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$\mathcal{MN}$-convergence and lim-inf$_{\mathcal{M}}$-convergence in partially ordered sets

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Abstract: In this paper, we first introduce the notion of $\mathcal{MN}$-convergence in posets as an unified form of O-convergence and O_2-convergence. Then, by studying the fundamental properties of $\mathcal{MN}$-topology which is determined by $\mathcal{MN}$-convergence according to the standard topological approach, an equivalent characterization to the $\mathcal{MN}$-convergence being topological is established. Finally, the lim-inf$_{\mathcal{M}}$-convergence in posets is further investigated, and a sufficient and necessary condition for lim-inf$_{\mathcal{M}}$-convergence to be topological is obtained.

Keywords: $\mathcal{MN}$-convergence, $\mathcal{MN}$-topology, lim-inf$_{\mathcal{M}}$-convergence, $\mathcal{M}$-topology

MSC: 54A20, 06A06

1 Introduction, Notations and Preliminaries

The concept of O-convergence in partially ordered sets (posets, for short) was introduced by Birkhoff [1], Frink [2] and Mcshane [3]. It is defined as follows: a net $(x_i)_{i\in I}$ in a poset P is said to *O-converge to* $x \in P$ if there exist subsets D and F of P such that

(1) D is directed and F is filtered;
(2) $\sup D = x = \inf F$;
(3) for every $d \in D$ and $e \in F$, $d \leqslant x_i \leqslant e$ holds eventually, i.e., there exists $i_0 \in I$ such that $d \leqslant x_i \leqslant e$ for all $i \geqslant i_0$.

As what has been showed in [4], the O-convergence (Note: in [4], the O-convergence is called order-convergence) in a general poset P may not be topological, i.e., it is possible that P can not be endowed with a topology such that the O-convergence and the associated topological convergence are consistent. Hence, much work has been done to characterize those special posets in which the O-convergence is topological. The most recent result in [5] shows that the O-convergence in a poset which satisfies Condition ($\triangle$) is topological if and only if the poset is $\mathcal{O}$-doubly continuous. This means that for a special class of posets, a sufficient and necessary condition for O-convergence being topological is obtained.

As a direct generalization of O-convergence, O_2-convergence in posets has been discussed in [11] from the order-theoretical point of view. It is defined as follows: a net $(x_i)_{i\in I}$ in a poset P is said to *O_2-converge to* $x \in P$ if there exist subsets A and B of P such that

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(1) $\sup A = x = \inf B$;
(2) for every $a \in A$ and $b \in B$, $a \leqslant x_i \leqslant b$ holds eventually.

In fact, the O_2-convergence is also not topological generally. To clarify those special posets in which the O_2-convergence is topological, Zhao and Li [6] showed that for any poset P satisfying Condition (*), O_2-convergence is topological if and only if P is α-doubly continuous. As a further result, Li and Zou [7] proved that the O_2-convergence in a poset P is topological if and only if P is O_2-doubly continuous. This result demonstrates the equivalence between the O_2-convergence being topological and the O_2-double continuity of a given poset.

On the other hand, Zhou and Zhao [8] have defined the lim-inf$_{\mathcal{M}}$-convergence in posets to generalize lim-inf-convergence and lim-inf$_2$-convergence [4]. They also found that the lim-inf$_{\mathcal{M}}$-convergence in a poset is topological if and only if the poset is $\alpha(\mathcal{M})$-continuous when some additional conditions are satisfied (see [8], Theorem 3.1). This result clarified some special conditions of posets under which the lim-inf$_{\mathcal{M}}$-convergence is topological. However, to the best of our knowledge, the equivalent characterization to the lim-inf$_{\mathcal{M}}$-convergence in general posets being topological is still unknown.

One goal of this paper is to propose the notion of $\mathcal{MN}$-convergence in posets which can unify O-convergence and O_2-convergence and search the equivalent characterization to the $\mathcal{MN}$-convergence being topological. More precisely,

(G11) Given a general poset P, we hope to clarify the order-theoretical condition of P which is sufficient and necessary for the $\mathcal{MN}$-convergence being topological.
(G12) Given a poset P satisfying such condition, we hope to provide a topology that can be equipped on P such that the $\mathcal{MN}$-convergence and the associated topological convergence agree.

Another goal is to look for the equivalent characterization to the lim-inf$_{\mathcal{M}}$-convergence being topological. More precisely,

(G21) Given a general poset P, we expect to present a sufficient and necessary condition of P which can precisely serve as an order-theoretical condition for the lim-inf$_{\mathcal{M}}$-convergence being topological.
(G22) Given a poset P satisfying such condition, we expect to give a topology on P such that the lim-inf$_{\mathcal{M}}$-convergence and the associated topological convergence are consistent.

To accomplish those goals, motivated by the ideal of introducing the Z-subsets system [9] for defining Z-continuous posets , we propose the notion of $\mathcal{MN}$-doubly continuous posets and define the $\mathcal{MN}$-topology on posets in Section 2. Based on the study of the basic properties of the $\mathcal{MN}$-topology, it is proved that the $\mathcal{MN}$-convergence in a poset P is topological if and only if P is an $\mathcal{MN}$-doubly continuous poset if and only if the $\mathcal{MN}$-convergence and the topological convergence with respect to $\mathcal{MN}$-topology are consistent. In Section 3, by introducing the notion of $\alpha^*(\mathcal{M})$-continuous posets and presenting the fundamental properties of $\mathcal{M}$-topology which is induced by the lim-inf$_{\mathcal{M}}$-convergence, we show that the lim-inf$_{\mathcal{M}}$-convergence in a poset P is topological if and only if P is an $\alpha^*(\mathcal{M})$-continuous poset if and only if the lim-inf$_{\mathcal{M}}$-convergence and the topological convergence with respect to $\mathcal{M}$-topology are consistent.

Some conventional notations will be used in the paper. Given a set$X$, $F \sqsubseteq X$ means that F is a finite subset of X. Given a topological space $(X, \mathcal{T})$ and a net $(x_i)_{i\in I}$ in X, we take $(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x$ to mean the net $(x_i)_{i\in I}$ converges to $x \in P$ with respect to the topology $\mathcal{T}$.

Let P be a poset and $x \in P$. $\uparrow x$ and $\downarrow x$ are always used to denote the principal filter $\{y \in P : y \geqslant x\}$ and the principal ideal $\{z \in P : z \leqslant x\}$ of P, respectively. Given a poset P and $A \subseteq P$, by writing $\sup A$ we mean that the least upper bound of A in P exists and equals to $\sup A \in P$; dually, by writing $\inf A$ we mean that the greatest lower bound of A in P exists and equals to $\inf A \in P$. And the set A is called an upper set if $A = \uparrow A = \{b \in P; (\exists a \in A)\ a \leqslant b\}$, the lower set is defined dually.

For a poset P, we succinctly denote

- $\mathcal{P}(P) = \{A : A \subseteq P\}$; $\mathcal{P}_0(P) = \mathcal{P}(P)/\{\emptyset\}$;
- $\mathcal{D}(P) = \{D \in \mathcal{P}(P) : D$ is a directed subset of $P\}$;

- $\mathcal{F}(P) = \{F \in \mathcal{P}(P) : F$ is a filtered subset of $P\}$;
- $\mathcal{L}(P) = \{L \in \mathcal{P}(P) : L \sqsubseteq P\}$; $\mathcal{L}_0(P) = \mathcal{L}(P)/\{\emptyset\}$;
- $\mathcal{S}_0(P) = \{\{x\} : x \in P\}$.

To make this paper self-contained, we briefly review the following notions:

Definition 1.1 ([5]). *Let P be a poset and $x, y, z \in P$. We say $y \ll_\mathcal{O} x$ if for every net $(x_i)_{i\in I}$ in P which O-converges to $x \in P$, $x_i \geqslant y$ holds eventually; dually, we say $z \rhd_\mathcal{O} x$ if for every net $(x_i)_{i\in I}$ in P which O-converges to $x \in P$, $x_i \leqslant z$ holds eventually.*

Definition 1.2 ([5]). *A poset P is said to be* $\mathcal{O}$-doubly continuous *if for every $x \in P$, the set $\{a \in P : a \ll_\mathcal{O} x\}$ is directed, the set $\{b \in P : b \rhd_\mathcal{O} x\}$ is filtered and* $\sup\{a \in P : a \ll_\mathcal{O} x\} = x = \inf\{b \in P : b \rhd_\mathcal{O} x\}$.

Condition ($\triangle$). *A poset P is said to satisfy* Condition($\triangle$) *if*
(1) for any $x, y, z \in P$, $x \ll_\mathcal{O} y \leqslant z$ implies $x \ll_\mathcal{O} z$;
(2) for any $w, s, t \in P$, $w \rhd_\mathcal{O} s \geqslant t$ implies $w \rhd_\mathcal{O} t$.

Definition 1.3 ([6]). *Let P be a poset and $x, y, z \in P$. We say $y \ll_\alpha x$ if for every net $(x_i)_{i\in I}$ in P which O_2-converges to $x \in P$, $x_i \geqslant y$ holds eventually; dually, we say $z \rhd_\alpha x$ if for every net $(x_i)_{i\in I}$ in P which O_2-converges to $x \in P$, $x_i \leqslant z$ holds eventually.*

Definition 1.4 ([7]). *A poset P is said to be O_2*-doubly continuous *if for every $x \in P$,*
(1) $\sup\{a \in P : a \ll_\alpha x\} = x = \inf\{b \in P : b \rhd_\alpha x\}$;
(2) for any $y, z \in P$ with $y \ll_\alpha x$ and $z \rhd_\alpha x$, there exist $A \sqsubseteq \{a \in P : a \ll_\alpha x\}$ and $B \sqsubseteq \{b \in P : b \rhd_\alpha x\}$ such that $y \ll_\alpha c$ and $z \rhd_\alpha c$ for each $c \in \bigcap\{\uparrow a \cap \downarrow b : a \in A \;\&\; b \in B\}$.

2 $\mathcal{MN}$-topology on posets

Based on the introduction of $\mathcal{MN}$-convergence in posets, the $\mathcal{MN}$-topology can be defined on posets. In this section, we first define the $\mathcal{MN}$-double continuity for posets. Then, we show the equivalence between the $\mathcal{MN}$-convergence being topological and the $\mathcal{MN}$-double continuity of a given poset.

A *PMN-space* is a triplet $(P, \mathcal{M}, \mathcal{N})$ which consists of a poset P and two subfamily $\mathcal{M}, \mathcal{N} \subseteq \mathcal{P}(P)$.

All *PMN*-spaces $(P, \mathcal{M}, \mathcal{N})$ considered in this section are assumed to satisfy the following conditions:
(C1) If P has the least element $\perp$, then $\{\perp\} \in \mathcal{M}$;
(C2) If P has the greatest element $\top$, then $\{\top\} \in \mathcal{N}$;
(C3) $\emptyset \notin \mathcal{M}$ and $\emptyset \notin \mathcal{N}$.

Definition 2.1. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. A net $(x_i)_{i\in I}$ in P is said to* $\mathcal{MN}$-converge *to $x \in P$ if there exist $M \in \mathcal{M}$ and $N \in \mathcal{N}$ satisfying:*
(MN1) $\sup M = x = \inf N$;
(MN2) $x_i \in \uparrow m \cap \downarrow n$ eventually for every $m \in M$ and every $n \in N$.
In this case, we will write $(x_i)_{i\in I} \stackrel{\mathcal{MN}}{\to} x$.

Remark 2.2. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space.*
(1) If $\mathcal{M} = \mathcal{D}(P)$ and $\mathcal{N} = \mathcal{F}(P)$, then a net $(x_i)_{i\in I} \stackrel{\mathcal{MN}}{\to} x \in P$ if and only if it O-converges to x. That is to say, O-convergence is a particular case of $\mathcal{MN}$-convergence.
(2) If $\mathcal{M} = \mathcal{N} = \mathcal{P}_0(P)$, then a net $(x_i)_{i\in I} \stackrel{\mathcal{MN}}{\to} x \in P$ if and only if it O_2-converges to x. That is to say, O_2-convergence is a special case of $\mathcal{MN}$-convergence.

(3) *If* $\mathcal{M} = \mathcal{N} = \mathcal{L}_0(P)$, *then a net* $(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x \in P$ *if and only if* $x_i = x$ *holds eventually.*

(4) *The* $\mathcal{MN}$*-convergent point of a net* $(x_i)_{i\in I}$ *in* P*, if exists, is unique.*

Indeed, suppose that $(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x_1$ *and* $(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x_2$. *Then there exist* $A_k \in \mathcal{M}$ *and* $B_k \in \mathcal{N}$ *such that* $\sup A_k = x_k = \inf B_k$ *and* $a_k \leqslant x_i \leqslant b_k$ *holds eventually for every* $a_k \in A_k$ *and* $b_k \in B_k$ $(k = 1, 2)$. *This implies that for any* $a_1 \in A_1, a_2 \in A_2, b_1 \in B_1$ *and* $b_2 \in B_2$*, there exists* $i_0 \in I$ *such that* $a_1 \leqslant x_{i_0} \leqslant b_2$ *and* $a_2 \leqslant x_{i_0} \leqslant b_1$. *Thus we have* $\sup A_1 = x_1 \leqslant \inf B_2 = x_2$ *and* $\sup A_2 = x_2 \leqslant \inf B_1 = x_1$. *Therefore* $x_1 = x_2$.

(5) *For any* $A \in \mathcal{M}$ *and* $B \in \mathcal{N}$ *with* $\sup A = \inf B = x \in P$*, we denote* $F^x_{(A,B)} = \{\bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \ \&\ b \in B_0\} : A_0 \sqsubseteq A \ \&\ B_0 \sqsubseteq B\}$[1]. *Let* $D^x_{(A,B)} = \{(d, D) \in P \times F^x_{(A,B)} : d \in D\}$ *and let the preorder* $\leq$ *on* $D^x_{(A,B)}$ *be defined by*

$$(\forall (d_1, D_1), (d_2, D_2) \in D^x_{(A,B)})\ (d_1, D_1) \leq (d_2, D_2) \Longleftrightarrow D_2 \subseteq D_1.$$

One can readily check that $(D^x_{(A,B)}, \leq)$ *is directed. Now if we take* $x_{(d,D)} = d$ *for every* $(d, D) \in D^x_{(A,B)}$*, then the net* $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}} \xrightarrow{\mathcal{MN}} x$ *because* $\sup A = \inf B = x$*, and* $a \leqslant x_{(d,D)} \leqslant b$ *holds eventually for any* $a \in A$ *and* $b \in B$.

(6) *Let* $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}}$ *be the net defined in (5) for any* $A \in \mathcal{M}$ *and* $B \in \mathcal{N}$ *with* $\sup A = \inf B = x \in P$. *If* $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}}$ *converges to* $p \in P$ *with respect to some topology* $\mathcal{T}$ *on the poset* P*, then for every open neighborhood* U_p *of* p*, there exist* $A_0 \sqsubseteq A$ *and* $B_0 \sqsubseteq B$ *such that*

$$\bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \ \&\ b \in B_0\} \subseteq U_p.$$

Indeed, suppose that $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}} \xrightarrow{\mathcal{T}} p$. *Then for every open neighborhood* U_p *of* p*, there exists* $(d_0, D_0) \in D^x_{(A,B)}$ *such that* $x_{(d,D)} = d \in U_p$ *for all* $(d, D) \geq (d_0, D_0)$. *Since* $(d, D_0) \geq (d_0, D_0)$ *for every* $d \in D_0$, $x_{(d,D)} = d \in U_p$ *for every* $d \in D_0$. *This shows* $D_0 \subseteq U_p$. *So, there exist* $A_0 \sqsubseteq A$ *and* $B_0 \sqsubseteq B$ *such that*

$$D_0 = \bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \ \&\ b \in B_0\} \subseteq U_p.$$

Given a *PMN*-space $(P, \mathcal{M}, \mathcal{N})$, we can define two new approximate relations $\ll^{\mathcal{N}}_{\mathcal{M}}$ and $\rhd^{\mathcal{N}}_{\mathcal{M}}$ on the poset P in the following definition.

Definition 2.3. *Let* $(P, \mathcal{M}, \mathcal{N})$ *be a PMN-space and* $x, y, z \in P$.

(1) *We define* $y \ll^{\mathcal{N}}_{\mathcal{M}} x$ *if for any* $A \in \mathcal{M}$ *and* $B \in \mathcal{N}$ *with* $\sup A = x = \inf B$*, there exist* $A_0 \sqsubseteq A$ *and* $B_0 \sqsubseteq B$ *such that*

$$\bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \ \&\ b \in B_0\} \subseteq \uparrow y.$$

(2) *Dually, we define* $z \rhd^{\mathcal{N}}_{\mathcal{M}} x$ *if for any* $M \in \mathcal{M}$ *and* $N \in \mathcal{N}$ *with* $\sup M = x = \inf N$*, there exist* $M_0 \sqsubseteq M$ *and* $N_0 \sqsubseteq N$ *such that*

$$\bigcap\{\uparrow m \cap \downarrow n : m \in M_0 \ \&\ n \in N_0\} \subseteq \downarrow z.$$

For convenience, given a *PMN*-space $(P, \mathcal{M}, \mathcal{N})$ and $x \in P$, we will briefly denote

- $\blacktriangledown^{\mathcal{N}}_{\mathcal{M}} x = \{y \in P : y \ll^{\mathcal{N}}_{\mathcal{M}} x\}$;
- $\blacktriangle^{\mathcal{N}}_{\mathcal{M}} x = \{z \in P : x \ll^{\mathcal{N}}_{\mathcal{M}} z\}$;
- $\triangledown^{\mathcal{N}}_{\mathcal{M}} x = \{a \in P : x \rhd^{\mathcal{N}}_{\mathcal{M}} a\}$;
- $\vartriangle^{\mathcal{N}}_{\mathcal{M}} x = \{b \in P : b \rhd^{\mathcal{N}}_{\mathcal{M}} x\}$.

Remark 2.4. *Let* $(P, \mathcal{M}, \mathcal{N})$ *be a PMN-space and* $x, y, z \in P$.

(1) *If there is no* $A \in \mathcal{M}$ *such that* $\sup A = x$*, then* $p \ll^{\mathcal{N}}_{\mathcal{M}} x$ *and* $p \rhd^{\mathcal{N}}_{\mathcal{M}} x$ *for all* $p \in P$*; similarly, if there is no* $B \in \mathcal{N}$ *such that* $\inf B = x$*, then* $p \ll^{\mathcal{N}}_{\mathcal{M}} x$ *and* $p \rhd^{\mathcal{N}}_{\mathcal{M}} x$ *for all* $p \in P$.

(2) *By Definition 2.3, one can easily check that if* P *has the least element* $\perp$*, then* $\perp \ll^{\mathcal{N}}_{\mathcal{M}} p$ *for every* $p \in P$*, and if* P *has the greatest element* $\top$*, then* $\top \rhd^{\mathcal{N}}_{\mathcal{M}} p$ *for every* $p \in P$.

1 From the logical point of view, we stipulate $\bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \ \&\ b \in B_0\} = P$ if $A_0 = \emptyset$ or $B_0 = \emptyset$.

(3) *The implications $y \ll_{\mathcal{M}}^{\mathcal{N}} x \Rightarrow x \leqslant y$ and $z \rhd_{\mathcal{M}}^{\mathcal{N}} x \Rightarrow z \geqslant x$ are not true necessarily. See the following example: let $\mathbb{R}$ be the set of all real numbers, in its ordinal order, and $\mathcal{M} = \mathcal{N} = \{\{n\} : n \in \mathbb{Z}\}$, where $\mathbb{Z}$ is the set of all integers. Then, by (1), we have $1 \ll_{\mathcal{M}}^{\mathcal{N}} 1/2$ and $0 \rhd_{\mathcal{M}}^{\mathcal{N}} 1/2$. But $1 \not\leqslant 1/2$ and $0 \not\geqslant 1/2$.*

(4) *Assume that $\sup A_0 = x = \inf B_0$ for some $A_0 \in \mathcal{M}$ and $B_0 \in \mathcal{N}$. Then it follows from Definition 2.3 that $y \ll_{\mathcal{M}}^{\mathcal{N}} x$ implies $y \leqslant x$ and $z \rhd_{\mathcal{M}}^{\mathcal{N}} x$ implies $z \geqslant x$. In particular, if $\mathcal{S}_0(P) \subseteq \mathcal{M}, \mathcal{N}$, then $b \ll_{\mathcal{M}}^{\mathcal{N}} a$ implies $b \leqslant a$ and $c \rhd_{\mathcal{M}}^{\mathcal{N}} a$ implies $c \geqslant a$ for any $a, b, c \in P$. More particularly, for any $p_1, p_2, p_3 \in P$, we have $p_1 \ll_{\mathcal{S}_0}^{\mathcal{S}_0} p_2 \iff p_1 \leqslant p_2$ and $p_3 \rhd_{\mathcal{S}_0}^{\mathcal{S}_0} p_2 \iff p_3 \geqslant p_2$.*

Proposition 2.5. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space and $x, y, z \in P$. Then*
(1) *$y \ll_{\mathcal{M}}^{\mathcal{N}} x$ if and only if for every net $(x_i)_{i \in I}$ that $\mathcal{MN}$-converges to x, $x_i \geqslant y$ holds eventually.*
(2) *$z \rhd_{\mathcal{M}}^{\mathcal{N}} x$ if and only if for every net $(x_i)_{i \in I}$ that $\mathcal{MN}$-converges to x, $x_i \leqslant z$ holds eventually.*

Proof. (1) Suppose $y \ll_{\mathcal{M}}^{\mathcal{N}} x$. If a net $(x_i)_{i \in I} \xrightarrow{\mathcal{MN}} x$, then there exist $A \in \mathcal{M}$ and $B \in \mathcal{N}$ such that $\sup A = x = \inf B$, and for any $a \in A$ and $b \in B$, there exists $i_a^b \in I$ such that $a \leqslant x_i \leqslant b$ for all $i \geqslant i_a^b$. According to Definition 2.3 (1), it follows that there exist $A_0 = \{a_1, a_2, \dots, a_n\} \sqsubseteq A$ and $B_0 = \{b_1, b_2, \dots, b_m\} \sqsubseteq B$ such that $x \in \bigcap\{\uparrow a_k \cap \downarrow b_j : 1 \le k \le n \;\&\; 1 \le j \le m\} \subseteq \uparrow y$. Take $i_0 \in I$ with that $i_0 \geqslant i_{a_k}^{b_j}$ for every $k \in \{1, 2, \dots, n\}$ and every $j \in \{1, 2, \dots, m\}$. Then $x_i \in \bigcap\{\uparrow a_k \cap \downarrow b_j : 1 \le k \le n \;\&\; 1 \le j \le m\} \subseteq \uparrow y$ for all $i \geqslant i_0$. This means $x_i \geqslant y$ holds eventually.

Conversely, suppose that for every net $(x_i)_{i \in I}$ that $\mathcal{MN}$-converges to x, $x_i \geqslant y$ holds eventually. For every $A \in \mathcal{M}$ and $B \in \mathcal{N}$ with $\sup A = x = \inf B$, consider the net $(x_{(d,D)})_{(d,D) \in D_{(A,B)}^x}$ defined in Remark 2.2 (5). By Remark 2.2 (5), the net $(x_{(d,D)})_{(d,D) \in D_{(A,B)}^x} \xrightarrow{\mathcal{MN}} x$. So, there exists $(d_0, D_0) \in D_{(A,B)}^x$ such that $x_{(d,D)} = d \geqslant y$ for all $(d, D) \ge (d_0, D_0)$. Since $(d, D_0) \ge (d_0, D_0)$ for all $d \in D_0$, $x_{(d,D_0)} = d \geqslant y$ for all $d \in D_0$. Thus,we have $D_0 \subseteq \uparrow y$. It follows from the definition of $D_{(A,B)}^x$ that there exist $A_0 \sqsubseteq A$ and $B_0 \sqsubseteq B$ such that $D_0 = \bigcap\{\uparrow a \cap \downarrow b : a \in A_0 \;\&\; b \in B_0\} \subseteq \uparrow y$. This shows $y \ll_{\mathcal{M}}^{\mathcal{N}} x$.

The proof of (2) can be processed similarly. □

Remark 2.6. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space.*
(1) *If $\mathcal{M} = \mathcal{D}(P)$ and $\mathcal{N} = \mathcal{F}(P)$, then $\ll_{\mathcal{D}}^{\mathcal{F}} = \ll_{\mathcal{O}}$ and $\rhd_{\mathcal{D}}^{\mathcal{F}} = \rhd_{\mathcal{O}}$.*
(2) *If $\mathcal{M} = \mathcal{N} = \mathcal{P}_0(P)$, then $\ll_{\mathcal{P}_0}^{\mathcal{P}_0} = \ll_{\alpha}$ and $\rhd_{\mathcal{P}_0}^{\mathcal{P}_0} = \rhd_{\alpha}$.*

Given a *PMN*-space $(P, \mathcal{M}, \mathcal{N})$, depending on the approximate relations $\ll_{\mathcal{M}}^{\mathcal{N}}$ and $\rhd_{\mathcal{M}}^{\mathcal{N}}$ on P. we can define the $\mathcal{MN}$-double continuity for the poset P.

Definition 2.7. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. The poset P is called an* $\mathcal{MN}$-doubly continuous poset *if for every $x \in P$, there exist $M_x \in \mathcal{M}$ and $N_x \in \mathcal{N}$ such that*
(A1) $M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$, $N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$ and $\sup M_x = x = \inf N_x$.
(A2) For any $y \in \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and $z \in \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$, $\bigcap\{\uparrow m \cap \downarrow n : m \in M_0 \;\&\; n \in N_0\} \subseteq \blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z$ for some $M_0 \sqsubseteq M_x$ and $N_0 \sqsubseteq N_x$.

By Remark 2.4 (4) and Definition 2.7, we have the following basic property about $\mathcal{MN}$-doubly continuous posets:

Proposition 2.8. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space and $x, y, z \in P$. If the poset P is an $\mathcal{MN}$-doubly continuous poset, then $y \ll_{\mathcal{M}}^{\mathcal{N}} x$ implies $y \leqslant x$ and $z \rhd_{\mathcal{M}}^{\mathcal{N}} x$ implies $z \geqslant x$.*

Example 2.9. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space.*
(1) *If $\mathcal{M} = \mathcal{N} = \mathcal{S}_0(P)$, then by Remark 2.4 (4), we have $\ll_{\mathcal{S}_0}^{\mathcal{S}_0} = \leqslant$ and $\rhd_{\mathcal{S}_0}^{\mathcal{S}_0} = \geqslant$. By Definition 2.7, one can easily check that P is an $\mathcal{S}_0\mathcal{S}_0$-doubly continuous poset.*
(2) *If $\mathcal{M} = \mathcal{N} = \mathcal{L}_0(P)$, then by Definition 2.3, we have $\ll_{\mathcal{L}_0}^{\mathcal{L}_0} = \leqslant$ and $\rhd_{\mathcal{L}_0}^{\mathcal{L}_0} = \geqslant$. It can be easily checked from Definition 2.7 that P is an $\mathcal{L}_0\mathcal{L}_0$-doubly continuous poset.*

(3) *Let $\mathcal{M} = \mathcal{D}(P)$ and $\mathcal{N} = \mathcal{F}(P)$. Then it is easy to check that if P is an $\mathcal{O}$-doubly continuous poset which satisfies Condition ($\triangle$), then it is a $\mathcal{DF}$-doubly continuous poset. Particularly, finite posets, chains and anti-chains, completely distributive lattices are all $\mathcal{DF}$-doubly continuous posets.*
(4) *Let $\mathcal{M} = \mathcal{N} = \mathcal{P}_0(P)$. Then the poset P is $\mathcal{P}_0\mathcal{P}_0$-double continuous if and only if it is O_2-double continuous. Thus, chains and finite posets are all $\mathcal{P}_0\mathcal{P}_0$-doubly continuous posets.*

Next, we are going to consider the $\mathcal{MN}$-topology on posets, which is induced by the $\mathcal{MN}$-convergence.

Definition 2.10. *Given a PMN-space $(P, \mathcal{M}, \mathcal{N})$, a subset U of P is called an* $\mathcal{MN}$-open set *if for every net $(x_i)_{i\in I}$ with that $(x_i)_{i\in I} \overset{\mathcal{MN}}{\longrightarrow} x \in U$, $x_i \in U$ holds eventually.*

Clearly, the family $\mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)$ consisting of all $\mathcal{MN}$-open subsets of P forms a topology on P. And this topology is called the $\mathcal{MN}$-*topology*.

Theorem 2.11. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. Then a subset U of P is an $\mathcal{MN}$-open set if and only if for every $M \in \mathcal{M}$ and $N \in \mathcal{N}$ with $\sup M = x = \inf N \in U$, we have*

$$\bigcap\{\uparrow m \cap \downarrow n : m \in M_0 \,\&\, n \in N_0\} \subseteq U$$

for some $M_0 \sqsubseteq M$ and $N_0 \sqsubseteq N$.

Proof. Suppose that U is an $\mathcal{MN}$-open subset of P. For every $M \in \mathcal{M}$ and $N \in \mathcal{N}$ with $\sup M = x = \inf N \in U$, let $(x_{(d,D)})_{(d,D)\in D^x_{(M,N)}}$ be the net defined in Remark 2.2 (5). Then the net $(x_{(d,D)})_{(d,D)\in D^x_{(M,N)}} \overset{\mathcal{MN}}{\longrightarrow} x$. By the definition of $\mathcal{MN}$-open set, the exists $(d_0, D_0) \in D^x_{(M,N)}$ such that $x_{(d,D)} = d \in U$ for all $(d, D) \geq (d_0, D_0)$. Since $(d, D_0) \geq (d_0, D_0)$ for all $d \in D_0$, $x_{(d,D_0)} = d \in U$ for every $d \in D_0$, and thus $D_0 \subseteq U$. It follows from the definition of the directed set $D^x_{(M,N)}$ that $D_0 = \bigcap\{\uparrow m \cap \downarrow n : m \in M_0 \,\&\, n \in N_0\} \subseteq U$ for some $M_0 \sqsubseteq M$ and some $N_0 \sqsubseteq N$.

Conversely, assume that U is a subset of P with the property that for any $M \in \mathcal{M}$ and $N \in \mathcal{N}$ with $\sup M = x = \inf N \in U$, there exist $M_0 = \{m_1, m_2, \ldots, m_k\} \sqsubseteq M$ and $N_0 = \{n_1, n_2, \ldots, n_l\} \sqsubseteq N$ such that $\bigcap\{\uparrow m_h \cap \downarrow n_j : 1 \leq h \leq k \,\&\, 1 \leq j \leq l\} \subseteq U$. Let $(x_i)_{i\in I}$ be a net that $\mathcal{MN}$-converges to $x \in U$. Then there exist $M \in \mathcal{M}$ and $N \in \mathcal{N}$ such that $\sup M = x = \inf N \in U$, and for every $m \in M$ and $n \in N$, $m \leqslant x_i \leqslant n$ holds eventually. This means that for every $m_h \in M_0$ and $n_j \in N_0$, there exists $i_{h,j} \in I$ such that $m_h \leqslant x_i \leqslant n_j$ for all $i \geq i_{h,j}$. Take $i_0 \in I$ such that $i_0 \geq i_{h,j}$ for all $h \in \{1, 2, \ldots, k\}$ and $j \in \{1, 2, \ldots, l\}$. Then $x_i \in \bigcap\{\uparrow m_h \cap \downarrow n_j : 1 \leq h \leq k \,\&\, 1 \leq j \leq l\} \subseteq U$ for all $i \geq i_0$. Therefore, U is an $\mathcal{MN}$-open subset of P. □

Proposition 2.12. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space in which P is an $\mathcal{MN}$-doubly continuous poset, and $y, z \in P$. Then $\blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z \in \mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)$.*

Proof. Suppose that $M \in \mathcal{M}$ and $N \in \mathcal{N}$ with $\sup M = \inf N = x \in \blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z$. Since P is an $\mathcal{MN}$-doubly continuous poset, there exist $M_x \in \mathcal{M}$ and $N_x \in \mathcal{N}$ satisfying condition (A1) and (A2) in Definition 2.7. This means that there exist $M_0 \sqsubseteq M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and $N_0 \sqsubseteq N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$ such that $\bigcap\{\uparrow m_0 \cap \downarrow n_0 : m_0 \in M_0 \,\&\, n_0 \in N_0\} \subseteq \blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z$. As $M_0 \sqsubseteq M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and $N_0 \sqsubseteq N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$, by Definition 2.3, there exist $M_{m_0} \sqsubseteq M$ and $N_{n_0} \sqsubseteq N$ such that $\bigcap\{\uparrow m \cap \downarrow n : m \in M_{m_0} \,\&\, n \in N_{n_0}\} \subseteq \uparrow m_0 \cap \downarrow n_0$ for every $m_0 \in M_0$ and $n_0 \in N_0$. Take $M_F = \bigcup\{M_{m_0} : m_0 \in M_0\}$ and $N_F = \bigcup\{N_{n_0} : n_0 \in N_0\}$. Then it is easy to check that $M_F \sqsubseteq M$, $N_F \sqsubseteq N$ and

$$\begin{aligned} x &\in \bigcap\{\uparrow a \cap \downarrow b : a \in M_F \,\&\, b \in N_F\} \\ &\subseteq \bigcap\{\uparrow m_0 \cap \downarrow n_0 : m_0 \in M_0 \,\&\, n_0 \in N_0\} \\ &\subseteq \blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z. \end{aligned}$$

So, it follows from Theorem 2.11 that $\blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z \in \mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)$. □

Lemma 2.13. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space in which P is an $\mathcal{MN}$-doubly continuous poset. Then a net*

$$(x_i)_{i\in I} \overset{\mathcal{MN}}{\longrightarrow} x \in P \Longleftrightarrow (x_i)_{i\in I} \overset{\mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)}{\longrightarrow} x.$$

Proof. From the definition of $\mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)$, it is easy to see that a net

$$(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x \in P \Longrightarrow (x_i)_{i\in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)} x.$$

To prove the Lemma, it suffices to show that a net $(x_i)_{i\in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)} x \in P$ implies $(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x$. Suppose a net $(x_i)_{i\in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)} x$. Since P is an $\mathcal{MN}$-doubly continuous poset, there exist $M_x \in \mathcal{M}$ and $N_x \in \mathcal{N}$ such that $M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$, $N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$ and $\sup M_x = x = \inf N_x$. By Proposition 2.12, $x \in \blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z \in \mathcal{O}_{\mathcal{M}}^{\mathcal{N}}(P)$ for every $y \in M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and every $z \in N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$, and hence $x_i \in \blacktriangle_{\mathcal{M}}^{\mathcal{N}} y \cap \triangledown_{\mathcal{M}}^{\mathcal{N}} z$ holds eventually for every $y \in M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and every $z \in N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$. It follows from Proposition 2.8 that $y \leqslant x_i \leqslant z$ holds eventually for every $y \in M_x$ and $z \in N_x$. Thus $(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x$. □

Lemma 2.14. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. If the $\mathcal{MN}$-convergence in P is topological, then P is $\mathcal{MN}$-doubly continuous.*

Proof. Suppose that the $\mathcal{MN}$-convergence in P is topological. Then there exists a topology $\mathcal{T}$ on P such that for every $x \in P$, a net $(x_i)_{i\in I} \xrightarrow{\mathcal{MN}} x$ if and only if $(x_i)_{i\in I} \xrightarrow{\mathcal{T}} x$. Define $I_x = \{(p, U) \in P\times\mathcal{N}(x) : p \in U\}$, where $\mathcal{N}(x)$ denotes the set of all open neighbourhoods of x in the topological space $(P, \mathcal{T})$, i.e., $\mathcal{N}(x) = \{U \in \mathcal{T} : x \in U\}$. Define the preorder $\preccurlyeq$ on I_x as follows:

$$(\forall (p_1, U_1), (p_2, U_2) \in I_x)\, (p_1, U_1) \preccurlyeq (p_2, U_2) \Longleftrightarrow U_2 \subseteq U_1.$$

Now one can easily see that I_x is directed. Let $x_{(p,U)} = p$ for every $(p, U) \in I_x$. Then it is straightforward to check that the net $(x_{(p,U)})_{(p,U)\in I_x} \xrightarrow{\mathcal{T}} x$, and thus $(x_{(p,U)})_{(p,U)\in I_x} \xrightarrow{\mathcal{MN}} x$. By Definition 2.1, there exist $M_x \in \mathcal{M}$ and $N_x \in \mathcal{N}$ such that $\sup M_x = x = \inf N_x$, and for every $m \in M_x$ and $n \in N_x$, there exists $(p_m^n, U_m^n) \in I_x$ such that $x_{(p,U)} = p \in {\uparrow} m \cap {\downarrow} n$ for all $(p, U) \succcurlyeq (p_m^n, U_m^n)$. Since $(p, U_m^n) \succcurlyeq (p_m^n, U_m^n)$ for every $p \in U_m^n$, $x_{(p,U_m^n)} = p \in {\uparrow} m \cap {\downarrow} n$ for every $p \in U_m^n$. This shows

$$(\forall m \in M_x, n \in N_x)\, (\exists U_m^n \in \mathcal{N}(x))\, x \in U_m^n \subseteq {\uparrow} m \cap {\downarrow} n. \tag{$*$}$$

For any $A \in \mathcal{M}$ and $B \in \mathcal{N}$ with $\sup A = x = \inf B$, let $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}}$ be the net defined as in Remark 2.2 (5). Then $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}} \xrightarrow{\mathcal{MN}} x$, and hence $(x_{(d,D)})_{(d,D)\in D^x_{(A,B)}} \xrightarrow{\mathcal{T}} x$. This implies, by Remark 2.2 (6), that there exist $A_0 \sqsubseteq A$ and $B_0 \sqsubseteq B$ satisfying

$$\begin{aligned} x &\in \bigcap \{{\uparrow} a \cap {\downarrow} b : a \in A_0 \,\&\, b \in B_0\} \\ &\subseteq U_m^n \subseteq {\uparrow} m \cap {\downarrow} n. \end{aligned}$$

Therefore, $m \in \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and $n \in \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$, and hence $M_x \subseteq \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and $N_x \subseteq \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$.

Let $y \in \blacktriangledown_{\mathcal{M}}^{\mathcal{N}} x$ and $z \in \vartriangle_{\mathcal{M}}^{\mathcal{N}} x$. Since $\sup M_x = x = \inf N_x$, by Definition 2.3, $\bigcap\{{\uparrow} m \cap {\downarrow} n : m \in M_1 \,\&\, n \in N_1\} \subseteq {\uparrow} y \cap {\downarrow} z$ for some $M_1 \sqsubseteq M_x$ and $N_1 \sqsubseteq N_x$. This concludes by Condition $(*)$ and the finiteness of sets M_1 and N_1 that $\bigcap\{U_m^n : m \in M_1 \,\&\, n \in N_1\} \in \mathcal{N}(x)$ and

$$\begin{aligned} x &\in \bigcap \{U_m^n : m \in M_1 \,\&\, n \in N_1\} \\ &\subseteq \bigcap \{{\uparrow} m \cap {\downarrow} n : m \in M_1 \,\&\, n \in N_1\} \\ &\subseteq {\uparrow} y \cap {\downarrow} z. \end{aligned}$$

Considering the net $(x_{(d,D)})_{(d,D)\in D^x_{(M_x,N_x)}}$ defined in Remark 2.2 (5), we have $(x_{(d,D)})_{(d,D)\in D^x_{(M_x,N_x)}} \xrightarrow{\mathcal{MN}} x$, and hence $(x_{(d,D)})_{(d,D)\in D^x_{(M_x,N_x)}} \xrightarrow{\mathcal{T}} x$. So, by Remark 2.2 (6), there exist $M_2 \sqsubseteq M_x$ and $N_2 \sqsubseteq N_x$ such that

$$\begin{aligned} x &\in \bigcap \{{\uparrow} m \cap {\downarrow} n : m \in M_2 \,\&\, n \in N_2\} \\ &\subseteq \bigcap \{U_m^n : m \in M_1 \,\&\, n \in N_1\} \\ &\subseteq {\uparrow} y \cap {\downarrow} z. \end{aligned}$$

Finally, we show $\bigcap\{\uparrow m \cap \downarrow n : m \in M_2 \,\&\, n \in N_2\} \subseteq \blacktriangle^{\mathcal{N}}_{\mathcal{M}} y \cap \triangledown^{\mathcal{N}}_{\mathcal{M}} z$. Let $(x_{(d,D)})_{(d,D)\in D^{x'}_{(M',N')}}$ be the net defined in 2.2 (5) for any $M' \in \mathcal{M}$ and $N' \in \mathcal{N}$ with $\sup M' = \inf N' = x' \in \bigcap\{\uparrow m \cap \downarrow n : m \in M_2 \,\&\, n \in N_2\}$. Then $(x_{(d,D)})_{(d,D)\in D^{x'}_{(M',N')}} \xrightarrow{\mathcal{M}\mathcal{N}} x'$, and thus $(x_{(d,D)})_{(d,D)\in D^{x'}_{(M',N')}} \xrightarrow{\mathcal{T}} x'$. This implies by Remark 2.2 (6) that there exist $M'_0 \sqsubseteq M'$ and $N'_0 \sqsubseteq N'$ satisfying

$$\begin{aligned} x' &\in \bigcap\{\uparrow m' \cap \downarrow n' : m \in M'_0 \,\&\, n \in N'_0\} \\ &\subseteq \bigcap\{U^n_m : m \in M_1 \,\&\, n \in N_1\} \\ &\subseteq \uparrow y \cap \downarrow z. \end{aligned}$$

Hence, we have $x' \in \blacktriangle^{\mathcal{N}}_{\mathcal{M}} y \cap \triangledown^{\mathcal{N}}_{\mathcal{M}} z$ by Definition 2.3. This shows $\bigcap\{\uparrow m \cap \downarrow n : m \in M_2 \,\&\, n \in N_2\} \subseteq \blacktriangle^{\mathcal{N}}_{\mathcal{M}} y \cap \triangledown^{\mathcal{N}}_{\mathcal{M}} z$. Therefore, it follows from Definition 2.7 that P is $\mathcal{M}\mathcal{N}$-doubly continuous. □

Combining Lemma 2.13 and Lemma 2.14, we obtain the following theorem.

Theorem 2.15. *Let $(P, \mathcal{M}, \mathcal{N})$ be a PMN-space. Then the following statements are equivalent:*
(1) P is an $\mathcal{M}\mathcal{N}$-doubly continuous poset.
(2) For any net $(x_i)_{i\in I}$ in P, $(x_i)_{i\in I} \xrightarrow{\mathcal{M}\mathcal{N}} x$ if and only if $(x_i)_{i\in I} \xrightarrow{\mathcal{O}^{\mathcal{N}}_{\mathcal{M}}(P)} x$.
(3) The $\mathcal{M}\mathcal{N}$-convergence in P is topological.

Proof. (1) ⇒ (2): By Lemma 2.13.
(2) ⇒ (3): It is clear.
(3) ⇒ (1): By Lemma 2.14. □

3 $\mathcal{M}$-topology induced by lim-inf$_{\mathcal{M}}$-convergence

In this section, the notion of lim-inf$_{\mathcal{M}}$-convergence is reviewed and the $\mathcal{M}$-topology on posets is defined. By exploring the fundamental properties of the $\mathcal{M}$-topology, those posets under which the lim-inf$_{\mathcal{M}}$-convergence is topological are precisely characterized.

By saying a *PM-space*, we mean a pair $(P, \mathcal{M})$ that contains a poset P and a subfamily $\mathcal{M}$ of $\mathcal{P}(P)$.

Definition 3.1 ([8]). *Let $(P, \mathcal{M})$ be a PM-space. A net $(x_i)_{i\in I}$ in P is said to* lim-inf$_{\mathcal{M}}$-converge *to $x \in P$ if there exists $M \in \mathcal{M}$ such that*
(M1)$x \leqslant \sup M$;
(M2)for every $m \in M$, $x_i \geqslant m$ holds eventually.
In this case, we write $(x_i)_{i\in I} \xrightarrow{\mathcal{M}} x$.

It is worth noting that both lim-inf-convergence and lim-inf$_2$-convergence [4] in posets are particular cases of lim-inf$_{\mathcal{M}}$-convergence.

Remark 3.2. *Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$.*
(1) Suppose that a net $(x_i)_{i\in I} \xrightarrow{\mathcal{M}} x$ and $y \leqslant x$. Then $(x_i)_{i\in I} \xrightarrow{\mathcal{M}} y$ by Definition 3.1. This concludes that the set of all lim-inf$_{\mathcal{M}}$-convergent points of the net $(x_i)_{i\in I}$ in P is a lower subset of P. Thus, the lim-inf$_{\mathcal{M}}$-convergent points of the net $(x_i)_{i\in I}$ need not be unique.
(2) If P has the least element $\perp$ and $\emptyset \in \mathcal{M}$, then we have $(x_i)_{i\in I} \xrightarrow{\mathcal{M}} \perp$ for every net $(x_i)_{i\in I}$ in P.

(3) *For every $M \in \mathcal{M}$ with $\sup M \geqslant x$, we denote $F_M^x = \{\bigcap\{\uparrow m : m \in M_0\} : M_0 \sqsubseteq M\}$*[2]*. Let $D_M^x = \{(d, D) \in P \times F_M^x : d \in D\}$ be in the preorder $\leqq$ defined by*

$$(\forall (d_1, D_1), (d_2, D_2) \in D_M^x)\ (d_1, D_1) \leqq (d_2, D_2) \Longleftrightarrow D_2 \subseteq D_1.$$

It is easy to see that the set D_M^x is directed. Take $x_{(d,D)} = d$ for every $(d, D) \in D_M^x$. Then, by Definition 3.1, one can straightforwardly check that the net $(x_{(d,D)})_{(d,D)\in D_M^x} \xrightarrow{\mathcal{M}} a$ for every $a \leqslant x$.

(4) *If the net $(x_{(d,D)})_{(d,D)\in D_M^x}$ defined in (3) converges to $p \in P$ with respect to some topology $\mathcal{T}$ on P, then for every open neighbourhood U_p of p, there exists $M_0 \sqsubseteq M$ such that $\bigcap\{\uparrow m : m \in M_0\} \subseteq U_p$.*

Definition 3.3 ([8]). *Let $(P, \mathcal{M})$ be a PM-space.*

(1) *For $x, y \in P$, define $y \ll_{\alpha(\mathcal{M})} x$ if for every net $(x_i)_{i\in I}$ that lim-inf$_{\mathcal{M}}$-converges to x, $x_i \geqslant y$ holds eventually.*

(2) *The poset P is said to be $\alpha(\mathcal{M})$-continuous if $\{x \in P : x \ll_{\alpha(\mathcal{M})} a\} \in \mathcal{M}$ and $a = \sup\{x \in P : x \ll_{\alpha(\mathcal{M})} a\}$ holds for every $a \in P$.*

Given a *PM*-space $(P, \mathcal{M})$, the approximate relation $\ll_{\alpha(\mathcal{M})}$ on the poset P can be equivalently characterized in the following proposition.

Proposition 3.4. *Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$. Then $y \ll_{\alpha(\mathcal{M})} x$ if and only if for every $M \in \mathcal{M}$ with $\sup M \geqslant x$, there exists $M_0 \sqsubseteq M$ such that*

$$\bigcap\{\uparrow m : m \in M_0\} \subseteq \uparrow y.$$

Proof. Suppose $y \ll_{\alpha(\mathcal{M})} x$. Let $(x_{(d,D)})_{(d,D)\in D_M^x}$ be the net defined in Remark 3.2 (3) for every $M \in \mathcal{M}$ with $\sup M = p \geqslant x$. Then the net $(x_{(d,D)})_{(d,D)\in D_M^x} \xrightarrow{\mathcal{M}} x$. By Definition 3.3 (1), there exists $(d_0, D_0) \in D_M^x$ such that $x_{(d,D)} = d \geqslant y$ for all $(d, D) \geqq (d_0, D_0)$. Since $(d, D_0) \geqq (d_0, D_0)$ for every $d \in D_0$, $x_{(d,D_0)} = d \geqslant y$ for every $d \in D_0$. So $D_0 \subseteq \uparrow y$. This shows that there exists $M_0 \sqsubseteq M$ such that $D_0 = \bigcap\{\uparrow m : m \in M_0\} \subseteq \uparrow y$.

Conversely, suppose that for every $M \in \mathcal{M}$ with $\sup M \geqslant x$, there exists $M_0 \sqsubseteq M$ such that $\bigcap\{\uparrow m : m \in M_0\} \subseteq \uparrow y$. Let $(x_i)_{i\in I}$ be a net that lim-inf$_{\mathcal{M}}$-converges to x. Then, by Definition 3.1, there exists $M \in \mathcal{M}$ such that $\sup M = p \geqslant x$, and for every $m \in M$, there exists $i_m \in I$ such that $x_i \geqslant m$ for all $i \geq i_m$. Take $i_0 \in I$ with that $i_0 \geq i_m$ for every $m \in M_0 \sqsubseteq M$, we have that $x_i \in \bigcap\{\uparrow m : m \in M_0\} \subseteq \uparrow y$ for all $i \geq i_0$. This shows that $x_i \geqslant y$ holds eventually. Thus, by Definition 3.3 (1), we have $y \ll_{\alpha(\mathcal{M})} x$. □

Remark 3.5. *Let $(P, \mathcal{M})$ be a PM-space and $x, y \in P$.*

(1) *If there is no $M \in \mathcal{M}$ such that $\sup M \geqslant x$, then $p \ll_{\alpha(\mathcal{M})} x$ for every $p \in P$. And, if the poset P has the least element $\perp$, then $\perp \ll_{\alpha(\mathcal{M})} p$ for every $p \in P$.*

(2) *The implication $y \ll_{\alpha(\mathcal{M})} x \Longrightarrow y \leqslant x$ may not be true. For example, let $P = \{0, 1, 2, \dots\}$ be in the discrete order $\leqslant$ defined by*

$$(\forall i, j \in P)\ i \leqslant j \Longleftrightarrow i = j.$$

And let $\mathcal{M} = \{\{2\}\}$. Then, it is easy to see from Remark 3.5 (1) that $0 \ll_{\alpha(\mathcal{M})} 1$ and $0 \not\leqslant 1$.

(3) *Assume the PM-space $(P, \mathcal{M})$ has the property that for every $p \in P$, there exists $M_p \in \mathcal{M}$ such that $\sup M_p = p$. Then, by Proposition 3.4, we have*

$$(\forall q, r \in P)\ q \ll_{\alpha(\mathcal{M})} r \Longrightarrow q \leqslant r.$$

For more interpretations of the approximate relation $\ll_{\alpha(\mathcal{M})}$ on posets, the readers can refer to Example 3.2 and Remark 3.3 in [8].

For simplicity, given a *PM*-space $(P, \mathcal{M})$ and $x \in P$, we will denote

– $\blacktriangledown_{\mathcal{M}} x = \{y \in P : y \ll_{\alpha(\mathcal{M})} x\}$;

[2] From the logical point of view, we stipulate $\bigcap\{\uparrow m : m \in M_0\} = P$ if $M_0 = \emptyset$.

– $\blacktriangle_{\mathcal{M}} x = \{z \in P : x \ll_{\alpha(\mathcal{M})} z\}$.

Based on the approximate relation $\ll_{\alpha(\mathcal{M})}$ on posets, the $\alpha^{\star}(\mathcal{M})$-continuity can be defined for posets in the following:

Definition 3.6. *Let $(P, \mathcal{M})$ be a PM-space. The poset P is called an $\alpha^{\star}(\mathcal{M})$-*continuous poset *if for every $x \in P$, there exists $M_x \in \mathcal{M}$ such that*
(O1) $\sup M_x = x$ *and* $M_x \subseteq \blacktriangledown_{\mathcal{M}} x$. *And,*
(O2)for every $y \in \blacktriangledown_{\mathcal{M}} x$, there exists $F \sqsubseteq M_x$ such that $\bigcap\{\uparrow f : f \in F\} \subseteq \blacktriangle_{\mathcal{M}} y$.

Noticing Remark 3.5 (3), we have the following proposition about $\alpha^{\star}(\mathcal{M})$-continuous posets.

Proposition 3.7. *Let $(P, \mathcal{M})$ be a PM-space in which the poset P is $\alpha^{\star}(\mathcal{M})$-continuous. Then*

$$(\forall x, y \in P)\; y \ll_{\alpha(\mathcal{M})} x \Longrightarrow y \leqslant x.$$

The following examples of $\alpha^{\star}(\mathcal{M})$-continuous posets can be formally checked by Definition 3.6.

Example 3.8. *Let $(P, \mathcal{M})$ be a PM-space.*
1. *If P is a finite poset, then P is an $\alpha^{\star}(\mathcal{M})$-continuous poset if and only if for every $x \in P$, there exists $M_x \in \mathcal{M}$ such that* $\sup M_x = x$.
2. *Let $\mathcal{M} = \mathcal{L}(P)$. Then P is an $\alpha^{\star}(\mathcal{L})$-continuous poset. This means that every poset is $\alpha^{\star}(\mathcal{L})$-continuous.*
3. *Let $\mathcal{M} = \mathcal{D}(P)$. Then we have $\ll = \ll_{\alpha(\mathcal{D})}$ (see Example 3.2 (1) in [8]). The poset P is a continuous poset if and only if it is an $\alpha^{\star}(\mathcal{D})$-continuous poset. In particular, finite posets, chains, anti-chains and completely distributive lattices are all $\alpha^{\star}(\mathcal{D})$-continuous.*
4. *Let $\mathcal{M} = \mathcal{P}(P)$. If P is a finite poset (resp. chain, anti-chain), then P is an $\alpha^{\star}(\mathcal{P})$-continuous poset.*

Proposition 3.9. *Let $(P, \mathcal{M})$ be a PM-space. If P is an $\alpha(\mathcal{M})$-continuous poset, and $\{y \in P : (\exists z \in P)\; y \ll_{\alpha(\mathcal{M})} z \ll_{\alpha(\mathcal{M})} a\} \in \mathcal{M}$ for every $a \in P$, then P is an $\alpha^{\star}(\mathcal{M})$-continuous poset.*

Proof. Suppose that P is an $\alpha(\mathcal{M})$-continuous poset, and $\{y \in P : (\exists z \in P)\; y \ll_{\alpha(\mathcal{M})} z \ll_{\alpha(\mathcal{M})} a\} \in \mathcal{M}$ for every $a \in P$. Take $M_a = \blacktriangledown_{\mathcal{M}} a$. Then it is easy to see that $\sup M_a = a$ and $M_a \subseteq \blacktriangledown_{\mathcal{M}} a$. By Remark 3.3 (4) in [8], we have $\sup\{y \in P : (\exists z \in P)\; y \ll_{\alpha(\mathcal{M})} z \ll_{\alpha(\mathcal{M})} a\} = a$. This implies, by Proposition 3.4 and Remark 3.5 (2), that for every $y \in \blacktriangledown_{\mathcal{M}} a$, there exist $\{y_1, y_2, \ldots, y_n\}, \{z_1, z_2, \ldots, z_n\} \sqsubseteq M_a = \blacktriangledown_{\mathcal{M}} a$ such that

$$\begin{aligned} &\bigcap\{\uparrow z_i : i \in \{1, 2, \ldots, n\}\} \\ \subseteq &\bigcap\{\uparrow y_i : i \in \{1, 2, \ldots, n\}\} \\ \subseteq &\uparrow y, \end{aligned}$$

and $y_i \ll_{\alpha(\mathcal{M})} z_i \ll_{\alpha(\mathcal{M})} a$ for every $i \in \{1, 2, \ldots, n\}$. Next, we show $\bigcap\{\uparrow z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \blacktriangle_{\mathcal{M}} y$. For every $M \in \mathcal{M}$ with $\sup M \geqslant b \in \bigcap\{\uparrow z_i : i \in \{1, 2, \ldots, n\}\}$, by Proposition 3.4, there exists $M_i \sqsubseteq M$ such that $\bigcap\{\uparrow m' : m' \in M_i\} \subseteq \uparrow y_i$ for every $i \in \{1, 2, \ldots, n\}$. Take $M_0 = \bigcup\{M_i : i \in \{1, 2, \ldots, n\}\}$. Then $M_0 \sqsubseteq M$ and

$$\begin{aligned} &\bigcap\{\uparrow m : m \in M_0\} \\ \subseteq &\bigcap\{\uparrow y_i : i \in \{1, 2, \ldots, n\}\} \\ \subseteq &\uparrow y. \end{aligned}$$

This shows $y \ll_{\alpha(\mathcal{M})} b$ for every $b \in \bigcap\{\uparrow z_i : i \in \{1, 2, \ldots, n\}\}$. Hence, $\bigcap\{\uparrow z_i : i \in \{1, 2, \ldots, n\}\} \subseteq \blacktriangle_{\mathcal{M}} y$. Thus P is an $\alpha^{\star}(\mathcal{M})$-continuous poset. □

The fact that an $\alpha^{\star}(\mathcal{M})$-continuous poset P in a *PM*-space $(P, \mathcal{M})$ may not be $\alpha(\mathcal{M})$-continuous can be demonstrated in the following example.

Example 3.10. *Let $(P, \mathcal{M})$ be the PM-space in which the poset $P = \mathbb{R}$ is the set of all real number with its usual order $\leqslant$ and $\mathcal{M} = \mathcal{S}_0(\mathbb{R})$. Then we have $\ll_{\alpha(\mathcal{S}_0)} = \leqslant$ by Proposition 3.4. It is easy to check, by Definition 3.6, that $\mathbb{R}$ is an $\alpha^\star(\mathcal{S}_0)$-continuous poset. But $\mathbb{R}$ is not an $\alpha(\mathcal{S}_0)$-continuous poset because $\blacktriangledown_{\mathcal{S}_0} x = \downarrow x \notin \mathcal{S}_0(P)$ for every $x \in \mathbb{R}$.*

We turn to consider the topology induced by the lim-inf$_{\mathcal{M}}$-convergence in posets.

Definition 3.11. *Let $(P, \mathcal{M})$ be a PM-space. A subset V of P is said to be* $\mathcal{M}$-open *if for every net $(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x \in V$, $x_i \in V$ holds eventually.*

Given a PM-space $(P, \mathcal{M})$, one can formally verify that the set of all $\mathcal{M}$-open subsets of P forms a topology on P. This topology is called the $\mathcal{M}$-*topology*, and denoted by $\mathcal{O}_{\mathcal{M}}(P)$.

The following Theorem is an order-theoretical characterization of $\mathcal{M}$-open sets.

Theorem 3.12. *Let $(P, \mathcal{M})$ be a PM-space. Then a subset V of P is $\mathcal{M}$-open if and only if it satisfies the following two conditions:*
(V1)$\uparrow V = V$, i.e., V is an upper set.
(V2)For every $M \in \mathcal{M}$ with $\sup M \in V$, there exists $M_0 \sqsubseteq M$ such that $\bigcap\{\uparrow m : m \in M_0\} \subseteq V$.

Proof. Suppose that V is an $\mathcal{M}$-open subset of P. By Remark 3.2 (1), it is easy to see that V is an upper set. Let $(x_{(d,D)})_{(d,D) \in D_M^x}$ be the net defined in Remark 3.2 (3) for every $M \in \mathcal{M}$ with $\sup M = x \in V$. Then $(x_{(d,D)})_{(d,D) \in D_M^x} \xrightarrow{\mathcal{M}} x \in V$. This implies, by Definition 3.11, that there exists $(d_0, D_0) \in D_M^x$ such that $x_{(d,D)} = d \in V$ for all $(d, D) \geqq (d_0, D_0)$. Since $(d, D_0) \geqq (d_0, D_0)$ for all $d \in D_0$, $x_{(d,D_0)} = d \in V$ for all $d \in D_0$. This shows $D_0 \subseteq V$. Thus there exists $M_0 \sqsubseteq M$ such that $D_0 = \bigcap\{\uparrow m : m \in M_0\} \subseteq V$.

Conversely, suppose V is a subset of P which satisfies Condition (V1) and (V2). Let $(x_i)_{i \in I}$ be a net that lim-inf$_{\mathcal{M}}$-converges to $x \in V$. Then there exists $M \in \mathcal{M}$ such that $\sup M = y \geqslant x \in V = \uparrow V$ (hence, $y \in V$), and for every $m \in M$, there exists $i_m \in I$ such that $x_i \geqslant m$ for all $i \geq i_m$. By Condition (V2), we have that $\bigcap\{\uparrow m : m \in M_0\} \subseteq V$ for some $M_0 \sqsubseteq M$. Take $i_0 \in I$ with that $i_0 \geq i_m$ for all $m \in M_0$. Then $x_i \in \bigcap\{\uparrow m : m \in M_0\} \subseteq V$ for all $i \geq i_0$. This shows that V is an $\mathcal{M}$-open set. □

Recall that given a topological space $(X, \mathcal{T})$ and a point $x \in P$, a family $\mathcal{B}(x)$ of open neighbourhoods of x is called *a base for the topological space* $(X, \mathcal{T})$ *at the point* x if for every neighbourhood V of x there exists an $U \in \mathcal{B}(x)$ such that $x \in U \subseteq V$.

If the poset P in a PM-space $(P, \mathcal{M})$ is an $\alpha^\star(\mathcal{M})$-continuous poset, we provide a base for the topological space $(P, \mathcal{O}_{\mathcal{M}}(P))$ at a point $x \in P$.

Proposition 3.13. *Let $(P, \mathcal{M})$ be a PM-space in which the poset P is $\alpha^\star(\mathcal{M})$-continuous. Then $\blacktriangle_{\mathcal{M}} x \in \mathcal{O}_{\mathcal{M}}(P)$ for every $x \in P$.*

Proof. One can readily see, by Proposition 3.4, that $\blacktriangle_{\mathcal{M}} x$ is an upper subset of P for every $x \in P$. For every $M \in \mathcal{M}$ with $\sup M = y \in \blacktriangle_{\mathcal{M}} x$, by Definition 3.6 (O1) there exists $M_y \in \mathcal{M}$ such that $M_y \subseteq \blacktriangledown_{\mathcal{M}} y$ and $\sup M_y = y$. Since $x \ll_{\alpha(\mathcal{M})} y$, by Definition 3.6 (O2), we have $\bigcap\{\uparrow m_i : i \in \{1, 2, \ldots, n\}\} \subseteq \blacktriangle_{\mathcal{M}} x$ for some $\{m_1, m_2, \ldots, m_n\} \sqsubseteq M_y$. Observing $\{m_1, m_2, \ldots, m_n\} \sqsubseteq M_y \subseteq \blacktriangledown_{\mathcal{M}} y$, we can conclude that there exists $M_i \sqsubseteq M$ such that $\bigcap\{\uparrow a : a \in M_i\} \subseteq \uparrow m_i$ for every $i \in \{1, 2, \ldots, n\}$. Let $M_0 = \bigcup\{M_i : i \in \{1, 2, \ldots, n\}\}$. Then $M_0 \sqsubseteq M$ and

$$\begin{aligned} &\bigcap\{\uparrow m : m \in M_0\} \\ \subseteq &\bigcap\{\uparrow m_i : i \in \{1, 2, \ldots, n\}\} \\ \subseteq &\blacktriangle_{\mathcal{M}} x. \end{aligned}$$

This shows, by Theorem 3.12, that $\blacktriangle_{\mathcal{M}} x \in \mathcal{O}_{\mathcal{M}}(P)$ for every $x \in P$. □

Proposition 3.14. *Let $(P, \mathcal{M})$ be a PM-space in which the poset P is $\alpha^\star(\mathcal{M})$-continuous and $x \in P$. Then $\{\bigcap\{\blacktriangle_{\mathcal{M}} a : a \in A\} : A \sqsubseteq \blacktriangledown_{\mathcal{M}} x\}$ is a base for the topological space $(P, \mathcal{O}_{\mathcal{M}}(P))$ at the point x.*

Proof. Clearly, by Proposition 3.13, we have $\bigcap\{\blacktriangle_{\mathcal{M}} a : a \in A\} \in \mathcal{O}_{\mathcal{M}}(P)$ for every $A \sqsubseteq \blacktriangledown_{\mathcal{M}} x$. Let $U \in \mathcal{O}_{\mathcal{M}}(P)$ and $x \in U$. Since P is an $\alpha^\star(\mathcal{M})$-continuous poset, there exists $M_x \in \mathcal{M}$ such that $M_x \subseteq \blacktriangledown_{\mathcal{M}} x$ and $\sup M_x = x \in U$. By Theorem 3.12, it follows that $\bigcap\{\uparrow m : m \in M_0\} \subseteq U$ for some $M_0 \sqsubseteq M_x \subseteq \blacktriangledown_{\mathcal{M}} x$. So, from Proposition 3.7, we have

$$\begin{aligned} x &\in \bigcap\{\blacktriangle_{\mathcal{M}} m : m \in M_0\} \\ &\subseteq \bigcap\{\uparrow m : m \in M_0\} \subseteq U. \end{aligned}$$

Thus, $\{\bigcap\{\blacktriangle_{\mathcal{M}} a : a \in A\} : A \sqsubseteq \blacktriangledown_{\mathcal{M}} x\}$ is a base for the topological space $(P, \mathcal{O}_{\mathcal{M}}(P))$ at the point x. □

In the rest, we are going to establish a characterization theorem which demonstrates the equivalence between the lim-inf$_{\mathcal{M}}$-convergence being topological and the $\alpha^\star(\mathcal{M})$-continuity of the poset in a given *PM*-space.

Lemma 3.15. *Let $(P, \mathcal{M})$ be a PM-space. If P is an $\alpha^\star(\mathcal{M})$-continuous poset, then a net*

$$(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x \in P \iff (x_i)_{i \in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}(P)} x.$$

Proof. By the definition of $\mathcal{O}_{\mathcal{M}}(P)$, it is easy to see that a net

$$(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x \in P \Longrightarrow (x_i)_{i \in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}(P)} x.$$

To prove the Lemma, we only need to show that a net $(x_i)_{i \in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}(P)} x \in P$ implies $(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x$. Suppose $(x_i)_{i \in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}(P)} x$. As P is an $\alpha^\star(\mathcal{M})$-continuous poset, there exists $M_x \in \mathcal{M}$ such that $M_x \subseteq \blacktriangledown_{\mathcal{M}} x$ and $\sup M_x = x$. By Proposition 3.13, we have $x \in \blacktriangle_{\mathcal{M}} y \in \mathcal{O}_{\mathcal{M}}(P)$ for every $y \in M_x \subseteq \blacktriangledown_{\mathcal{M}} x$. Hence, $x_i \in \blacktriangle_{\mathcal{M}} y$ holds eventually. This implies, by Proposition 3.7, that $x_i \in \blacktriangle_{\mathcal{M}} y \subseteq \uparrow y$ holds eventually. By the definition of lim-inf$_{\mathcal{M}}$-convergence, we have $(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x$. □

In the converse direction, we have the following Lemma.

Lemma 3.16. *Let $(P, \mathcal{M})$ be a PM-space. If the lim-inf$_{\mathcal{M}}$-convergence in P is topological, then P is an $\alpha^\star(\mathcal{M})$-continuous poset.*

Proof. Suppose that the lim-inf$_{\mathcal{M}}$-convergence in P is topological. Then there exists a topology $\mathcal{T}$ such that for every $x \in P$, a net

$$(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x \iff (x_i)_{i \in I} \xrightarrow{\mathcal{T}} x.$$

Define $I_x = \{(p, V) \in P \times \mathcal{N}(x) : p \in V\}$, where $\mathcal{N}(x)$ is the set of all open neighbourhoods of x, namely, $\mathcal{N}(x) = \{V \in \mathcal{T} : x \in V\}$. Define also the preorder $\preceq$ on I_x as follows:

$$(\forall (p_1, V_1), (p_2, V_2) \in I_x)\, (p_1, V_1) \preceq (p_2, V_2) \iff V_2 \subseteq V_1.$$

It is easy to see that I_x is directed. Now, let $x_{(p,V)} = p$ for every $(p, V) \in I_x$. Then one can readily check that the net $(x_{(p,V)})_{(p,V) \in I_x} \xrightarrow{\mathcal{T}} x$, and hence $(x_{(p,V)})_{(p,V) \in I_x} \xrightarrow{\mathcal{M}} x$. This means that there exists $M_x \in \mathcal{M}$ such that $\sup M_x \geqslant x$, and for every $m \in M_x$, there exists $(p_m, V_m) \in I_x$ with that $x_{(p,V)} = p \geqslant m$ for all $(p, V) \succeq (p_m, V_m)$. Since $(p, V_m) \succeq (p_m, V_m)$ for all $p \in V_m$, we have $x_{(p,V_m)} = p \geqslant m$ for all $p \in V_m$. This shows

$$(\forall m \in M_x)\, (\exists V_m \in \mathcal{N}(x))\, x \in V_m \subseteq \uparrow m. \tag{$\star\star$}$$

Next we prove $M_x \subseteq \blacktriangledown_{\mathcal{M}} x$. For every $m \in M_x$ and every $M \in M$ with $\sup M \geqslant x$, let $(x_{(d,D)})_{(d,D) \in D_M^x}$ be the net defined in Remark 3.2 (3). Then the net $(x_{(d,D)})_{(d,D) \in D_M^x} \xrightarrow{\mathcal{M}} x$, and thus $(x_{(d,D)})_{(d,D) \in D_M^x} \xrightarrow{\mathcal{T}} x$. It follows from

Remark 3.2 (4) that there exists $M_0 \sqsubseteq M$ such that $x \in \bigcap\{\uparrow a : a \in M_0\} \subseteq V_m$. By Condition (**), we have $x \in \bigcap\{\uparrow a : a \in M_0\} \subseteq V_m \subseteq \uparrow m$. So, $m \ll_{\alpha(\mathcal{M})} x$. This shows $M_x \subseteq \blacktriangledown_{\mathcal{M}} x$.

Let $y \in \blacktriangledown_{\mathcal{M}} x$. Then there exists $\{m_1, m_2, \ldots, m_n\} \sqsubseteq M_x$ such that $\bigcap\{\uparrow m_i : i \in \{1, 2, \ldots, n\}\} \subseteq \uparrow y$ as $M_x \in \mathcal{M}$ and $\sup M_x \geqslant x$. By Condition (**), it follows that $\bigcap\{V_{m_i} : i \in \{1, 2, \ldots, n\}\} \subseteq \bigcap\{\uparrow m_i : i \in \{1, 2, \ldots, n\}\} \subseteq \uparrow y$. Considering the net $(x_{(d,D)})_{(d,D) \in D^x_{M_x}}$ defined in Remark 3.2 (3), we have $(x_{(d,D)})_{(d,D) \in D^x_{M_x}} \xrightarrow{\mathcal{M}} x$, and hence $(x_{(d,D)})_{(d,D) \in D^x_{M_x}} \xrightarrow{\mathcal{T}} x$. This implies, by Remark 3.2 (4), that

$$\begin{aligned} &\bigcap\{\uparrow b : b \in M_{00}\} \\ \subseteq &\bigcap\{V_{m_i} : i \in \{1, 2, \ldots, n\}\} \\ \subseteq &\bigcap\{\uparrow m_i : i \in \{1, 2, \ldots, n\}\} \subseteq \uparrow y \end{aligned} \qquad (***)$$

for some $M_{00} \sqsubseteq M_x$. Finally, we show $\bigcap\{\uparrow b : b \in M_{00}\} \subseteq \blacktriangle_{\mathcal{M}} y$. For every $x' \in \bigcap\{\uparrow b : b \in M_{00}\}$ and every $M' \in \mathcal{M}$ with $\sup M' \geqslant x'$, let $(x_{(d,D)})_{(d,D) \in D^{x'}_{M'}}$ be the net defined in Remark 3.2 (3). Then $(x_{(d,D)})_{(d,D) \in D^{x'}_{M'}} \xrightarrow{\mathcal{M}} x'$, and thus $(x_{(d,D)})_{(d,D) \in D^{x'}_{M'}} \xrightarrow{\mathcal{T}} x'$. It follows from Condition (***) and Remark 3.2 (4) that there exists $M'_0 \sqsubseteq M'$ such that

$$\begin{aligned} &\bigcap\{\uparrow a' : a' \in M'_0\} \\ \subseteq &\bigcap\{V_{m_i} : i \in \{1, 2, \ldots, n\}\} \\ \subseteq &\bigcap\{\uparrow m_i : i \in \{1, 2, \ldots, n\}\} \subseteq \uparrow y. \end{aligned}$$

This shows $x' \in \blacktriangle_{\mathcal{M}} y$, and thus $\bigcap\{\uparrow b : b \in M_{00}\} \subseteq \blacktriangle_{\mathcal{M}} y$. Therefore, P is an $\alpha^{\star}(\mathcal{M})$-continuous poset. □

Combining Lemma 3.15 and Lemma 3.16, we deduce the following result.

Theorem 3.17. *Let $(P, \mathcal{M})$ be a PM-space. The following statements are equivalent:*
(1) P is an $\alpha^{\star}(\mathcal{M})$-continuous poset.
(2) For any net $(x_i)_{i \in I}$ in P, $(x_i)_{i \in I} \xrightarrow{\mathcal{M}} x \in P \Longleftrightarrow (x_i)_{i \in I} \xrightarrow{\mathcal{O}_{\mathcal{M}}(P)} x$.
(3) The lim-inf$_{\mathcal{M}}$-convergence in P is topological.

Proof. (1) ⇒ (2): By Lemma 3.15.
(2) ⇒ (3): Clear.
(3) ⇒ (1): By Lemma 3.16. □

Corollary 3.18 ([8]). *Let $(P, \mathcal{M})$ be a PM-space with $\mathcal{S}_0(P) \subseteq \mathcal{M} \subseteq \mathcal{P}(P)$. Suppose $\blacktriangledown_{\mathcal{M}} a \in \mathcal{M}$ and $\{y \in P : (\exists z \in P)\, y \ll_{\alpha(\mathcal{M})} z \ll_{\alpha(\mathcal{M})} a\} \in \mathcal{M}$ holds for every $a \in P$. Then the lim-inf$_{\mathcal{M}}$-convergence in P is topological if and only if P is $\alpha(\mathcal{M})$-continuous.*

Proof. (⟹): To show the $\alpha(\mathcal{M})$-continuity of P, it suffices to prove $\sup \blacktriangledown_{\mathcal{M}} a = a$ for every $a \in P$. Since the lim-inf$_{\mathcal{M}}$-convergence in P is topological, by Theorem 3.17, P is an $\alpha^{\star}(\mathcal{M})$-continuous poset. This implies that there exists $M_a \in \mathcal{M}$ such that $\sup M_a \subseteq \blacktriangledown_{\mathcal{M}} a$ and $\sup M_a = a$ for every $a \in P$. By Proposition 3.7, we have $\blacktriangledown_{\mathcal{M}} a \subseteq \downarrow a$. So $\sup \blacktriangledown_{\mathcal{M}} a = a$.
(⟸): By Proposition 3.9 and Theorem 3.17. □

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