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Stability and convergence of a local discontinuous Galerkin finite element method for the general Lax equation

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Abstract: In this paper we develop and analyze the local discontinuous Galerkin (LDG) finite element method for solving the general Lax equation. The local discontinuous Galerkin method has the flexibility for arbitrary h and p adaptivity, and allows for hanging nodes. By choosing the numerical fluxes carefully we prove stability and give an error estimate. Finally some numerical examples are computed to show the convergence order and excellent numerical performance of proposed method.

Keywords: Lax equation, Local discontinuous Galerkin method, Stability analysis, Error estimates

MSC: 35S10, 65M12

1 Introduction

In this paper we develop a local discontinuous Galerkin (LDG) method for general Lax equation

$$\begin{aligned} &u_t + \alpha u u_{xxx} + 2\alpha u_x u_{xx} + \frac{3}{10}\alpha^2 u^2 u_x + u_{xxxxx} = 0, \\ &u(x,0) = u_0(x). \end{aligned} \tag{1}$$

where α are arbitrary nonzero and real parameters. We do not pay attention to boundary condition; hence the solution is considered to be either periodic or compactly supported.

There are only a few numerical works in the literature to solve the Lax equation. Xu and Shu [1] simulate the solutions of the Kawahara equation, the generalized Kawahara equation and Ito's fifth-order mKdV equation. The general Lax equation discussed in our paper is different from the class of fifth-order equation in [1]. The general Lax equation (1) is an important mathematical model with wide applications in quantum mechanics, nonlinear optics, and describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice [2-4]. Typical examples are widely used in various fields such as solid state physics, plasma physics, fluid physics and quantum field theory. It is well known that the Lax equation is completely integrable equation, and it has many sets of conservation laws [5, 6]. For numerical study, Abdul-Majid Wazwaz [7] revealed solitons and periodic solutions for the fifth-order nonlinear KdV equation using the sine-cosine and the tanh methods. In [8] Cesar A studied the periodic and soliton solutions for the Lax equation using a generalization of extended tanh method. The DG method is beneficial for parallel computing

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and has high-order accuracy. Meanwhile, it is flexibility and efficiency in terms of mesh and shape functions. To our best knowledge, this is the first provably stable finite element method for the Lax equation.

The paper is organized as follows. In Section 2, notation and some preliminaries are described and cited. In Section 3, we discuss the LDG scheme for the general Lax equation, and prove the cell entropy inequality and L^2 stability by choosing the fluxes carefully in Section 4. In Section 5 we give an error estimate, and some numerical experiments to illustrate the accuracy and capability of the method are given in Section 6. Concluding remarks are provided in Section 7.

2 Notations and auxiliary results

In this section we introduce notations and definitions to be used later in the paper and also present some auxiliary results.

2.1 Basic notations

We denote the mesh in $[a, b]$ by $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ for $j = 1, 2\cdots N$. The center of the cell is $x_j = \frac{1}{2}(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})$, and the mesh size is denoted $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, with $h = \max_{1\le j\le N} h_j$ being the maximum mesh size. We assume that the mesh is regular, namely, that the ratio between the maximum and the minimum mesh sizes stays bounded during mesh refinements. We define the piecewise-polynomial space V_h as the space of polynomials of the degree up to k in each cell I_j, i.e.

$$V_h = \{v : v \in P^k(I_j), x \in I_j, j = 1, 2, \cdots N\}.$$

Note that functions in V_h are allowed to have discontinuities across element interfaces.

The solution of the numerical scheme is denoted by u_h, which belongs to the finite element space V_h. We denote by $(u_h)^+_{j+\frac{1}{2}}$ and $(u_h)^-_{j+\frac{1}{2}}$ the values of u_h at $x_{j+\frac{1}{2}}$ from the right cell I_{j+1} and from the left cell I_j, respectively. $[u_h]$ is used to denote $u_h^+ - u_h^-$, i.e. the jump of u_h at cell interfaces.

Lemma 2.1 ([9]). *For any piecewise smooth function $\omega \in L^2(\Omega)$, on each cell boundary point we define*

$$\beta(\widehat{f};\omega) = \beta(\widehat{f};\omega^-,\omega^+) = \begin{cases} [\omega]^{-1}(f(\bar{\omega}) - \widehat{f}(\omega)) & if [\omega] \neq 0, \\ \frac{1}{2}|f'(\bar{\omega})| & if [\omega] = 0, \end{cases} \tag{2}$$

where $\widehat{f}(\omega^-,\omega^+)$ is a monotone numerical flux consistent with the given flux f. Then $\beta(\widehat{f};\omega)$ is non-negative and bounded.

In the present paper we use C to denote a positive constant which may have a different value in each occurrence. The usual notation of norms in Sobolev spaces will be used. For any integer $s \ge 0$, let $H^s(\Omega)$ represent the well-known Sobolev space equipped with the norm $\|\cdot\|_s$, which consists of functions with (distributional) derivatives of order not greater than s in $L^2(\Omega)$. Next, let the scalar inner product on L^2 be denoted by $(\cdot,\cdot)$, and the associated norm by $\|\cdot\|$.

2.2 Projection

We will give the projection in one dimension $[a, b]$, denoted by $\mathbb{P}$, i.e.,for each j,

$$\int_{I_j} (\mathbb{P}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^k(I_i), \tag{3}$$

and special projection $\mathbb{P}^{\pm}$, i.e.,for each j,

$$\begin{aligned}
&\int_{I_j} (\mathbb{P}^{+}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j),\\
\text{and}\ \ &\mathbb{P}^{+}\omega(x^{+}_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}}),\\
&\int_{I_j} (\mathbb{P}^{-}\omega(x) - \omega(x))v(x) = 0, \forall v \in P^{k-1}(I_j),\\
\text{and}\ \ &\mathbb{P}^{-}\omega(x^{-}_{j+\frac{1}{2}}) = \omega(x_{j+\frac{1}{2}}).
\end{aligned} \tag{4}$$

There are some approximation results for the projection in [10–12]

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{\frac{1}{2}}\|\omega^e\|_{\tau_h} \le Ch^{k+1}. \tag{5}$$

where $\omega^e = \mathbb{P}\omega - \omega$ or $\omega^e = \mathbb{P}^{\pm}\omega - \omega$, and

$$\|\omega^e\|_{\tau_h} = (\frac{1}{2}\sum_{i=1}^{N}[((\omega^e)^+)^2_{i-\frac{1}{2}} + ((\omega^e)^-)^2_{i+\frac{1}{2}}])^{\frac{1}{2}}.$$

The positive constant C, solely depending on ω, is independent of h. τ_h denotes the set of boundary points of all elements I_j.

3 LDG Scheme

In this section, we define our LDG method for the general Lax equation (1), written in the following form:

$$u_t + (\frac{\alpha}{2}u^2)_{xxx} - (\frac{\alpha}{2}u_x^2)_x + (\frac{\alpha^2}{10}u^3)_x + u_{xxxxx} = 0. \tag{6}$$

To define the local discontinuous Galerkin method, we rewrite equation (1) as a first-order system:

$$\begin{aligned}
&f(u) = \frac{\alpha^2}{10}u^3,\ \ B(z) = \frac{\alpha}{2}z^2,\ \ b(z) = B'(z) = \alpha z,\ \ p = (b(z)u)_x,\\
&z - u_x = 0,\ \ q - z_x = 0,\ \ v - q_x = 0,\ \ w - v_x = 0,\\
&u_t + f(u)_x + p_x - B(z)_x + w_x = 0.
\end{aligned} \tag{7}$$

Now we can use the local discontinuous Galerkin method to equation (1), resulting in the following scheme: find $u_h, p_h, w_h, v_h, q_h, z_h \in V_h$, such that for all test functions $\rho, \phi, \varphi, \eta, \xi, \theta \in V_h$

$$\begin{aligned}
&\int_{I_j} (u_h)_t\rho dx - \int_{I_j} f(u_h)\rho_x dx - \int_{I_j} p_h\rho_x dx + \int_{I_j} B(z_h)\rho_x dx\\
&\quad - \int_{I_j} w_h\rho_x dx + (\widehat{f(u_h)}\rho^-)_{j+\frac{1}{2}} - (\widehat{f(u_h)}\rho^+)_{j-\frac{1}{2}}\\
&\quad + (\widehat{p}_h\rho^-)_{j+\frac{1}{2}} - (\widehat{p}_h\rho^+)_{j-\frac{1}{2}} - (\widehat{B(z_h)}\rho^-)_{j+\frac{1}{2}}\\
&\quad + (\widehat{B(z_h)}\rho^+)_{j-\frac{1}{2}} + (\widehat{w}_h\rho^-)_{j+\frac{1}{2}} - (\widehat{w}_h\rho^+)_{j-\frac{1}{2}} = 0,
\end{aligned} \tag{8a}$$

$$\int_{I_j} p_h\phi dx + \int_{I_j} b(z_h)u_h\phi_x dx - (\widehat{b(z_h)}\widetilde{u}_h\phi^-)_{j+\frac{1}{2}} + (\widehat{b(z_h)}\widetilde{u}_h\phi^+)_{j-\frac{1}{2}} = 0, \tag{8b}$$

$$\int_{I_j} w_h\varphi dx + \int_{I_j} v_h\varphi_x dx - (\widehat{v}_h\varphi^-)_{j+\frac{1}{2}} + (\widehat{v}_h\varphi^+)_{j-\frac{1}{2}} = 0, \tag{8c}$$

$$\int_{I_j} v_h\eta dx + \int_{I_j} q_h\eta_x dx - (\widehat{q}_h\eta^-)_{j+\frac{1}{2}} + (\widehat{q}_h\eta^+)_{j-\frac{1}{2}} = 0, \tag{8d}$$

$$\int_{I_j} q_h \xi dx + \int_{I_j} z_h \xi_x dx - (\widehat{z_h}\xi^-)_{j+\frac{1}{2}} + (\widehat{z_h}\xi^+)_{j-\frac{1}{2}} = 0, \tag{8e}$$

$$\int_{I_j} z_h \theta dx + \int_{I_j} u_h \theta_x dx - (\widehat{u_h}\theta^-)_{j+\frac{1}{2}} + (\widehat{u_h}\theta^+)_{j-\frac{1}{2}} = 0. \tag{8f}$$

The "hat" terms in (8) in the cell boundary terms from integration by parts are the so-called "numerical fluxes", which are single valued functions defined on the edges and should be designed based on different guiding principles for different PDEs to ensure stability. It turns out that we can take the simple choices such that

$$\begin{aligned} &\widehat{u_h} = u_h^+, \ \ \widehat{z_h} = z_h^+ - \tau_1[v_h], \ \ \widehat{v_h} = v_h^- - \tau_2[z_h], \ \ \widehat{p_h} = p_h^-, \ \ \widehat{w_h} = w_h^-, \\ &\widehat{q_h} = q_h^-, \ \ \widehat{b(z_h)} = \frac{B(z_h^+) - B(z_h^-)}{z_h^+ - z_h^-}, \ \ \widetilde{u_h} = u_h^+, \ \ \widehat{B(z_h)} = B(z_h^-), \end{aligned} \tag{9}$$

where $\tau_1, \tau_2 > 0$. Some dissipation terms in the flux of $\widehat{z_h}, \widehat{v_h}$ will give a control on the boundary terms. We have omitted the half-integer indices $j - \frac{1}{2}$ as all quantities in (8) are computed at the same points (i.e. the interfaces between the cells). The flux $\widehat{f}$ is a monotone flux. Examples of monotone fluxes which are suitable for the local discontinuous Galerkin methods can be found in, e.g., [13]. For example, one could use the Lax-Friedriches flux, which is given by

$$\widehat{f}^{LF}(w^-, w^+) = \frac{1}{2}(f(w^-) + f(w^+) - \alpha(w^+ - w^-)), \ \alpha = \max_w |f'(w)|. \tag{10}$$

We remark that the choice for the fluxes (9) is not unique. In fact the crucial part is taking $\widehat{u_h}$ and $\widehat{p_h}$, $\widehat{w_h}$ from opposite sides, taking $\widehat{v_h}$ except the dissipation term and $\widehat{z_h}$ except the dissipation term from opposite sides, taking $\widehat{B(z_h)}$ and $\widetilde{u_h}$ from opposite sides, and $\widehat{q_h} = q_h^-$ [14–16].

With such a choice of fluxes we can get the theoretical results of the L^2 stability.

4 Stability analysis

Theorem 4.1 (cell entropy inequality). *For periodic or compactly supported boundary conditions, the solution u_h to the semi-discrete LDG scheme (8) satisfies the following cell entropy inequality*

$$\frac{1}{2}\frac{d}{dt}\int_{I_j} u_h^2 dx + \Phi_{j+\frac{1}{2}} - \Phi_{j-\frac{1}{2}} \le 0. \tag{11}$$

Proof. Choosing the test function $\theta = -w_h$ in (8f), we obtain

$$-\int_{I_j} z_h w_h dx - \int_{I_j} u_h (w_h)_x dx + (\widehat{u_h} w_h^-)_{j-\frac{1}{2}} - (\widehat{u_h} w_h^+)_{j-\frac{1}{2}} = 0. \tag{12}$$

Since (8) holds for any test functions in V_h, we can choose

$$\rho = u_h, \phi = z_h, \varphi = z_h, \eta = -q_h, \xi = v_h, \theta = -p_h \tag{13}$$

Then we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{I_j} u_h^2 dx - \int_{I_j} f(u_h)(u_h)_x dx - \int_{I_j} p_h (u_h)_x dx + \int_{I_j} B(z_h)(u_h)_x dx \\ &\quad - \int_{I_j} w_h (u_h)_x dx + (\widehat{f(u_h)} u_h^-)_{j+\frac{1}{2}} - (\widehat{f(u_h)} u_h^+)_{j-\frac{1}{2}} \\ &\quad + (\widehat{p_h} u_h^-)_{j+\frac{1}{2}} - (\widehat{p_h} u_h^+)_{j-\frac{1}{2}} - (\widehat{B(z_h)} u_h^-)_{j+\frac{1}{2}} \end{aligned}$$

$$+(\widehat{B(z_h)}u_h^+)_{j-\frac{1}{2}} + (\widehat{w_h}u_h^-)_{j+\frac{1}{2}} - (\widehat{w_h}u_h^+)_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} p_h z_h dx + \int_{I_j} b(z_h)u_h(z_h)_x dx - (\widehat{b(z_h)}\widetilde{u_h}z_h^-)_{j+\frac{1}{2}} + (\widehat{b(z_h)}\widetilde{u_h}z_h^+)_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} w_h z_h dx + \int_{I_j} v_h(z_h)_x dx - (\widehat{v_h}z_h^-)_{j+\frac{1}{2}} + (\widehat{v_h}z_h^+)_{j-\frac{1}{2}} = 0, \tag{14}$$
$$-\int_{I_j} v_h q_h dx - \int_{I_j} q_h(q_h)_x dx + (\widehat{q_h}q_h^-)_{j+\frac{1}{2}} - (\widehat{q_h}q_h^+)_{j-\frac{1}{2}} = 0,$$
$$\int_{I_j} q_h v_h dx + \int_{I_j} z_h(v_h)_x dx - (\widehat{z_h}v_h^-)_{j+\frac{1}{2}} + (\widehat{z_h}v_h^+)_{j-\frac{1}{2}} = 0,$$
$$-\int_{I_j} z_h p_h dx - \int_{I_j} u_h(p_h)_x dx + (\widehat{u_h}p_h^-)_{j+\frac{1}{2}} - (\widehat{u_h}p_h^+)_{j-\frac{1}{2}} = 0.$$

Summing up Eqs. (12) and (14), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{I_j} u_h^2 dx - F(u_h^-)_{j+\frac{1}{2}} + F(u_h^+)_{j-\frac{1}{2}} + (\widehat{f(u_h)}u_h^-)_{j+\frac{1}{2}} - (\widehat{f(u_h)}u_h^+)_{j-\frac{1}{2}}\\
&+(B(z_h^-)u_h^-)_{j+\frac{1}{2}} - (B(z_h^+)u^+)_{j-\frac{1}{2}} - (\widehat{B(z_h)}u_h^-)_{j+\frac{1}{2}} + (\widehat{B(z_h)}u_h^+)_{j-\frac{1}{2}}\\
&-(\widehat{b(z_h)}\widetilde{u_h}z_h^-)_{j+\frac{1}{2}} + (\widehat{b(z_h)}\widetilde{u_h}z_h^+)_{j-\frac{1}{2}} - (p_h^-u_h^-)_{j+\frac{1}{2}} + (p_h^+u_h^+)_{j-\frac{1}{2}}\\
&+(\widehat{p_h}u_h^-)_{j+\frac{1}{2}} - (\widehat{p_h}u_h^+)_{j-\frac{1}{2}} + (\widehat{u_h}p_h^-)_{j+\frac{1}{2}} - (\widehat{u_h}p_h^+)_{j-\frac{1}{2}}\\
&-(w_h^-u_h^-)_{j+\frac{1}{2}} + (w_h^+u_h^+)_{j-\frac{1}{2}} + (\widehat{w_h}u_h^-)_{j+\frac{1}{2}} - (\widehat{w_h}u_h^+)_{j-\frac{1}{2}}\\
&+(\widehat{u_h}w_h^-)_{j+\frac{1}{2}} - (\widehat{u_h}w_h^+)_{j-\frac{1}{2}} + (v_h^-z_h^-)_{j+\frac{1}{2}} - (v_h^+z_h^+)_{j-\frac{1}{2}}\\
&-(\widehat{v_h}z_h^-)_{j+\frac{1}{2}} + (\widehat{v_h}z_h^+)_{j-\frac{1}{2}} - (\widehat{z_h}v_h^-)_{j+\frac{1}{2}} + (\widehat{z_h}v_h^+)_{j-\frac{1}{2}}\\
&-\frac{1}{2}(q_h^-)^2_{j+\frac{1}{2}} + \frac{1}{2}(q_h^+)^2_{j-\frac{1}{2}} + (\widehat{q_h}q_h^-)_{j+\frac{1}{2}} - (\widehat{q_h}q_h^+)_{j-\frac{1}{2}} = 0.
\end{aligned}\tag{15}$$

where $F(u_h) = \int^{u_h} f(s)ds$. We introduce a short-hand notation

$$\begin{aligned}
\Phi_{j+\frac{1}{2}} = &-F(u_h^-)_{j+\frac{1}{2}} + (\widehat{f(u_h)}u_h^-)_{j+\frac{1}{2}} + (B(z_h^-)u^-)_{j+\frac{1}{2}} - (\widehat{B(z_h)}u^-)_{j+\frac{1}{2}}\\
&-(\widehat{b(z_h)}\widetilde{u_h}z_h^-)_{j+\frac{1}{2}} - (p_h^-u_h^-)_{j+\frac{1}{2}} + (\widehat{p_h}u_h^-)_{j+\frac{1}{2}} + (\widehat{u_h}p_h^-)_{j+\frac{1}{2}}\\
&-(w_h^-u_h^-)_{j+\frac{1}{2}} + (\widehat{w_h}u_h^-)_{j+\frac{1}{2}} + (\widehat{u_h}w_h^-)_{j+\frac{1}{2}} + (v_h^-z_h^-)_{j+\frac{1}{2}}\\
&-(\widehat{v_h}z_h^-)_{j+\frac{1}{2}} - (\widehat{z_h}v_h^-)_{j+\frac{1}{2}} - \frac{1}{2}(q_h^-)^2_{j+\frac{1}{2}} + (\widehat{q_h}q_h^-)_{j+\frac{1}{2}}
\end{aligned}\tag{16}$$

Then we have

$$\frac{1}{2}\frac{d}{dt}\int_{I_j} u_h^2 dx + \Phi_{j+\frac{1}{2}} - \Phi_{j-\frac{1}{2}} + \Theta_{j-\frac{1}{2}} = 0. \tag{17}$$

and the extra term Θ is given by

$$\begin{aligned}
\Theta_{j-\frac{1}{2}} = &([F(u_h)] - \widehat{f(u_h)}[u_h] - [B(z_h)u_h] + \widehat{B(z_h)}[u_h] + \widehat{b(z_h)}\widetilde{u_h}[z_h]\\
&+[u_hp_h] - \widehat{p_h}[u_h] - \widehat{u_h}[p_h] + [u_hw_h] - \widehat{w_h}[u_h] - \widehat{u_h}[w_h]\\
&-[v_hz_h] + \widehat{v_h}[z_h] + \widehat{z_h}[v_h] + [\frac{q_h^2}{2}] - \widehat{q_h}[q_h])_{j-\frac{1}{2}} + \tau_1[v_h]^2 + \tau_2[z_h]^2.
\end{aligned}\tag{18}$$

With the definition (9) of the numerical fluxes and after some algebraic manipulation, we easily obtain

$$-[B(z_h)u_h] + \widehat{B(z_h)}[u_h] + \widehat{b(z_h)}\widetilde{u_h}[z_h] = 0$$
$$[u_hp_h] - \widehat{p_h}[u_h] - \widehat{u_h}[p_h] = 0$$

$$[u_h w_h] - \widehat{w_h}[u_h] - \widehat{u_h}[w_h] = 0$$
$$-[v_h z_h] + \widehat{v_h}[z_h] + \widehat{z_h}[v_h] = 0$$
$$[\frac{q_h^2}{2}] - \widehat{q_h}[q_h] = \frac{1}{2}[q_h]^2$$

and hence

$$\Theta_{j-\frac{1}{2}} = ([F(u_h)] - \widehat{f(u_h)}[u_h])_{j-\frac{1}{2}} \geq 0. \tag{19}$$

where the last inequality follows from the monotonicity of the flux (10). This finishes the proof of the cell entropy inequality. □

Summing up over I_j, we obtain the following L^2 stability of numerical solution.

Theorem 4.2 (L^2 stability)**.** *The solution u to the semi-discrete LDG scheme (8) satisfies the following L^2 stability*

$$\frac{1}{2}\frac{d}{dt}\int_a^b u_h^2 dx \leq 0.$$

5 Error estimates

We state the main error estimates of the semi-discrete LDG scheme (8). We have the following theorem.

Theorem 5.1. *Let u be the exact solution of the problem (1), which is sufficiently smooth with bounded derivatives. Let u_h be the numerical solution of the semi-discrete LDG scheme (8). For rectangular triangulations of $I_i \times J_j$, if the finite element space V_h is the piecewise polynomials of degree $k \geq 2$, then for small enough h there holds the following error estimates*

$$\|u - u_h\| \leq Ch^k. \tag{20}$$

Proof. First we would like to make an a priori assumption that, for small enough h, there holds [17]

$$\|u - u_h\| \leq h. \tag{21}$$

Suppose that the interpolation property (5) is satisfied, then the a priori assumption (21) implies that

$$\|u - u_h\|_\infty \leq Ch^{\frac{1}{2}}, \quad \|\mathbb{Q}u - u_h\|_\infty \leq Ch^{\frac{1}{2}}, \tag{22}$$

where $\mathbb{Q} = \mathbb{P}$ or $\mathbb{Q} = \mathbb{P}^\pm$ is the projection operator.

Notice that the equations (8) are also satisfied when the numerical solutions $u_h, p_h, w_h, v_h, q_h, z_h$ are replaced by the exact solutions u, p, w, v, q, z. We then obtain the cell error equation

$$\begin{aligned}
&\int_{I_j} (u - u_h)_t \rho dx - \int_{I_j} (f(u) - f(u_h))\rho_x dx - \int_{I_j} (p - p_h)\rho_x dx + \int_{I_j} (B(z) - B(z_h))\rho_x dx \\
&\quad - \int_{I_j} (w - w_h)\rho_x dx + ((f(u) - \widehat{f(u_h)})\rho^-)_{j+\frac{1}{2}} - ((f(u) - \widehat{f(u_h)})\rho^+)_{j-\frac{1}{2}} \\
&\quad + ((p - \widehat{p_h})\rho^-)_{j+\frac{1}{2}} - ((p - \widehat{p_h})\rho^+)_{j-\frac{1}{2}} - ((B(z) - \widehat{B(z_h)})\rho^-)_{j+\frac{1}{2}} \\
&\quad + ((B(z) - \widehat{B(z_h)})\rho^+)_{j-\frac{1}{2}} + ((w - \widehat{w_h})\rho^-)_{j+\frac{1}{2}} - ((w - \widehat{w_h})\rho^+)_{j-\frac{1}{2}} + \int_{I_j} (p - p_h)\phi dx
\end{aligned}$$

$$\begin{aligned}
&+\int_{I_j}(b(z)u-b(z_h))u_h\phi_x dx-((b(z)u-\widehat{b(z_h)}\widetilde{u_h})\phi^-)_{j+\frac{1}{2}}+((b(z)u-\widehat{b(z_h)}\widetilde{u_h})\phi^+)_{j-\frac{1}{2}}\\
&+\int_{I_j}(w-w_h)\varphi dx+\int_{I_j}(v-v_h)\varphi_x dx-((v-\widehat{v_h})\varphi^-)_{j+\frac{1}{2}}+((v-\widehat{v_h})\varphi^+)_{j-\frac{1}{2}}\\
&+\int_{I_j}(v-v_h)\eta dx+\int_{I_j}(q-q_h)\eta_x dx-((q-\widehat{q_h})\eta^-)_{j+\frac{1}{2}}+((q-\widehat{q_h})\eta^+)_{j-\frac{1}{2}}\\
&+\int_{I_j}(q-q_h)\xi dx+\int_{I_j}(z-z_h)\xi_x dx-((z-\widehat{z_h})\xi^-)_{j+\frac{1}{2}}+((z-\widehat{z_h})\xi^+)_{j-\frac{1}{2}}\\
&+\int_{I_j}(z-z_h)\theta dx+\int_{I_j}(u-u_h)\theta_x dx-((u-\widehat{u_h})\theta^-)_{j+\frac{1}{2}}+((u-\widehat{u_h})\theta^+)_{j-\frac{1}{2}}=0.
\end{aligned} \tag{23}$$

Define

$$\begin{aligned}
&\mathfrak{A}_j(u-u_h,p-p_h,w-w_h,v-v_h,q-q_h,z-z_h;\rho,\phi,\varphi,\eta,\xi,\theta)\\
&=\int_{I_j}(u-u_h)_t\rho dx-\int_{I_j}(p-p_h)\rho_x dx-\int_{I_j}(w-w_h)\rho_x dx\\
&\quad+((p-\widehat{p_h})\rho^-)_{j+\frac{1}{2}}-((p-\widehat{p_h})\rho^+)_{j-\frac{1}{2}}+((w-\widehat{w_h})\rho^-)_{j+\frac{1}{2}}\\
&\quad-((w-\widehat{w_h})\rho^+)_{j-\frac{1}{2}}+\int_{I_j}(p-p_h)\phi dx+\int_{I_j}(w-w_h)\varphi dx\\
&\quad+\int_{I_j}(v-v_h)\varphi_x dx-((v-\widehat{v_h})\varphi^-)_{j+\frac{1}{2}}+((v-\widehat{v_h})\varphi^+)_{j-\frac{1}{2}}\\
&\quad+\int_{I_j}(v-v_h)\eta dx+\int_{I_j}(q-q_h)\eta_x dx-((q-\widehat{q_h})\eta^-)_{j+\frac{1}{2}}+((q-\widehat{q_h})\eta^+)_{j-\frac{1}{2}}\\
&\quad+\int_{I_j}(q-q_h)\xi dx+\int_{I_j}(z-z_h)\xi_x dx-((z-\widehat{z_h})\xi^-)_{j+\frac{1}{2}}+((z-\widehat{z_h})\xi^+)_{j-\frac{1}{2}}\\
&\quad+\int_{I_j}(z-z_h)\theta dx+\int_{I_j}(u-u_h)\theta_x dx-((u-\widehat{u_h})\theta^-)_{j+\frac{1}{2}}+((u-\widehat{u_h})\theta^+)_{j-\frac{1}{2}},
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\mathfrak{H}_j(f,u,u_h;\rho)\\
&=\int_{I_j}(f(u)-f(u_h))\rho_x dx-((f(u)-\widehat{f(u_h)})\rho^-)_{j+\frac{1}{2}}+((f(u)-\widehat{f(u_h)})\rho^+)_{j-\frac{1}{2}},
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
&\mathfrak{R}_j(B,b;z,u,z_h,u_h;\rho,\phi)\\
&=-\int_{I_j}(B(z)-B(z_h))\rho_x dx+((B(z)-\widehat{B(z_h)})\rho^-)_{j+\frac{1}{2}}-((B(z)-\widehat{B(z_h)})\rho^+)_{j-\frac{1}{2}}\\
&\quad-\int_{I_j}(b(z)u-b(z_h))u_h\phi_x dx+((b(z)u-\widehat{b(z_h)}\widetilde{u_h})\phi^-)_{j+\frac{1}{2}}-((b(z)u-\widehat{b(z_h)}\widetilde{u_h})\phi^+)_{j-\frac{1}{2}}.
\end{aligned} \tag{26}$$

Summing over j, the error equation (23) becomes

$$\begin{aligned}
&\sum_{j=1}^{N}\mathfrak{A}_{ij}(u-u_h,p-p_h,w-w_h,v-v_h,q-q_h,z-z_h;\rho,\phi,\varphi,\eta,\xi,\theta)\\
&=\mathfrak{H}_j(f,u,u_h;\rho)+\mathfrak{R}_j(B,b;z,u,z_h,u_h;\rho,\phi).
\end{aligned} \tag{27}$$

Denoting

$$
\begin{aligned}
e_u &= u - u_h = \mathbb{P}^+ u - u_h - (\mathbb{P}^+ u - u) = \mathbb{P}^+ e_u - (\mathbb{P}^+ u - u),\\
e_p &= p - p_h = \mathbb{P}p - p_h - (\mathbb{P}p - p) = \mathbb{P}e_p - (\mathbb{P}p - p),\\
e_w &= w - w_h = \mathbb{P}w - w_h - (\mathbb{P}w - w) = \mathbb{P}e_w - (\mathbb{P}w - w),\\
e_v &= v - v_h = \mathbb{P}v - v_h - (\mathbb{P}v - v) = \mathbb{P}e_v - (\mathbb{P}v - v),\\
e_q &= q - q_h = \mathbb{P}q - q_h - (\mathbb{P}q - q) = \mathbb{P}e_q - (\mathbb{P}q - q),\\
e_z &= z - z_h = \mathbb{P}z - z_h - (\mathbb{P}z - z) = \mathbb{P}e_z - (\mathbb{P}z - z).
\end{aligned}
$$

Take the test fuctions

$$
\rho = \mathbb{P}^+ e_u, \phi = \mathbb{P}e_z, \varphi = \mathbb{P}e_z, \eta = -\mathbb{P}e_q, \xi = \mathbb{P}e_v, \theta = -(\mathbb{P}e_w + \mathbb{P}e_p),
$$

from the linearity of $\mathfrak{A}_j$ we obtain the energy equality

$$
\begin{aligned}
&\sum_{j=1}^{N}\mathfrak{A}_j(\mathbb{P}^+ e_u, \mathbb{P}e_p, \mathbb{P}e_w, \mathbb{P}e_v, \mathbb{P}e_q, \mathbb{P}e_z; \mathbb{P}^+ e_u, \mathbb{P}e_z, \mathbb{P}e_z, -\mathbb{P}e_q, \mathbb{P}e_v, -(\mathbb{P}e_w + \mathbb{P}e_p))\\
&= \sum_{j=1}^{N}\mathfrak{A}_j(\mathbb{P}^+ u - u, \mathbb{P}p - p, \mathbb{P}w - w, \mathbb{P}v - v, \mathbb{P}q - q, \mathbb{P}z - z; \mathbb{P}^+ e_u, \mathbb{P}e_z, \mathbb{P}e_z,\\
&\quad - \mathbb{P}e_q, \mathbb{P}e_v, -(\mathbb{P}e_w + \mathbb{P}e_p)) + \mathfrak{H}_j(f, u, u_h; \mathbb{P}^+ e_u)\\
&\quad + \mathfrak{R}_j(B, b; z, u, z_h, u_h; \mathbb{P}^+ e_u, \mathbb{P}e_z).
\end{aligned}
\tag{28}
$$

First we consider the left-hand side of the energy equation (28).

Lemma 5.2. *The following equation holds*

$$
\begin{aligned}
&\sum_{j=1}^{N}\mathfrak{A}_j(\mathbb{P}^+ e_u, \mathbb{P}e_p, \mathbb{P}e_w, \mathbb{P}e_v, \mathbb{P}e_q, \mathbb{P}e_z; \mathbb{P}^+ e_u, \mathbb{P}e_z, \mathbb{P}e_z, -\mathbb{P}e_q, \mathbb{P}e_v, -(\mathbb{P}e_w + \mathbb{P}e_p))\\
&= \frac{1}{2}\frac{d}{dt}\int_a^b (\mathbb{P}^+ e_u)^2 dx + \sum_{j=1}^{N}\Big(\frac{1}{2}[\mathbb{P}e_q]^2 + \tau_1[\mathbb{P}e_v]^2 + \tau_2[\mathbb{P}e_z]^2\Big)_{j-\frac{1}{2}}.
\end{aligned}
\tag{29}
$$

The proof is by the same argument as that used for the stability result in Section 4.

Lemma 5.3. *There exist numerical entropy fluxes $\mathcal{B}_{j+\frac{1}{2}}$, such that the following equation holds*

$$
\begin{aligned}
&\mathfrak{A}_j(\mathbb{P}^+ u - u, \mathbb{P}p - p, \mathbb{P}w - w, \mathbb{P}v - v, \mathbb{P}q - q, \mathbb{P}z - z;\\
&\quad \mathbb{P}^+ e_u, \mathbb{P}e_z, \mathbb{P}e_z, -\mathbb{P}e_q, \mathbb{P}e_v, -(\mathbb{P}e_w + \mathbb{P}e_p))\\
&\le \int_{I_j} \frac{1}{2}(\mathbb{P}^+ e_u)^2 dx + \mathcal{B}_{j+\frac{1}{2}} + \mathcal{B}_{j-\frac{1}{2}}\\
&\quad + \varepsilon([\mathbb{P}^+ e_u]^2 + [\mathbb{P}e_z]^2 + [\mathbb{P}e_q]^2 + [\mathbb{P}e_v]^2)_{j-\frac{1}{2}}.
\end{aligned}
\tag{30}
$$

Proof. As to the first term of right-hand side in (28), we first write out all the terms

$$
\begin{aligned}
&\mathfrak{A}_j(\mathbb{P}^+u-u,\mathbb{P}p-p,\mathbb{P}w-w,\mathbb{P}v-v,\mathbb{P}q-q,\mathbb{P}z-z;\mathbb{P}^+e_u,\mathbb{P}e_z,\\
&\quad\mathbb{P}e_z,-\mathbb{P}e_q,\mathbb{P}e_v,-(\mathbb{P}e_w+\mathbb{P}e_p))\\
&=\int_{I_j}(\mathbb{P}^+u-u)_t\mathbb{P}^+e_u dx-\int_{I_j}(\mathbb{P}p-p)(\mathbb{P}^+e_u)_x dx-\int_{I_j}(\mathbb{P}w-w)(\mathbb{P}^+e_u)_x dx\\
&\quad+((\mathbb{P}p-p)^-(\mathbb{P}^+e_u)^-)_{j+\frac12}-((\mathbb{P}p-p)^-(\mathbb{P}^+e_u)^+)_{j-\frac12}+((\mathbb{P}w-w)^-(\mathbb{P}^+e_u)^-)_{j+\frac12}\\
&\quad-((\mathbb{P}w-w)^-(\mathbb{P}^+e_u)^+)_{j-\frac12}+\int_{I_j}(\mathbb{P}p-p)\mathbb{P}e_z dx+\int_{I_j}(\mathbb{P}w-w)\mathbb{P}e_z dx\\
&\quad+\int_{I_j}(\mathbb{P}v-v)(\mathbb{P}e_z)_x dx-((\mathbb{P}v-v)^-(\mathbb{P}e_z)^-)_{j+\frac12}+((\mathbb{P}v-v)^-(\mathbb{P}e_z)^+)_{j-\frac12}\\
&\quad+\int_{I_j}(\mathbb{P}v-v)(-\mathbb{P}e_q)dx+\int_{I_j}(\mathbb{P}q-q)(-\mathbb{P}e_q)_x dx-((\mathbb{P}q-q)^-(-\mathbb{P}e_q)^-)_{j+\frac12}\\
&\quad+((\mathbb{P}q-q)^-(-\mathbb{P}e_q)^+)_{j-\frac12}+\int_{I_j}(\mathbb{P}q-q)\mathbb{P}e_v dx+\int_{I_j}(\mathbb{P}z-z)(\mathbb{P}e_v)_x dx\\
&\quad-((\mathbb{P}z-z)^+(\mathbb{P}e_v)^-)_{j+\frac12}+((\mathbb{P}z-z)^+(\mathbb{P}e_v)^+)_{j-\frac12}+\int_{I_j}(z-z_h)(-(\mathbb{P}e_w+\mathbb{P}e_p))dx\\
&\quad+\int_{I_j}(\mathbb{P}^+u-u)(-(\mathbb{P}e_w+\mathbb{P}e_p))_x dx-((\mathbb{P}^+u-u)^+(-(\mathbb{P}e_w+\mathbb{P}e_p))^-)_{j+\frac12}\\
&\quad+((\mathbb{P}^+u-u)^+(-(\mathbb{P}e_w+\mathbb{P}e_p))^+)_{j-\frac12}.
\end{aligned}
\tag{31}
$$

Using the property of the special projection $\mathbb{P}$, $\mathbb{P}^-$ and $\mathbb{P}^+$, the expression (31) becomes

$$
\begin{aligned}
&\mathfrak{A}_j(\mathbb{P}^+u-u,\mathbb{P}p-p,\mathbb{P}w-w,\mathbb{P}v-v,\mathbb{P}q-q,\mathbb{P}z-z;\mathbb{P}^+e_u,\mathbb{P}e_z,\\
&\quad\mathbb{P}e_z,-\mathbb{P}e_q,\mathbb{P}e_v,-(\mathbb{P}e_w+\mathbb{P}e_p))\\
&=\int_{I_j}(\mathbb{P}^+u-u)_t\mathbb{P}^+e_u dx+((\mathbb{P}p-p)^-(\mathbb{P}^+e_u)^-)_{j+\frac12}-((\mathbb{P}p-p)^-(\mathbb{P}^+e_u)^+)_{j-\frac12}\\
&\quad+((\mathbb{P}w-w)^-(\mathbb{P}^+e_u)^-)_{j+\frac12}-((\mathbb{P}w-w)^-(\mathbb{P}^+e_u)^+)_{j-\frac12}\\
&\qquad-((\mathbb{P}v-v)^-(\mathbb{P}e_z)^-)_{j+\frac12}+((\mathbb{P}v-v)^-(\mathbb{P}e_z)^+)_{j-\frac12}\\
&\qquad-((\mathbb{P}q-q)^-(-\mathbb{P}e_q)^-)_{j+\frac12}+((\mathbb{P}q-q)^-(-\mathbb{P}e_q)^+)_{j-\frac12}\\
&\qquad-((\mathbb{P}z-z)^+(\mathbb{P}e_v)^-)_{j+\frac12}+((\mathbb{P}z-z)^+(\mathbb{P}e_v)^+)_{j-\frac12}\\
&\qquad\le\int_{I_j}\frac12(\mathbb{P}^+e_u)^2dx+\mathcal{B}_{j+\frac12}+\mathcal{B}_{j-\frac12}\\
&\qquad+\varepsilon([\mathbb{P}^+e_u]^2+[\mathbb{P}e_z]^2+[\mathbb{P}e_q]^2+[\mathbb{P}e_v]^2)_{j-\frac12}
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
\mathcal{B}=&(\mathbb{P}p-p)^-(\mathbb{P}^+e_u)^-+(\mathbb{P}w-w)^-(\mathbb{P}^+e_u)^--(\mathbb{P}v-v)^-(\mathbb{P}e_z)^-\\
&-(\mathbb{P}q-q)^-(-\mathbb{P}e_q)^--(\mathbb{P}z-z)^+(\mathbb{P}e_v)^-.
\end{aligned}
\tag{33}
$$

Then we finish the proof of Lemma 5.3. □

The estimate for the second term of right-hand side in (28) is given in the following lemma.

Lemma 5.4 ([18]). *Suppose that the interpolation property (5) is satisfied; then we have the following estimate for*

$$\sum_{j=1}^{N}\mathfrak{H}_j(f;u,u_h;\mathbb{P}^+e_u)\le\sum_{j=1}^{N}-\frac14\beta(\widehat{f};u_h)_{j-\frac12}[\mathbb{P}^+e_u]_{j-\frac12}^2+C\int_a^b(\mathbb{P}^+e_u)^2dx+Ch^{2k+1}. \tag{34}$$

For the proof of this lemma, we refer readers to Lemmas 3.4 and 3.5 in [18]. For the linear flux $f(u) = cu$, this a priori assumption is unnecessary, hence the result in Theorem 5.1 holds for any $k \geq 0$.

The estimate for the final term of right-hand side in (28) is given in the following lemma.

Lemma 5.5 ([17]). *Suppose that the interpolation property (5) is satisfied; then we have the following estimate for*

$$\begin{aligned}
&|\sum_{j=1}^{N}\mathfrak{R}_j(B,b;z,u,z_h,u_h;\mathbb{P}^+e_u,\mathbb{P}e_z)| \\
&\quad\leq \sum_{j=1}^{N}(|b'(z)u(\mathbb{P}z-z)[\mathbb{P}e_z]| + |b(z)(\mathbb{P}z-z)^-[\mathbb{P}^+e_u]|)_{j-\frac{1}{2}} \\
&\qquad + \frac{1}{2}\|\mathbb{P}^+e_u\|^2 + C\|\mathbb{P}e_z\|^2 + Ch^{2k+2}.
\end{aligned} \tag{35}$$

For the proof of this lemma, we refer readers to Lemmas 4.6 in [17].

Plugging (29), (30), (34) and (35) into the equality (28), we can obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^b(\mathbb{P}^+e_u)^2dx + \sum_{j=1}^{N}(\frac{1}{2}[\mathbb{P}e_q]^2 + \tau_1[\mathbb{P}e_v]^2 + \tau_2[\mathbb{P}e_z]^2)_{j-\frac{1}{2}} + \sum_{j=1}^{N}\frac{1}{4}\beta(\widehat{f};u_h)_{j-\frac{1}{2}}[\mathbb{P}^+e_u]^2_{j-\frac{1}{2}} \\
\leq{}& C\int_a^b\frac{1}{2}(\mathbb{P}^+e_u)^2dx + \sum_{j=1}^{N}\varepsilon([\mathbb{P}^+e_u]^2 + [\mathbb{P}e_z]^2 + [\mathbb{P}e_q]^2 + [\mathbb{P}e_v]^2)_{j-\frac{1}{2}} + Ch^{2k+1} + Ch^{2k},
\end{aligned} \tag{36}$$

the fact that the initial error

$$\|u(\cdot,0) - u_h(\cdot,0)\| \leq Ch^{k+1}, \tag{37}$$

and the interpolating property (5) finally give us the error estimate (20).

To complete the proof, let us verify the a priori assumption (21). For $k \geq 1$, we can consider h small enough so that $Ch^k < \frac{1}{2}h$, where C is the constant in (20) determined by the final time T. Then, if $t^* = \sup\{t : \|u-u_h\| \leq h\}$, we would have $\|u(t^*) - u_h(t^*)\| = h$ by continuity if t^* is finite. On the other hand, our proof implies that (20) holds for $t \leq t^*$, in particular $\|u(t^*) - u_h(t^*)\| \leq Ch^k < \frac{1}{2}h$. This is a contradiction if $t^* < T$. Hence $t^* \geq T$ and our a priori assumption (21) is justified. □

6 Numerical examples

In this section, we perform numerical experiments of the local discontinuous Galerkin method applied to the general Lax equation. We use the third order Runge-Kutta method and time steps are suitably adjusted in order to show a dominant spatial accuracy. All the computations were performed in double precision. This is not the most efficient method for the time discretization to our LDG scheme. However, we will not address the issue of time discretization efficiency in this paper. We have verified that the results shown are numerically convergent in all cases with the aid of successive mesh refinements.

Example 6.1. *We consider the standard Lax equation (1) with $\alpha = 10$ in $I = [-5, 5]$, the exact solution is of the form*

$$u(x,t) = -1 + 3\,\text{sech}^2(\frac{\sqrt{2}}{2}(x - 14t))). \tag{38}$$

The L^2 and L^1 errors and the numerical orders of accuracy at time $t = 0.0001$ are contained in Table 1. We can see that the method with P^k elements gives $(k+1)$-th order of accuracy in both L^2 and L^1 norms.

Example 6.2. *In this example, we test the scheme for the standard Lax equation with* $\alpha = 10$ *in* $I = [-10, 10]$. *We take the soliton solutions of the form*

$$u(x,t) = \frac{1}{2}\sqrt{c}\,\mathrm{sech}^2\Big(\frac{1}{2}\sqrt[4]{c}(x-ct)\Big). \tag{39}$$

We choose the constants $c = 16$. The L^2 and L^1 errors and the numerical orders of accuracy for u at time $t = 0.0001$ with uniform meshes are contained in Table 2. Periodic boundary conditions are used. We can see that the method with P^k elements gives a uniform $(k+1)$-th order of accuracy for u in both norms.

Example 6.3. *In this example, we test the scheme for the Lax equation with* $\alpha = 20$. *The solutions are of the form*

$$u(x,t) = \frac{5\sqrt{-\lambda}}{\alpha}\,\mathrm{sech}^2\Big(\frac{\sqrt[4]{-\lambda}}{2}(x+\lambda t)\Big). \tag{40}$$

We choose the constants $\lambda = -16$. The L^2 and L^1 errors and the numerical orders of accuracy for u at time $t = 0.0001$ with uniform meshes are contained in Table 3. Periodic boundary conditions are used. We can see that the method with P^k elements gives a uniform $(k+1)$-th order of accuracy for u in both norms.

Table 1. Accuracy test for Lax equation with the exact solution (38). Periodic boundary condition in $[-5, 5]$. Uniform meshes with N cells at final time $T = 0.0001$

	N	L^2-error	order	L^1-error	order
	30	0.250025095519879	-	0.505252806535136	-
	35	0.214456885190837	1.00	0.430576720947450	1.04
P^0	40	0.187734914641832	1.00	0.378541378730034	0.96
	45	0.166927435692666	1.00	0.335344338561188	1.03
	50	0.150268168708905	1.00	0.302686728389871	0.97
	30	1.879039441453699E-002	-	3.286545740223012E-002	-
	35	1.517793447466294E-002	1.39	2.648611321788866E-002	1.40
P^1	40	1.175185014958489E-002	1.92	2.051650636107464E-002	1.91
	45	9.301100063281460E-003	1.99	1.623421704551745E-002	1.99
	50	7.543588974233264E-003	1.99	1.316444163568550E-002	1.99
	30	5.437341211792110E-003	-	9.452351537362624E-003	-
	35	2.586643947316112E-003	4.82	4.388740273521469E-003	4.98
P^2	40	1.634177482372270E-003	3.44	2.801872130043346E-003	3.36
	45	1.131356506174137E-003	3.12	1.911467182749563E-003	3.25
	50	8.272791940860751E-004	2.97	1.442019800374107E-003	2.67

7 Conclusion

We have discussed the application of local discontinuous Galerkin methods to solve the general Lax equation. We prove stability and give an error estimate. Numerical examples for general Lax equation are given to illustrate the accuracy and capability of the methods. Although not addressed in this paper, the method is flexible for general geometry, unstructured meshes and h-p adaptivity, and has excellent parallel efficiency. These results indicate that the LDG method is a good tool for solving such nonlinear equations in mathematical physics.

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Table 2. Accuracy test for Lax equation with the exact solution (39) choosing $c = 16$. Periodic boundary condition in $[-10, 10]$. Uniform meshes with N cells at time $t = 0.0001$

	N	L^2-error	order	L^1-error	order
P^0	45	0.262562836987084	-	0.442363863105017	-
	50	0.236725114862793	0.98	0.407435575492229	0.78
	55	0.215485037022766	0.99	0.363985847157329	1.18
	60	0.197724527583503	0.99	0.338792804920091	0.82
	65	0.182656542818417	0.99	0.308965096338155	1.15
	70	0.169714225008466	0.99	0.290019731047581	0.85
P^1	45	3.880589332295393E-002	-	5.917128300448593E-002	-
	50	3.206033525121685E-002	1.81	4.853508770943189E-002	1.88
	55	2.671598323277074E-002	1.91	3.977612736077588E-002	2.09
	60	2.262802545703084E-002	1.91	3.386667345536551E-002	1.85
	65	1.940528673241487E-002	1.92	2.904070662670210E-002	1.92
	70	1.682010624218387E-002	1.93	2.485815847211677E-002	2.10
P^2	45	1.506187123768599E-002	-	2.442327109945212E-002	-
	50	1.019076038429557E-002	3.71	1.683162047008102E-002	3.53
	55	6.815580704424561E-003	4.22	1.089819034720809E-002	4.56
	60	4.952438129993914E-003	3.67	7.619139588922541E-003	4.11
	65	3.872358100972614E-003	3.07	5.450917599415668E-003	4.18
	70	3.191727848648557E-003	2.61	4.358328765569666E-003	3.02

Table 3. Accuracy test for Lax equation with the exact solution (40) choosing $\lambda = -16$. Periodic boundary condition in $[-5, 5]$. Uniform meshes with N cells at time $t = 0.0001$

	N	L^2-error	order	L^1-error	order
P^0	30	9.885941626628193E-002	-	0.169366079037095	-
	35	8.485448377283446E-002	0.99	0.143501208768824	1.08
	40	7.431484018739452E-002	0.99	0.126755684746074	0.93
	45	6.609867111817130E-002	0.99	0.111915629567131	1.06
	50	5.951526640818351E-002	1.00	0.101306503082584	0.95
P^1	30	1.019279381151275E-002	-	1.535758890466971E-002	-
	35	8.340246396509980E-003	1.30	1.239027721246629E-002	1.39
	40	6.482260809629124E-003	1.89	9.583787194704823E-003	1.92
	45	5.146246954540782E-003	1.96	7.574137237557694E-003	2.00
	50	4.183737195964606E-003	1.97	6.140217793902676E-003	2.00
P^2	30	2.704100139391892E-003	-	3.924975265062904E-003	-
	35	1.514846548183961E-003	3.76	2.201195114825479E-003	3.75
	40	8.785309061922388E-004	4.08	1.282439894082946E-003	4.05
	45	5.986846947394631E-004	3.26	7.895185077491700E-004	4.11
	50	4.319620667412239E-004	3.10	5.491810801293127E-004	3.45

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