hence $\langle \{d\} \cup F \rangle \subsetneq D_a(F)$.

(ii). Let $F, Q \in \mathcal{F}_i(L)$ and $a \in L$. If $x \in \langle \{a\} \cup Q \cup F \rangle$ then $x \geq a \odot q \odot f$, for some $q \in Q, f \in F$. We obtain successively $(a^{**})^n \rightarrow x^{**} \stackrel{(r_4)}{\geq} (a^{**})^n \rightarrow (a \odot q \odot f)^{**} \stackrel{(r_4)}{\geq} a^{**} \rightarrow (a \odot q \odot f)^{**} \stackrel{(r_{14})}{\leq} a \rightarrow (a \odot q \odot f)^{**} = a \rightarrow [(a \odot q \odot f)^* \rightarrow 0] \stackrel{(r_5)}{=} [a \odot (a \odot q \odot f)^*] \rightarrow 0 \stackrel{(r_5)}{=} [a \odot (a \rightarrow (q \odot f)^*)] \rightarrow 0 \geq (q \odot f)^* \rightarrow 0 = (q \odot f)^{**} \geq (q \odot f) \in \langle Q \cup F \rangle$, that is $x \in D_a(\langle Q \cup F \rangle)$. Thus $\langle \{a\} \cup Q \cup F \rangle \subset D_a(\langle Q \cup F \rangle)$.

For the converse we consider the residuated lattice L from (i), let $Q = \{1, m\}$ and $F = \{1, m, f\}$, then $\langle \{b\} \cup Q \cup F \rangle = \{x \in L : x \geq b \odot q \odot t \text{ for some } q \in Q, t \in F\} = \{b, c, d, e, f, m, 1\}$. But $D_b(\langle Q \cup F \rangle) = \{x \in L : (d^{**})^n \rightarrow x^{**} \in \langle Q \cup F \rangle\} = \{L\}$, hence $\langle \{b\} \cup Q \cup F \rangle \subset D_b(\langle Q \cup F \rangle)$. $\square$

Clearly, if $a \in L$ and $J, F \in \mathcal{F}_i(L)$, then $\langle J \cup F \rangle \in \mathcal{F}_i(L)$ and $D_a(\langle J \cup F \rangle) \in \mathcal{F}_i(L)$, too.

Theorem 4.8. *Let $a \in L$. The lattice $(D_a(\mathcal{F}_i(L)), \subseteq)$ is a complete Brouwerian lattice (hence distributive).*

Proof. Clearly, if $\{X_i : i \in I\}$ is a family of filters from L , then the infimum of this family is $\bigwedge_i D_a(X_i) = \bigcap_{i \in I} D_a(X_i)$ and the supremum is $\bigvee_i D_a(X_i) = \bigcup_{i \in I} D_a(X_i)$, that is the lattice $(D_a(\mathcal{F}_i(L)), \bigvee_i, \bigwedge_i, \subseteq, \{1\}, L)$ is complete.

By Lemma 3.27 we have that an upset X of L is compact iff $X \subseteq D_a(\uparrow \{x_{i_1}, x_{i_2}, \dots, x_{i_n}\})$, for some $x_{i_1}, x_{i_2}, \dots, x_{i_n} \in X$ and $i \in I$. Since any filter is an upset we deduce that the compact elements of $D_a(\mathcal{F}_i(L))$ are exactly the upsets generated by the principal filters of L , which are filters, too.

Following Theorem 3.12, (xi) we deduce that the lattice $(D_a(\mathcal{F}_i(L)), \subseteq)$ is distributive. $\square$

Following Lemma 4.7 and Theorem 3.1 from [18] we obtain:

Lemma 4.9. *Consider J a filter of L and $a \in L$. For each $F \in \Lambda$, let $\overline{D_a(F)} = D_a(\langle J \cup F \rangle)/J$. Then $\overline{\Lambda} = \{D_a(\langle J \cup F \rangle)/J \in L/J : F \in \Lambda\}$ gives a uniform topology on L/J , where $U_{\overline{D_a(F)}} = \{([x], [y]) \in L/J \times L/J : [x] \equiv_{\overline{D_a(F)}} [y]\}$. That is, L/J is a topological space with the uniform topology $\tau_{\overline{\Lambda}}$ induced by $\overline{\Lambda}$.*

For a residuated lattice L and a topology τ defined on the set L , the pair (L, τ) is called a *topological residuated lattice* if the operations $\wedge, \vee, \odot$ and $\rightarrow$ are continuous with respect to τ .

In the same manner as Theorem 3.2 from [18] the following result can be proved:

Theorem 4.10. *For each $x \in L$ and $F \in \Lambda$, $\pi(U_F[x]) = U_{\overline{D_a(F)}}[[x]]$. Hence each $U_{\overline{D_a(F)}}[[x]]$ is open in the quotient topology and $\tau_{\overline{\Lambda}} \subseteq \tau$. Moreover, $\tau_{\Lambda^*} \subseteq \tau_{\overline{\Lambda}} \subseteq \tau$.*

Following Lemma 4.7, Lemma 4.9, Theorem 4.10 and Theorem 3.3 from [18] we obtain:

Theorem 4.11. *$\tau = \tau_{\Lambda^*} = \tau_{\overline{\Lambda}}$ in L/J . Moreover, if J is a filter, then $(L/J, \tau_{\overline{\Lambda}})$ is a topological residuated lattice.*

We recall that for any non-empty set X of L , we denote by $X^c = L \setminus X$. In the same manner as Theorem 3.9 and Theorem 3.10, from [18] the following result can be proved:

Theorem 4.12. *$(L/J, \tau_{\overline{\Lambda}})$ is Hausdorff iff J is closed and $(L/J, \tau_{\overline{\Lambda}})$ is discrete iff J is open.*

Theorem 4.13. *Let J be a filter of L . If J is closed relative to the uniform topology $\tau_{\overline{\Lambda}}$, then $(L/J, \tau_{\overline{\Lambda}})$ is a regular space.*

Proof. Consider J a filter of L such that J is closed relative to the uniform topology $\tau_{\overline{\Lambda}}$, then $(L/J, \tau_{\overline{\Lambda}})$ is Hausdorff. Since every Hausdorff and locally compact topological space is regular, we have to prove that $(L/J, \tau_{\overline{\Lambda}})$ is locally compact.

Let $[x] \in L/J$. Since L is a locally compact space by uniform topology induced by $\overline{\Lambda}$, there exists an open neighborhood O of x and a compact set K such that $x \in O \subseteq K$. Since π is open and continuous map, we have that $\pi(O)$ is open and $\pi(K)$ is compact such that $[x] \in \pi(O) \subseteq \pi(K)$. Thus L/J is locally compact at $[x]$ and hence is locally compact. $\square$

5 Some Classes of Semitopological Residuated Lattices

By definition the class of BL-algebras is a subclass of divisible residuated lattices. In [11], O. Zahiri and R. A. Borzooei studied the (semi)topological algebras and for a BL-algebra L they proved that (see Theorem 3.17, [11]) $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological BL-algebra and $(L, \{\rightarrow\}, \tau_a)$ is a right semitopological BL-algebra, for an element $a \in L$. We prove that their result works in the case of divisible residuated lattices, which is a larger class than BL-algebras.

We notice the following rules of calculus:

Lemma 5.1. *If L is a residuated lattice, then for $a, p, q \in L$ and $m \geq 1$ we have:*

- (ε_1) $(a^{**})^m \rightarrow (q \vee p)^{**} = [(a^{**})^m \rightarrow (q \vee p)^{**}]^{**}$;
- (ε_2) $q \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}] = 1$;
- (ε_3) $((a^{**})^m \rightarrow (p^{**})) \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}] = 1$;
- (ε_4) $[q \wedge ((a^{**})^m \rightarrow (p^{**}))] \rightarrow [((a^{**})^m \rightarrow (q^{**})) \wedge ((a^{**})^m \rightarrow (p^{**}))] = 1$;
- (ε_5) $[q \odot ((a^{**})^m \wedge (p^{**}))] \rightarrow [(q \odot p)^{**}] = 1$.

Proof. (ε_1). Following (r_{35}) we have $a^m \odot (q \vee p) = (a^{**})^m \rightarrow (q \vee p)^{**}$, by (r_{36}) we obtain $a^m \odot (q \vee p) = (a^m \odot (q \vee p))^{**} = [(a^{**})^m \rightarrow (q \vee p)^{**}]^{**}$. Hence $(a^{**})^m \rightarrow (q \vee p)^{**} = [(a^{**})^m \rightarrow (q \vee p)^{**}]^{**}$.

The rule of calculus (ε_1) holds if instead of $(q \vee p)$ we have $(q \wedge p)$, $(q \odot p)$, $(q \rightarrow p)$ or p .

(ε_2). Since $q \leq q \vee p \leq (q \vee p)^{**} \leq (a^{**})^m \rightarrow (q \vee p)^{**}$, we obtain $q \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}] = 1$.

(ε_3). Since $p^{**} \leq (q \vee p)^{**}$ and by (r_4), we obtain $(a^{**})^m \rightarrow p^{**} \leq (a^{**})^m \rightarrow (q \vee p)^{**}$, that is $((a^{**})^m \rightarrow (p^{**})) \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}] = 1$.

(ε_4). Since $q \leq q^{**} \leq (a^{**})^m \rightarrow (q^{**})$, then $[q \wedge ((a^{**})^m \rightarrow (p^{**}))] \leq [((a^{**})^m \rightarrow (q^{**})) \wedge ((a^{**})^m \rightarrow (p^{**}))]$. Hence $[q \wedge ((a^{**})^m \rightarrow (p^{**}))] \rightarrow [((a^{**})^m \rightarrow (q^{**})) \wedge ((a^{**})^m \rightarrow (p^{**}))] = 1$.

(ε_5). Since $q \leq q^{**}$, $(a^{**})^m \wedge (p^{**}) \leq (p^{**})$, then $q \odot ((a^{**})^m \wedge (p^{**})) \leq (q^{**}) \odot (p^{**})$. By (r_{18}), $(q^{**}) \odot (p^{**}) \leq (q \odot p)^{**}$, then $[(q^{**}) \odot (p^{**})] \rightarrow (q \odot p)^{**} = 1$. Hence $[q \odot ((a^{**})^m \wedge (p^{**}))] \rightarrow [(q \odot p)^{**}] = 1$. $\square$

Definition 5.2. *Let τ be a topology on the divisible residuated lattice L . If $(L, \{\ast_i\}, \tau)$, where $\{\ast_i\} \subseteq \{\vee, \wedge, \odot, \rightarrow\}$ is a (semi)topological algebra, then $(L, \{\ast_i\}, \tau)$ is a (semi)topological divisible residuated lattice. For simplicity, if $\{\ast_i\} \subseteq \{\vee, \wedge, \odot, \rightarrow\}$, we consider (L, τ) instead of $(L, \{\vee, \wedge, \odot, \rightarrow\}, \tau)$.*

Theorem 5.3. *Let L be a divisible residuated lattice and $a \in L$. Then:*

- (i) $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological divisible residuated lattice;
- (ii) $(L, \{\rightarrow\}, \tau_a)$ is a right semitopological divisible residuated lattice.

Proof. (i). Consider q an arbitrary element of L .

(1). We prove that $(L, \{\vee\}, \tau_a)$ is a semitopological divisible residuated lattice.

Consider the map $\varphi_q : L \rightarrow L$, defined by $\varphi_q(x) = q \vee x$, for any $x \in L$. Following Proposition 3.24 the set $\beta_a = \{D_a(\uparrow x) \mid x \in L\}$ is a base for the topology τ_a on L . Then it suffices to prove that $\varphi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\varphi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \varphi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \vee p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \vee p)^{**}, \text{ for some } n \in \mathbb{N}\}$.

Consider the set $A = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \vee p)^{**}, \text{ for some } n \in \mathbb{N}\}$ and we show that the set A is an upset of L . Let $p \in A$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \rightarrow (q \vee p)^{**}$. By (r_4), $(a^{**})^n \rightarrow (q \vee p)^{**} \leq (a^{**})^n \rightarrow (q \vee s)^{**}$, and so $s \in A$. Hence A is an upset. Now, we prove that $D_a(A) = A$, then we deduce that $A \in \tau_a$, that is $\varphi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence φ_q is a continuous map.

Let $p \in D_a(A)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \rightarrow (p^{**}) \in A$, that is $x \leq (a^{**})^n \rightarrow (q \vee ((a^{**})^m \rightarrow (p^{**})))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$$[(a^{**})^n \rightarrow (q \vee ((a^{**})^m \rightarrow (p^{**})))^{**}] \rightarrow$$

$$\begin{aligned}
& [(a^{**})^{m+n} \rightarrow (q \vee p)^{**}] \stackrel{(r_5)}{=} \\
& [(a^{**})^n \rightarrow (q \vee ((a^{**})^m \rightarrow (p^{**})))^{**}] \rightarrow \\
& [(a^{**})^n \rightarrow ((a^{**})^m \rightarrow (q \vee p)^{**})] \stackrel{(r_{39})}{\geq} \\
& [q \vee ((a^{**})^m \rightarrow (p^{**}))]^{**} \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}] \stackrel{(\varepsilon_1)}{=} \\
& [q \vee ((a^{**})^m \rightarrow (p^{**}))]^{**} \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}]^{**} \stackrel{(r_{22})}{=} \\
& ([q \vee ((a^{**})^m \rightarrow (p^{**}))] \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}])^{**} \stackrel{(r_{22})}{=} \\
& ([q \vee ((a^{**})^m \rightarrow (p^{**}))] \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}])^{**} \stackrel{(r_7)}{=} \\
& ([q \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}]] \wedge [((a^{**})^m \rightarrow (p^{**})) \rightarrow \\
& [(a^{**})^m \rightarrow (q \vee p)^{**}]]^{**} \stackrel{(\varepsilon_2)}{=} \\
& (1 \wedge [((a^{**})^m \rightarrow (p^{**})) \rightarrow [(a^{**})^m \rightarrow (q \vee p)^{**}]])^{**} \stackrel{(\varepsilon_3)}{=} \\
& (1 \wedge 1)^{**} = 1^{**} = 1.
\end{aligned}$$

By (r_2) , $(a^{**})^n \rightarrow (q \vee ((a^{**})^m \rightarrow (p^{**})))^{**} \leq (a^{**})^{m+n} \rightarrow (q \vee p)^{**}$.

We deduce that $x \leq (a^{**})^n \rightarrow (q \vee ((a^{**})^m \rightarrow (p^{**})))^{**} \leq (a^{**})^{m+n} \rightarrow (q \vee p)^{**}$, hence $x \leq (a^{**})^{m+n} \rightarrow (q \vee p)^{**}$, that is $p \in A$. Following Theorem 3.12 (ii), we deduce that $A = D_a(A)$, that is φ_q is a continuous map.

(2). We prove that $(L, \{\wedge\}, \tau_a)$ is a semitopological divisible residuated lattice.

Consider the map $\phi_q : L \rightarrow L$, defined by $\phi_q(x) = q \wedge x$, for any $x \in L$. Following Proposition 3.24 the set $\beta_a = \{D_a(\uparrow x) \mid x \in L\}$ is a base for the topology τ_a on L . Then it suffices to prove that $\phi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\phi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \phi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \wedge p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \wedge p)^{**}, \text{ for some } n \in \mathbb{N}\}$.

Consider the set $B = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \wedge p)^{**}, \text{ for some } n \in \mathbb{N}\}$ and we show that the set B is an upset of L . Let $p \in B$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \rightarrow (q \wedge p)^{**}$. By (r_4) , $(a^{**})^n \rightarrow (q \wedge p)^{**} \leq (a^{**})^n \rightarrow (q \wedge s)^{**}$, and so $s \in B$. Hence B is an upset. Now, we prove that $D_a(B) = B$, then we deduce that $B \in \tau_a$, that is $\phi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence ϕ_q is a continuous map.

Let $p \in D_a(B)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \rightarrow (p^{**}) \in B$, that is $x \leq (a^{**})^n \rightarrow (q \wedge ((a^{**})^m \rightarrow (p^{**})))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$$\begin{aligned}
& [(a^{**})^n \rightarrow (q \wedge ((a^{**})^m \rightarrow (p^{**})))^{**}] \rightarrow [(a^{**})^{m+n} \rightarrow (q \wedge p)^{**}] \stackrel{(r_5)}{=} \\
& [(a^{**})^n \rightarrow (q \wedge ((a^{**})^m \rightarrow (p^{**})))^{**}] \rightarrow [(a^{**})^n \rightarrow ((a^{**})^m \rightarrow (q \wedge p)^{**})] \stackrel{(r_{39})}{\geq} \\
& [q \wedge ((a^{**})^m \rightarrow (p^{**}))]^{**} \rightarrow [(a^{**})^m \rightarrow (q \wedge p)^{**}] \stackrel{(\varepsilon_1)}{=} \\
& [q \wedge ((a^{**})^m \rightarrow (p^{**}))]^{**} \rightarrow [(a^{**})^m \rightarrow (q \wedge p)^{**}]^{**} \stackrel{(r_{22})}{=} \\
& ([q \wedge ((a^{**})^m \rightarrow (p^{**}))] \rightarrow [(a^{**})^m \rightarrow (q \wedge p)^{**}])^{**} \stackrel{(r_9)}{=} \\
& ([q \wedge ((a^{**})^m \rightarrow (p^{**}))] \rightarrow [((a^{**})^m \rightarrow (q^{**})) \wedge ((a^{**})^m \rightarrow (p^{**}))])^{**} \stackrel{(\varepsilon_4)}{=} 1^{**} = 1.
\end{aligned}$$

By (r_2) , $(a^{**})^n \rightarrow (q \wedge ((a^{**})^m \rightarrow (p^{**})))^{**} \leq (a^{**})^{m+n} \rightarrow (q \wedge p)^{**}$.

We deduce that $x \leq (a^{**})^n \rightarrow (q \wedge ((a^{**})^m \rightarrow (p^{**})))^{**} \leq (a^{**})^{m+n} \rightarrow (q \wedge p)^{**}$, hence $x \leq (a^{**})^{m+n} \rightarrow (q \wedge p)^{**}$, that is $p \in B$. Following Theorem 3.12 (ii), we deduce that $B = D_a(B)$, that is ϕ_q is a continuous map.

(3). We prove that $(L, \{\odot\}, \tau_a)$ is a semitopological divisible residuated lattice.

Consider the map $\psi_q : L \rightarrow L$, defined by $\psi_q(x) = q \odot x$, for any $x \in L$. Following Proposition 3.24 the set $\beta_a = \{D_a(\uparrow x) \mid x \in L\}$ is a base for the topology τ_a on L . Then it suffices to prove that $\psi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\psi_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \psi_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \odot p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\} =$

$$\{p \in L \mid x \leq (a^{**})^n \rightarrow (q \odot p)^{**}, \text{ for some } n \in \mathbb{N}\}.$$

Consider the set $C = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \odot p)^{**}, \text{ for some } n \in \mathbb{N}\}$ and we show that the set C is an upset of L . Let $p \in C$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \rightarrow (q \odot p)^{**}$. By (r_4) ,

$(a^{**})^n \rightarrow (q \odot p)^{**} \leq (a^{**})^n \rightarrow (q \odot s)^{**}$, and so $s \in C$. Hence C is an upset. Now, we prove that $D_a(C) = C$, then we deduce that $C \in \tau_a$, that is $\psi_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence ψ_q is a continuous map.

Let $p \in D_a(C)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \rightarrow (p^{**}) \in C$, that is $x \leq (a^{**})^n \rightarrow (q \odot ((a^{**})^m \rightarrow (p^{**})))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$$\begin{aligned} & [(a^{**})^n \rightarrow (q \odot ((a^{**})^m \rightarrow (p^{**})))^{**}] \rightarrow [(a^{**})^{m+n} \rightarrow (q \odot p)^{**}] \stackrel{(r_5)}{=} \\ & [(a^{**})^n \rightarrow (q \odot ((a^{**})^m \rightarrow (p^{**})))^{**}] \rightarrow [(a^{**})^n \rightarrow ((a^{**})^m \rightarrow (q \odot p)^{**})] \stackrel{(r_{39})}{\geq} \\ & [q \odot ((a^{**})^m \rightarrow (p^{**}))]^{**} \rightarrow [(a^{**})^m \rightarrow (q \odot p)^{**}] \stackrel{(\varepsilon_1)}{=} \\ & [q \odot ((a^{**})^m \rightarrow (p^{**}))]^{**} \rightarrow [(a^{**})^m \rightarrow (q \odot p)^{**}]^{**} \stackrel{(r_{22})}{=} \\ & ([q \odot ((a^{**})^m \rightarrow (p^{**}))]) \rightarrow [(a^{**})^m \rightarrow (q \odot p)^{**}]^{**} \stackrel{(r_2)}{=} \\ & ([q \odot (a^{**})^m \odot ((a^{**})^m \rightarrow (p^{**}))]) \rightarrow [(q \odot p)^{**}]^{**} \stackrel{(i_1)}{=} \\ & ([q \odot ((a^{**})^m \wedge (p^{**}))]) \rightarrow [(q \odot p)^{**}]^{**} \stackrel{(\varepsilon_5)}{=} 1^{**} = 1. \end{aligned}$$

By (r_2) , $(a^{**})^n \rightarrow (q \odot ((a^{**})^m \rightarrow (p^{**})))^{**} \leq (a^{**})^{m+n} \rightarrow (q \odot p)^{**}$.

We deduce that $x \leq (a^{**})^n \rightarrow (q \odot ((a^{**})^m \rightarrow (p^{**})))^{**} \leq (a^{**})^{m+n} \rightarrow (q \odot p)^{**}$, hence $x \leq (a^{**})^{m+n} \rightarrow (q \odot p)^{**}$, that is $p \in C$. Following Theorem 3.12 (ii), we deduce that $C = D_a(C)$, that is ψ_q is a continuous map.

Since $\vee, \wedge, \odot$ are commutative and from (1), (2) and (3) we deduce that $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological divisible residuated lattice.

(ii). We prove that $(L, \{\rightarrow\}, \tau_a)$ is a right semitopological divisible residuated lattice.

Consider the map $\omega_q : L \rightarrow L$, defined by $\omega_q(x) = q \rightarrow x$, for any $x \in L$. Following Proposition 3.24 the set $\beta_a = \{D_a(\uparrow x) \mid x \in L\}$ is a base for the topology τ_a on L . Then it suffices to prove that $\omega_q^{-1}(D_a(\uparrow x)) \in \tau_a$, for any $x \in L$.

Let $x \in L$, then we obtain successively $\omega_q^{-1}(D_a(\uparrow x)) = \{p \in L \mid \omega_q(p) \in D_a(\uparrow x)\} = \{p \in L \mid (a^{**})^n \rightarrow (q \rightarrow p)^{**} \in \uparrow x, \text{ for some } n \in \mathbb{N}\} = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \rightarrow p)^{**}, \text{ for some } n \in \mathbb{N}\}$.

Consider the set $D = \{p \in L \mid x \leq (a^{**})^n \rightarrow (q \rightarrow p)^{**}, \text{ for some } n \in \mathbb{N}\}$ and we show that the set D is an upset of L . Let $p \in D$ and $p \leq s$, for some $s \in L$. Then there is $n \in \mathbb{N}$ such that $x \leq (a^{**})^n \rightarrow (q \rightarrow p)^{**}$. By (r_4) , $(a^{**})^n \rightarrow (q \rightarrow p)^{**} \leq (a^{**})^n \rightarrow (q \rightarrow s)^{**}$, and so $s \in D$. Hence D is an upset. Now, we prove that $D_a(D) = D$, then we deduce that $D \in \tau_a$, that is $\omega_q^{-1}(D_a(\uparrow x)) \in \tau_a$, hence ω_q is a continuous map.

Let $p \in D_a(D)$. Then there is $m \in \mathbb{N}$ such that $(a^{**})^m \rightarrow (p^{**}) \in D$, that is $x \leq (a^{**})^n \rightarrow (q \rightarrow ((a^{**})^m \rightarrow (p^{**})))^{**}$, for some $n \in \mathbb{N}$.

We obtain successively

$$\begin{aligned} & (a^{**})^n \rightarrow [q \rightarrow ((a^{**})^m \rightarrow (p^{**}))]^{**} \stackrel{(r_{22})}{=} (a^{**})^n \rightarrow [(q^{**}) \rightarrow ((a^{**})^m \rightarrow (p^{**}))]^{**} \stackrel{(\varepsilon_1)}{=} \\ & (a^{**})^n \rightarrow [(q^{**}) \rightarrow ((a^{**})^m \rightarrow (p^{**}))] \stackrel{(r_5)}{=} (a^{**})^n \rightarrow [((q^{**}) \odot (a^{**})^m) \rightarrow (p^{**})] \stackrel{(r_5)}{=} \\ & (a^{**})^n \rightarrow [(a^{**})^m \rightarrow ((q^{**}) \rightarrow (p^{**}))] \stackrel{(r_5)}{=} [(a^{**})^n \odot (a^{**})^m] \rightarrow [(q^{**}) \rightarrow (p^{**})] = \\ & (a^{**})^{n+m} \rightarrow [(q^{**}) \rightarrow (p^{**})] \stackrel{(r_{22})}{=} (a^{**})^{n+m} \rightarrow (q \rightarrow p)^{**}. \end{aligned}$$

We deduce that $x \leq (a^{**})^n \rightarrow (q \rightarrow ((a^{**})^m \rightarrow (p^{**})))^{**} = (a^{**})^{n+m} \rightarrow (q \rightarrow p)^{**}$, hence $x \leq (a^{**})^{n+m} \rightarrow (q \rightarrow p)^{**}$, that is $p \in D$. Following Theorem 3.12 (ii), we deduce that $D = D_a(D)$, that is ω_q is a continuous map.

Since $\rightarrow$ is not commutative we deduce that $(L, \{\rightarrow\}, \tau_a)$ is a right semitopological divisible residuated lattice. $\square$

We recall ([12]) that a residuated lattice L with double-negation property ($x^{**} = x$, for all $x \in L$) is called *involutive*. It is known that the classes of divisible residuated lattices and involutive residuated lattices are different (see [12]). The following result is an easy consequence of Lemma 2.2, Lemma 5.1 and Theorem 5.3.

Corollary 5.4. *Let L be an involutive residuated lattice and $a \in L$. Then:*

- (i) $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological involutive residuated lattice;
- (ii) $(L, \{\rightarrow\}, \tau_a)$ is a right semitopological involutive residuated lattice.

In [3] it was proved that L is an MV-algebra iff L is an involutive BL-algebra. Therefore, following Theorem 5.3 and Corollary 5.4 we deduce that:

Corollary 5.5. *Let L be an MV-algebra and $a \in L$. Then:*

- (i) $(L, \{\vee, \wedge, \odot\}, \tau_a)$ is a semitopological MV-algebra;
- (ii) $(L, \{\rightarrow\}, \tau_a)$ is a right semitopological MV-algebra.

Corollary 5.6. *Let L be a divisible residuated lattice. For any filter F of L and $a, x \in L$, then:*

- (i) $D_a(\uparrow x)/F = D_{a/F}(\uparrow x/F)$;
- (ii) The topologies $\tau_{a/F}$ and $\bar{\tau}_a$ on L/J are the same, where $\bar{\tau}_a$ is the quotient topology of L/F .

Proof. (i). By Theorem 4.3 (i) we get $D_a(\uparrow x)/F \subseteq D_{a/F}(\uparrow x/F)$, then it remains to prove $D_{a/F}(\uparrow x/F) \subseteq D_a(\uparrow x)/F$. Let $u/F \in D_{a/F}(\uparrow x/F)$. Then there exists $n \in \mathbb{N}$ such that $x \rightarrow [(a^{**})^n \rightarrow u^{**}] \in F$. Let $f = x \rightarrow [(a^{**})^n \rightarrow u^{**}]$. Then by (r_5) we obtain $x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u^{**}]) = f \rightarrow [x \rightarrow ((a^{**})^n \rightarrow u^{**})] = f \rightarrow f = 1$ and so by (r_{15}) and (r_{22}) we get $1 = x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u^{**}]) \leq x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u^{**}]^{**}) = x \rightarrow ((a^{**})^n \rightarrow [f^{**} \rightarrow u^{**}]) = x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u]^{**})$. Hence $f \rightarrow u \in D_a(\uparrow x)$. Since $f \in F$, then $f/F = 1/F$ and so $(f \rightarrow u)/F = f/F \rightarrow u/F = 1/F \rightarrow u/F = (1 \rightarrow u)/F = u/F$. That is $(f \rightarrow u) \equiv_F u$. Therefore, $u/F \in D_a(\uparrow x)/F$.

(ii). By Proposition 3.24 the set $\{D_{a/F}(\uparrow x/F) : x/F \in L/F\}$ is a base for the topology $\tau_{a/F}$ on L/F . Moreover, the set $\{D_a(\uparrow x)/F : x \in L\}$ is a base for the topology $\bar{\tau}_a$. Now, the proof of (ii) is straightforward by (i). $\square$

Corollary 5.7. *Let L be an involutive residuated lattice. For any filter F of L and $a, x \in L$, then:*

- (i) $D_a(\uparrow x)/F = D_{a/F}(\uparrow x/F)$;
- (ii) The topologies $\tau_{a/F}$ and $\bar{\tau}_a$ on L/F are the same, where $\bar{\tau}_a$ is the quotient topology of L/F .

Proof. (i). By Theorem 4.3 (i) we get $D_a(\uparrow x)/F \subseteq D_{a/F}(\uparrow x/F)$, then it remains to prove $D_{a/F}(\uparrow x/F) \subseteq D_a(\uparrow x)/F$. Let $u/F \in D_{a/F}(\uparrow x/F)$. Then there exists $n \in \mathbb{N}$ such that $x \rightarrow [(a^{**})^n \rightarrow u^{**}] \in F$. Let $f = x \rightarrow [(a^{**})^n \rightarrow u^{**}]$. Then by (r_5) we obtain $x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u^{**}]) = f \rightarrow [x \rightarrow ((a^{**})^n \rightarrow u^{**})] = f \rightarrow f = 1$ and so by the involutive property (that is, $x = x^{**}$, for all $x \in L$) we get $1 = x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u^{**}]) = x \rightarrow ((a^{**})^n \rightarrow [f \rightarrow u]^{**})$. Hence $f \rightarrow u \in D_a(\uparrow x)$. Since $f \in F$, then $f/F = 1/F$ and so $(f \rightarrow u)/F = f/F \rightarrow u/F = 1/F \rightarrow u/F = (1 \rightarrow u)/F = u/F$. That is $(f \rightarrow u) \equiv_F u$. Therefore, $u/F \in D_a(\uparrow x)/F$.

(ii). By Proposition 3.24 the set $\{D_{a/F}(\uparrow x/F) : x/F \in L/F\}$ is a base for the topology $\tau_{a/F}$ on L/F . Moreover, the set $\{D_a(\uparrow x)/F : x \in L\}$ is a base for the topology $\bar{\tau}_a$. Now, the proof of (ii) is straightforward by (i). $\square$

6 Conclusions

In Borzooei and Zahiri (2014)[11] the definition of double complemented elements for any filter F in BL-algebras, initially, introduced by A. Borumand Saeid and S. Motamed (2009)[10], was generalized to the concept of $D_\gamma(F)$, for any upset F of the BL-algebra L . Their aim was to show that any BL-algebra L with that topology is a semitopological BL-algebra, so in that way they could construct many semitopological BL-algebras. Our goal is to extend the study in the case of residuated lattices. Therefore, we work on a special type of topology induced by a modal and closure operator denoted by $D_a(X)$, for an upset X of a residuated lattice L , where a is an element of L .

We discuss briefly the properties and applications of the operator $D_a(\cdot)$ in residuated lattices. We obtain some of the important topological aspects of these structures such as connectivity and compactness. We study some properties of quotient topologies on residuated lattices by considering two types of quotient topologies denoted by $\tau_{a/F}$ and $\bar{\tau}_a$. We study the uniform topology $\tau_{\bar{A}}$ and we obtain important characterizations as $(L/J, \tau_{\bar{A}})$ becomes a Hausdorff space iff J is closed relative to the uniform topology. Also, we study some properties of the direct product of residuated lattices.

Finally, we apply our results on classes of residuated lattices such as *divisible residuated lattices*, *MV-algebras* and *involutive residuated lattices* and we find that any of this subclasses of residuated lattices with respect to these topologies form semitopological algebras.

In this way we can construct many semitopological algebras on residuated lattices.

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