DE GRUYTER Open Math. 2018; 16: 1061–1076

Open Mathematics

Research Article

Yu Long Fu, Xiao Long Xin*, and Jun Tao Wang

State maps on semihoops

https://doi.org/10.1515/math-2018-0089
Received March 20, 2018; accepted June 22, 2018.

Abstract: In this paper, we introduce the notion of state maps from a semihoop H_1 to another semihoop H_2, which is a generalization of internal states (or state operators) on a semihoop H. Also we give a type of special state maps from a semihoop H_1 to H_1, which is called internal state maps (or IS-maps). Then we give some examples and basic properties of (internal) state maps on semihoops. Moreover, we discuss the relations between state maps and internal states on other algebras. Then we introduce several kinds of filters by state maps on semihoops, called SM-filters, state filters and dual state filters, respectively, and discuss the relations among them. Furthermore we introduce and study the notion of prime SM-filters on semihoops. Finally, using SM-filter, we characterize two kinds of state semihoops.

Keywords: Semihoop, State map, SM-filter, Prime SM-filter

MSC: 06F35, 03G25, 08A72

1 Introduction

Residuated structures arise in many areas of mathematics, and are particularly common among algebras associated with logical systems. The essential ingredients are a partial order $\leq$, a binary operation of associative and commutative multiplication $\odot$ that respects the partial order, and a binary (left-)residuation operation $\rightarrow$ characterized by $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$. Semihoops [14] are very important and basic residuated structures in which the community of many-valued logicians got interested in the last years, as they are building blocks for several interesting structures being the algebraic semantics for relevant many-valued logics such as basic fuzzy logic (BL, for short). Apart from their logic interest, semihoops have interesting algebraic properties and include kinds of important classes of algebras: Hoops which were originally introduced by Bosbach [6, 7] under the name of complementary semigroups and Brouwerian semilattices-the models of the conjunction-implication fragment of the intuitionistic propositional calculus. A semihoop is called a hoop if $x \odot (x \rightarrow y) = y \odot (y \rightarrow x)$ and a semihoop does not satisfy the divisibility condition $x \wedge y = x \odot (x \rightarrow y)$. Therefore, semihoops are the most fundamental fuzzy structures. It will play an important role in studying fuzzy logics and the related algebraic structures.

In order to measure the average truth-value of propositions in Lukasiewicz logic, Mundici [24] presented an analogue of probability measure, called a state, as averaging process for formulas in Łukasiewicz logic. States on MV-algebras have been deeply investigated. Consequently, the notion of states has been extended to other logical algebras such as BL-algebras [25], MTL-algebras [20, 21], R_0-algebras [22] and residuated lattices [12, 19, 23, 26].

Yu Long Fu: School of Cyber Engnieering, Xidian University, Xi'an, 710071, China, E-mail: yulongf@gmail.com
***Corresponding Author: Xiao Long Xin:** School of Mathematics, Northwest University, Xi'an, 710127, China, E-mail: xlxin@nwu.edu.cn
Jun Tao Wang: School of Mathematics, Northwest University, Xi'an, 710127, China, E-mail: wjt@stumail.nwu.edu.cn

Since MV-algebras with state are not universal algebras, they do not automatically induce an assertional logic. Flaminio and Montagna [15, 16] presented an algebraizable logic using a probabilistic approach, and its equivalent algebraic semantics is precisely the variety of state MV-algebras. We recall that a state MV-algebra is an MV-algebra whose language is extended by adding an operator(also called an internal state), whose properties are inspired by ones of states with the addition property. State MV-algebras generalize, for example, Hajek's approach [17] to fuzzy logic with modality Pr (interpreted as probably) which has the following semantic interpretation: The probability of an event a is presented as the truth value of $Pr(a)$. On the other hand, if s is a state, then $s(a)$ is interpreted as the average appearance of the many valued event a. Consequently, the notion of internal states has also been extended to other algebraic structures. For example, the concept of a state BL-algebra was introduced by Ciungu et al.[11], as an extension of the concept of a state MV-algebra. Subsequently, the concept of internal states was extended by Dvurečenskij et al.[13] to Rℓ-monoids (not necessarily commutative). More generally, the state residuated lattices were introduced by He and Xin [18].

We observed that the states and internal state on MV-algebras, BL-algebras, BCK-algebras and residuated lattices are maps from an algebra X to $[0, 1]$ and X to X, respectively. From the viewpoint of universal algebras, it is meaningful to study a state map from an algebra X to another algebra Y. In particular, if $Y = [0, 1]$, a state can be seen as a state map from X to $[0, 1]$, and if $X = Y$, a state operator can also be seen as a state map from $X \to X$. Based on this idea, we can conclude that a state map is not only a generalization of internal states but also preserves the usual properties of states. Therefore, it is meaningful to introduce state map to the more general fuzzy structures semihoops and providing an algebraic foundation for reasoning about probabilities of fuzzy events in a new way. This is the motivation for us to investigate state maps on semihoop.

This paper is structured in five sections. In order to make the paper as self-contained as possible, we recapitulate in Section 2 the definition of semihoops, and review their basic properties that will be used in the remainder of the paper. In Section 3, we introduce the notion of state maps (or simply, S-maps), which is a generalization of states on semihoops. Also, we give a characterization of two kinds of semihooops. In Section 4, we discuss the relations between state maps on semihoops and internal states on other algebras, respectively. In Section 5, we introduce several kinds of filters by state maps on semihoops, called SM-filters, state filters and dual state filters, respectively, and discuss the relations among them. Using SM-filter, we characterize two kinds of state semihoops.

2 Preliminaries

In this section, we summarize some definitions and results about semihoop, which will be used in the following sections of the paper.

Definition 2.1 ([14]). *An algebra $(H, \odot, \to, \wedge, 1)$ of type (2,2,2,0) is called a semihoop if it satisfies the following conditions:*
(1) $(H, \wedge, 1)$ is a $\wedge$-semilattice with upper bound 1,
(2) $(H, \odot, 1)$ is a commutative monoid,
(3) $(x \odot y) \to z = x \to (y \to z)$, for all $x, y, z \in H$.

In what follows, by H we denote the universe of a semihoop $(H, \odot, \to, \wedge, 1)$. For any $x \in H$ and a natural number n, we define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for $n \geq 1$.

On a semihoop $(H, \odot, \to, \wedge, 1)$ we define $x \leq y$ iff $x \to y = 1$. It is easy to check that $\leq$ is a partial order relation on H and for all $x \in H$, $x \leq 1$. A semihoop H is bounded if there exists an element $0 \in H$ such that $0 \leq x$ for all $x \in H$. In a bounded semihoop $(H, \odot, \to, \wedge, 0, 1)$, we define the negation $* : x^* = x \to 0$ for all $x \in L$. If $x^{**} = x$, for all $x \in H$, then the bounded semihoop H is said to have the Double Negation Property, or (DNP) for short. We define a relation $\perp$ on H by $x \perp y$ iff $y^{**} \leq x^*$. If $x \odot x = x$, that is, $x^2 = x$ for all $x \in H$, then

the semihoop H is said to be idemopent. A semihoop H is called a hoop if $x \odot (x \to y) = y \odot (y \to x)$ for all $x, y \in H$. Also, in every hoop H, $x \wedge y = x \odot (x \to y)$ for all $x, y \in H$, see [14].

Proposition 2.2 ([14, 30]). *In any semihoop $(H, \odot, \to, \wedge, 1)$, the following properties hold: for all $x, y, z \in H$,*
(1) $x \odot y \leq z$ iff $x \leq y \to z$,
(2) $x \odot y \leq x \wedge y$, $x \leq y \to x$,
(3) $1 \to x = x$, $x \to 1 = 1$,
(4) $x \odot (x \to y) \leq y$,
(5) If $x \leq y$, then $y \to z \leq x \to z$, $z \to x \leq z \to y$ and $x \odot z \leq y \odot z$,
(6) $x \leq (x \to y) \to y$,
(7) $((x \to y) \to y) \to y = x \to y$,
(8) $x \to (y \to z) = y \to (x \to z)$,
(9) $x \to y \leq (z \to x) \to (z \to y)$, $x \to y \leq (y \to z) \to (x \to z)$,
(10) $x \to (x \wedge y) = x \to y$,
(11) $x \odot y = x \odot (x \to x \odot y)$.

Proposition 2.3 ([4, 30]). *In a bounded semihoop $(H, \odot, \to, \wedge, 0, 1)$, the following properties hold: for all $x, y, z \in H$,*
(1) $1^* = 0$, $0^* = 1$,
(2) $x \leq x^{**}$, where $x^{**} = (x^*)^*$,
(3) $x \odot x^* = 0$, $x^{***} = x^*$,
(4) $x \leq y$ implies $y^* \leq x^*$,
(5) $x \to y \leq y^* \to x^*$,
(6) $(x \to y^{**})^{**} = x \to y^{**}$,
(7) $x^{**} \odot y^{**} \leq (x \odot y)^{**}$,
(8) $(x^{**} \odot y)^* = (x \odot y)^*$.

Proposition 2.4 ([14]). *Let $(H, \odot, \to, \wedge, 1)$ be a semihoop and for all $x, y \in H$, we define $x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Then the following conditions are equivalent:*
(1) $\sqcup$ is an associative operation on H,
(2) $x \leq y$ implies $x \sqcup z \leq y \sqcup z$ for all $x, y, z \in H$,
(3) $x \sqcup (y \wedge z) \leq (x \sqcup y) \wedge (x \sqcup z)$ for all $x, y, z \in H$,
(4) $\sqcup$ is the join operation on H.

Definition 2.5 ([14]). *A semihoop is called a $\sqcup$-semihoop if it satisfies one of the equivalent conditions of Proposition 2.4.*

Proposition 2.6 ([30]). *In a $\sqcup$-semihoop, the following properties hold: for all $x, y, z \in H$,*
(1) $x \odot (y \sqcup z) = (x \odot y) \sqcup (x \odot z)$,
(2) $x \sqcup (y \odot z) \geq (x \sqcup y) \odot (x \sqcup z)$,
(3) $x \sqcup y^n \geq (x \sqcup y)^n$ and $x^m \sqcup y^n \geq (x \sqcup y)^{mn}$ for any natural numbers m, n.

Proof. The proofs are easy, and we hence omit the details. □

Definition 2.7 ([1, 2]). *Let $(H, \to, \odot, 1)$ be a hoop. H is called:*
(1) a basic hoop if $(x \to y) \to z \leq ((y \to x) \to z) \to z$ for any $x, y, z \in H$.
(2) a Wajsberg hoop $(x \to y) \to y = (y \to x) \to x$ for any $x, y \in H$.
(3) a Gödel hoop if $x \odot x = x$ for any $x \in H$.

Proposition 2.8 ([3]). *Let $(H, \odot, \to, 1)$ be a bounded hoop. Then*
(1) bounded basic hoops are definitionally equivalent to BL-algebras.
(2) bounded Wajsberg hoops are definitionally equivalent to MV-algebras.

Let $(H, \odot, \to, \wedge, 1)$ be a semihoop. A nonempty set F of H is called a filter of H if it satisfies: (1) $x, y \in F$ implies $x \odot y \in F$; (2) $x \in F$, $y \in H$ and $x \leq y$ imply $y \in F$. A filter F of H is called a proper filter if $F \neq H$. A proper filter F of H is called a maximal filter if it is not contained in any proper filter of H. A nonempty set F of H is a filter of H if and only if $1 \in F$ and if $x, x \to y \in F$, then $y \in F$. A proper filter F of a semihoop H is called a prime filter of H, if for any filters F_1, F_2 of H such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$. For more details about filters in semihoops, see [4].

Definition 2.9 ([4, 28]). *Let $(H, \to, \odot, \wedge, 1)$ be a semihoop. H is called:*
(1) a simple semihoop if it has exactly two filters: $\{1\}$ and H.
(2) a local semihoop if it has only one maximal filter.

3 State maps on semihoops

In this section, we introduce the notion of state maps on a semihoop and investigate some related properties of state maps.

Definition 3.1. *Let $(X, \odot_1, \to_1, \wedge_1, 1_1)$ and $(Y, \odot_2, \to_2, \wedge_2, 1_2)$ be two semihoops. A map $\sigma : X \to Y$ is called a state map from X to Y, which is denoted simply by S-map, if it is satisfies the following conditions:*
(SM1) $x \leq_1 y$ implies $\sigma(x) \leq_2 \sigma(y)$;
(SM2) $\sigma(x \to_1 y) = \sigma((x \to_1 y) \to_1 y) \to_2 \sigma(y)$;
(SM3) $\sigma(x \odot_1 y) = \sigma(x) \odot_2 \sigma(x \to_1 (x \odot_1 y))$;
(SM4) $\sigma(x) \odot_2 \sigma(y) \in \sigma(X)$;
(SM5) $\sigma(x) \wedge_2 \sigma(y) \in \sigma(X)$;
(SM6) $\sigma(x) \to_2 \sigma(y) \in \sigma(X)$.
for all $x, y \in X$.

The pair (X, Y, σ) is said to be a S-map semihoop. Moreover, if $X = Y$ and $\sigma^2 = \sigma$, then σ is called an internal state map on X, simply IS-map on X, in this case, (H, σ) is said to be an IS-map semihoop.

Now, we present some examples for S-maps on semihoops.

Example 3.2. *Let H_1 and H_2 be two semihoops. Then the map 1_{H_1}, defined by $1_{H_1}(x) = 1_2$ for all $x \in H_1$, is a S-map from H_1 to H_2.*

Example 3.3. *Let H be a semihoop. One can check that id_H is a S-map on H.*

Example 3.4. *Let $H_1 = \{0_1, a_1, b_1, c_1, 1_1\}$ and $H_2 = \{0_2, a_2, b_2, c_2, 1_2\}$, where $0_1 \leq a_1 \leq b_1, c_1 \leq 1_1$ and $0_2 \leq a_2 \leq b_2 \leq c_2 \leq 1_2$. Define operations $\odot_i$ and $\to_i$ for $i = 1, 2$ as follows:*

$\to_1$	0_1	a_1	b_1	c_1	1_1
0_1	1_1	1_1	1_1	1_1	1_1
a_1	0_1	1_1	1_1	1_1	1_1
b_1	0_1	c_1	1_1	c_1	1_1
c_1	0_1	b_1	b_1	1_1	1_1
1_1	0_1	a_1	b_1	c_1	1_1

$\odot_1$	0_1	a_1	b_1	c_1	1_1
0_1	0_1	0_1	0_1	0_1	0_1
a_1	0_1	a_1	a_1	a_1	a_1
b_1	0_1	a_1	b_1	a_1	1_1
c_1	0_1	a_1	a_1	c_1	1_1
1_1	0_1	a_1	b_1	c_1	1_1

$\to_2$	0_2	a_2	b_2	c_2	1_2
0_2	1_2	1_2	1_2	1_2	1_2
a_2	0_2	1_2	1_2	1_2	1_2
b_2	0_2	c_2	1_2	1_2	1_2
c_2	0_2	b_2	b_2	1_2	1_2
1_2	0_2	a_2	b_2	c_2	1_2

$\odot_2$	0_2	a_2	b_2	c_2	1_2
0_2	0_2	0_2	0_2	0_2	0_2
a_2	0_2	a_2	a_2	a_2	a_2
b_2	0_2	a_2	a_2	a_2	b_2
c_2	0_2	a_2	a_2	c_2	c_2
1_2	0_2	a_2	b_2	c_2	1_2

Then $(H_1, \to_1, \odot_1, \wedge_1, 1_1)$ *and* $(H_2, \to_2, \odot_2, \wedge_2, 1_2)$ *are semihoops. Now, we define a map* $\sigma : H_1 \to H_2$ *as follows:*

$$\sigma(x) = \begin{cases} 0_2, & x = 0_1 \\ a_2, & x = a_1, b_1 \\ 1_2, & x = c_1, 1_1 \end{cases}.$$

One can check that σ *is a S-map from* H_1 *to* H_2.

Example 3.5. *Let* $H = [0, 1]$ *be the real interval. If for* $x, y \in H$, *we define* $x \odot y = max\{0, x + y - 1\}$ *and* $x \to y = min\{1, 1 - x + y\}$, *then* $(H, \odot, \to, 0, 1)$ *becomes a hoop, and hence it is a semihoop. Now we define* $\sigma : H_1 \to H$ *as follows:*

$$\sigma(x) = \begin{cases} 0, & x = 0_1; \\ \frac{1}{2}, & x = a_1, b_1; \\ 1, & x = c_1, 1_1 \end{cases}$$

where H_1 *is given in Example 3.4. One can easily check that* σ *is a S-map from* H_1 *to* H.

Next, we present some properties of S-maps on semihoops.

Proposition 3.6. *Let* H_i, $i = 1, 2$ *be semihoops and* σ *be a S-map from* H_1 *to* H_2. *Then we have: for any* $x, y \in H_1$,
(1) $\sigma(1_1) = 1_2$;
(2) $\sigma(x \odot_1 y) \geq \sigma(x) \odot_2 \sigma(y)$;
(3) $\sigma(x \to_1 y) \leq_2 \sigma(x) \to_2 \sigma(y)$ *and if* $x \leq_1 y$, *then* $\sigma(x \to_1 y) = \sigma(x) \to_2 \sigma(y)$;
(4) $\sigma(H_1)$ *is a subalgebra of* H_2.

Proof. (1) Applying (SM2), we have $\sigma(1_1) = \sigma(0_1 \to_1 0_1) = \sigma((0_1 \to_1 0_1) \to_1 0_1) \to_2 \sigma(0_1) = \sigma(1_1 \to_1 0_1) \to_2 \sigma(0_1) = \sigma(0_1) \to_2 \sigma(0_1) = 1_2$.
(2) From $x \odot_1 y \leq x \odot_1 y$, we get $y \leq_1 x \to_1 (x \odot_1 y)$ by Proposition 2.2(1). By (SM1), we have $\sigma(y) \leq_2 \sigma(x \to_1 (x \odot_1 y))$. Applying (SM3), we get $\sigma(x \odot_1 y) = \sigma(x) \odot_2 \sigma(x \to_1 (x \odot_1 y)) \geq_2 \sigma(x) \odot_2 \sigma(y)$.
(3) By (SM2), we deduce $\sigma(x \to_1 y) = \sigma((x \to_1 y) \to_1 y) \to_2 \sigma(y) \leq_2 \sigma(x) \to_2 \sigma(y)$ by (5) and (6) of Proposition 2.2. If $x \leq_1 y$, then $\sigma(x) \leq_2 \sigma(y)$. This means $\sigma(x) \to_2 \sigma(y) = 1$. Moreover, $\sigma(x \to_1 y) = \sigma((x \to_1 y) \to_1 y) \to_2 \sigma(y) = \sigma(1_1 \to_1 y) \to_2 \sigma(y) = \sigma(y) \to_2 \sigma(y) = 1_2$. Thus $\sigma(x \to_1 y) = \sigma(x) \to_2 \sigma(y)$.
(4) It follows from (SM4), (SM5), (SM6) and (1). □

Definition 3.7. *Let* H_1 *and* H_2 *be two bounded semihoops. A S-map* σ *from* H_1 *to* H_2 *is called a regular if it satisfies* $\sigma(0_1) = 0_2$.

Note that the S-map σ given in Example 3.2 is not regular and the S-map σ given in Example 3.4 is regular.

In the following we give some characterizations for a S-map becoming regular.

Theorem 3.8. *Let* H_i, $i = 1, 2$ *be two bounded Wajsberg semihoops and* σ *be a S-map from* H_1 *to* H_2. *Then the following are equivalent:*
(1) σ *is regular,*
(2) $\sigma(x^{*_1}) = (\sigma(x))^{*_2}$ *for any* $x, y \in H_1$,
(3) $x \perp_1 y$ *implies* $\sigma(x) \perp_2 \sigma(y)$ *for any* $x, y \in H_1$.

Proof. (1)⇒(2) By (1) and (SM2), we get $\sigma(x^{*_1}) = \sigma(x \to_1 0_1) = \sigma((x \to_1 0_1) \to_1 0_1) \to_2 \sigma(0_1) = \sigma(x) \to_2 0_2 = (\sigma(x))^{*_2}$.

(2)⇒(3) Suppose that $x \perp_1 y$. Then $y^{*_1*_1} \leq_1 x^{*_1}$, it follows that $\sigma(y^{*_1*_1}) \leq_1 \sigma(x^{*_1})$. By (2) we have $(\sigma(y))^{*_2*_2} \leq_2 (\sigma(x))^*$. Hence we have $\sigma(x) \perp_2 \sigma(y)$.
(3)⇒(1) Since $0_1^{*_1*_1} = 1^{*_1}$, we get $1_1 \perp_1 0_1$. By (3) we have $\sigma(1_1) \perp_2 \sigma(0_1)$, and so $\sigma(0_1)^{*_2*_2} \leq_2 \sigma(1_1)^{*_2}$. From Proposition 3.6(1), $\sigma(0_1)^{*_2*_2} \leq_2 \sigma(1_1)^{*_2} = 1_2^{*_2} = 0_2$, and hence $\sigma(0_1)^{*_2*_2} = 0_2$. It follows that $\sigma(0_1)^{*_2*_2*_2} = 1_2$. By Proposition 2.2(7), $\sigma(0_1)^{*_2*_2*_2} = \sigma(0_1)^{*_2} = 1_2$, that is, $\sigma(0_1) \to_2 0_2 = 1_2$. This shows that $\sigma(0_1) \leq_2 0_2$, so $\sigma(0_1) = 0_2$. □

Proposition 3.9. *Let H be a semihoop and σ be an IS-map on H. Then we have: for any $x, y \in H$,*
(1) $\sigma(1) = 1$;
(2) $\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y)$;
(3) $\sigma(x \to y) \leq \sigma(x) \to \sigma(y)$ and if $x \leq y$, then $\sigma(x \to y) = \sigma(x) \to \sigma(y)$;
(4) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$;
(5) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$;
(6) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$;
(7) $\sigma(H) = Fix(\sigma)$, where $Fix(\sigma) = \{x \in H \mid \sigma(x) = x\}$;
(8) $\sigma(H)$ is a subalgebra of H;
(9) $Ker(\sigma)$ is a filter of H, where $Ker(\sigma) = \{x \in H | \sigma(x) = 1\}$.

Proof. (1) It follows from Proposition 3.6(1).
(2) It follows from Proposition 3.6(2).
(3) It follows from Proposition 3.6(3).
(4) From (SM4), we have $\sigma(x) \odot \sigma(y) = \sigma(z)$ for some $z \in H$. Hence $\sigma(\sigma(x) \odot \sigma(y)) = \sigma^2(z) = \sigma(z) = \sigma(x) \odot \sigma(y)$ by the definition of the IS-maps.
(5) It is similar to (4).
(6) It is similar to (1).
(7) Let $\in \sigma(H)$. Then $x = \sigma(z)$ for some $z \in H$. Hence $\sigma(x) = \sigma^2(z) = \sigma(z) = x$. So $x \in Fix(\sigma)$. Conversely assume $x \in Fix(\sigma)$. Then $x = \sigma(x) \in \sigma(H)$. This shows that (7) is true.
(8) It follows from (1), (2), (3) and (4).
(9) It is straightforward. □

Next, we consider properties of IS-map to characterize two kinds of semihoops. The following results and the next one are proved in [30], where (SM2) replace by (SM2')$\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$. We can show the same results without the identity (SM2').

Theorem 3.10. *Let H be a semihoop. Then the following are equivalent:*
(1) H is a hoop;
(2) every IS-map σ on H satisfies $\sigma(x) \odot \sigma(x \to y) = \sigma(y) \odot \sigma(y \to x)$ for all $x, y \in H$.

Proof. The proof is similar to that of He et al [30].(Theorem 4.7). □

Theorem 3.11. *Let H be a semihoop. Then the following are equivalent:*
(1) H is idemopent;
(2) every IS-map σ on H satisfies $\sigma(x \wedge y) = \sigma(y) \odot \sigma(y) = \sigma(x) \odot \sigma(x \to y)$ for all $x, y \in H$.

Proof. The proof is similar to that of He et al [30].(Theorem 4.8). □

Here, we give relations between IS-map and Riečan states on semihoops.

Definition 3.12 ([30]). *Let H be a bounded semihoop. A Riečan state on H is a founction $s : H \longrightarrow [0, 1]$ such that the following conditions hold: for all $x, y \in H$,*
(1) $s(1) = 1$,
(2) if $x \perp y$, then $s(x + y) = s(x) + s(y)$.

Let H be a semihoop, σ be an IS-map on H and s be a Riečan state on H. Then s is called σ-compatible if $\sigma(x) = \sigma(y) \Rightarrow s(x) = s(y)$ for all $x, y \in H$.

We denote by $RS[H]$ and $RS_\sigma[H]$ the set of all Riečan states and σ-compatible Riečan states on H, respectively.

Theorem 3.13. *Let H be a semihoop and σ be an IS-map on H. Then there is a one-to-one correspondence between σ-compatible Riečan states on H and Riečan states on $\sigma(H)$.*

Proof. (1) Suppose that s is a Riečan state on $\sigma(H)$. Define a mapping $\varphi : RS[\sigma(H)] \to RS_\sigma[H]$ as follows: $\varphi(s)(x) := s(\sigma(x))$ for all $x \in H$. We will prove that $\varphi(s)$ is a Riečan state on H. Clearly, $\varphi(s)(1) = s(\sigma(1)) = s(1) = 1$. Next, we will show that $\varphi(s)(x + y) = \varphi(s)(x) + \varphi(s)(y)$ when $x \perp y$. In order to do this, we prove that $\sigma(x + y) = \sigma(x) + \sigma(y)$ for $x \perp y$. Now, suppose that $x \perp y$. From Theorem 3.8(3), we have $\sigma(x) \perp \sigma(y)$. Then $\sigma(x) + \sigma(y) = (\sigma(x))^* \to (\sigma(y))^{**}$. Moreover, $\sigma(x + y) = \sigma(x^* y^{**}) = \sigma(x^*) \to \sigma(x^* \wedge y^{**})$. Since $x \perp y$, then $y^{**} \leq x^*$. It follows that $\sigma(x + y) = \sigma(x^*) \to \sigma(y^{**}) = (\sigma(x))^* \to (\sigma(y))^{**} = \sigma(x) + \sigma(y)$. Now, we prove that $\varphi(s)(x + y) = \varphi(s)(x) + \varphi(s)(y)$ when $x \perp y$. Since $\sigma(x + y) = \sigma(x) + \sigma(y)$ for $x \perp y$, we have that $\varphi(s)(x + y) = s(\sigma(x + y)) = s(\sigma(x) + \sigma(y)) = s(\sigma(x)) + s(\sigma(y)) = \varphi(s)(x) + \varphi(s)(y)$. Therefore, $\varphi(s)$ is a Riečan state on H. Moreover, let $\sigma(x) = \sigma(y)$ for all $x, y \in H$, then $\varphi(s)(x) = s(\sigma(x)) = s(\sigma(y)) = \varphi(s)(y)$. Thus, $\varphi(s)$ is a σ-compatible state on H. Therefore, the mapping φ is well defined.
(2) Assume that s is a σ-compatible Riečan state on H. The mapping $\psi : RS_\sigma[H] \to RS[\sigma(H)]$ is defined by $\psi(s)(\sigma(x)) := s(x)$ for all $x \in H$. Let $\sigma(x) = \sigma(y)$, then $s(x) = s(y)$ for all $x, y \in H$. Now, we show that $\psi(s)$ is a Riečan state on $\sigma(H)$. Let $\sigma(x) \perp \sigma(y)$. Then $\sigma(\sigma(x) + \sigma(y)) = \sigma((\sigma(x))^* \to (\sigma(y))^{**}) = \sigma(\sigma(x^*) \to \sigma(y^{**})) = \sigma(x)^* \to \sigma(y)^{**} = \sigma(x) + \sigma(y)$. Based on this, we have that $\psi(s)(\sigma(x) + \sigma(y)) = \psi(s)(\sigma(\sigma(x) + \sigma(y))) = s(\sigma(x) + \sigma(y)) = s(\sigma(x)) + s(\sigma(y)) = \psi(s)(\sigma(\sigma(x))) + \psi(s)(\sigma(\sigma(y))) = \psi(s)(\sigma(x)) + \psi(s)(\sigma(y))$. Moreover, $\psi(s)(\sigma(1)) = s(1) = 1$. That means that $\psi(s)$ is a Riečan state on $\sigma(H)$. Therefore, ψ is a mapping of $RS_\sigma[H]$ into $RS[\sigma(H)]$.
(3) Let s_1, s_2 be σ-compatible states on H and $\psi(s_1) = \psi(s_2)$. Then we have $\psi(s_1)(\sigma(x)) = \psi(s_2)(\sigma(x))$, which implies $s_1(x) = s_2(x)$ for all $x \in H$. Thus, $s_1 = s_2$. Now, suppose that s is a Riečan state on $\sigma(H)$, then we have that $(\psi(\varphi(s))(\sigma(x)) = \varphi(s)(x) = s(\sigma(x))$. Therefore, ψ is a bijective mapping from $RS_\sigma[H]$ onto $RS[\sigma(H)]$ and $\psi^{-1} = \varphi$. □

4 Relations between state maps on semihoops and states on other algebras

Definition 4.1 ([10]). *A Bosbach state on a bounded pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 0, 1)$ is a function $s : A \to [0, 1]$ such that the following conditions hold: for any $x, y \in A$:*
(B1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;
(B2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;
(B3) $s(0) = 0$ and $s(1) = 1$.

Proposition 4.2 ([10]). *Let A be a bounded pseudo-hoop and s be a Bosbach state on A. Then for all $x, y \in A$ the following properties hold:*
(1) $y \leq x$ implies $s(y) \leq s(x)$ and $s(x \to y) = s(x \rightsquigarrow y) = 1 - s(x) + s(y)$;
(2) $s(x^-) = s(x^\sim) = 1 - s(x)$, where $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$.

Definition 4.3. *A state-morphism map on a bounded hoop A is a function $m : A \to [0, 1]$ such that:*
(SM1) $m(0) = 0$;
(SM2) $m(x \to y) = min\{1, 1 - m(x) + m(y)\}$.

Proposition 4.4. *Every state-morphism map on a bounded hoop A is a Bosbach state on A.*

Proof. Let m be a state-morphism map on A. Then $m(1) = m(0^-) = m(0 \to 0) = min\{1, 1-m(0)+m(0) = 1\}$. (B3) holds. Consider $m(x) + m(x \to y) = m(x) + min\{1, 1 - m(x) + m(y) = min\{1 + m(x), 1 + m(y)\} = m(y) + m(y \to x)$. Hence (B1) is true. Since A is a hoop, (B2) is true, too. Combining the above arguments we get that m is a Bosbach state on A. □

Proposition 4.5. *Every state-morphism on a bounded hoop A is a state map from A to the hoop $H = ([0, 1], \odot, \to, 0, 1)$ given in Example 3.5.*

Proof. Assume m is a state-morphism map on a bounded hoop A. By Propositions 4.2 and 4.4, (SM1) holds.

Now we check (SM2). Let $x, y \in A$. By definition of 4.3, we have

$m((x \to y) \to y) \to m(y)$
$= min\{1, 1 - m((x \to y) \to y) + m(y))\}$
$= min\{1, 1 - (1 - m(x \to y) + m(y)) + m(y))\}$
$= m(x \to y)$.

For (SM3), we have

$m(x) \odot m(x \to x \odot y)$
$= 0 \vee (m(x) + m(x \to x \odot y) - 1)$
$= 0 \vee (m(x) + (1 - m(x) + m(x \odot y)) - 1)$
$= 0 \vee m(x \odot y)$
$= m(x \odot y)$.

For (SM4), we have $m(x) \odot m(y) = max\{0, m(x) + m(y) - 1\} = 1 - m(y) + m(x) = min\{1, 1 - m(y) + m(x)\} = m(y \to x)$ and hence $m(x) \odot m(y) \in m(A)$. This shows that (SM4) holds.

Note that $m(x) \to m(y) = min\{1, 1 - m(x) + m(y)\} = m(x \to y)$. It follows that $m(x) \to m(y) \in m(A)$, that is (SM6).

For (SM5), we have $m(x) \wedge m(y) = m(x) \odot (m(x) \to m(y))$. From (SM4) and (SM6), we get that (SM5) holds. □

Definition 4.6 ([30]). *A state semihoop is a pair (H, σ) where H is a bounded semihoop and $\sigma : H \to H$ is a mapping, called state operator, such that for any $x, y \in H$ the following conditions are satisfied:*
(1) $\sigma(0) = 0$;
(2) $x \leq y$ implies $\sigma(x) \leq \sigma(y)$;
(3) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$;
(4) $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \to x \odot y)$;
(5) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$;
(6) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

Theorem 4.7. *Let H be a bounded semihoop and $\sigma : H \to H$ be a mapping on H preserving $\to$. Then the following conditions are equivalent:*
(1) (H, σ) is an IS-map semihoop;
(2) (H, σ) is a state semihoop.

Proof. $(1) \Rightarrow (2)$ If H is a bounded semihoop and $\sigma : H \to H$ is a mapping on H preserving $\to$. Then $\sigma(x \to y) = \sigma(x \to x \wedge y) = \sigma(x) \to \sigma(x \wedge y)$. From proposition 3.9 and definition 4.6, we can obtain that (H, σ) a state semihoop.

$(2) \Rightarrow (1)$ Let (H, σ) be a state semihoop and σ preserving $\to$. We only need to prove that (SM2) holds. Since $((x \to y) \to y) \to y = x \to y$, so we have $\sigma((x \to y) \to y) \to \sigma(y) = \sigma(((x \to y) \to y) \to y) = \sigma(x \to y)$. Thus σ is an IS-map on H and hence (H, σ) is an IS-map semihoop. □

Inspired by Ciungu's state BL-algebras [11], He and Xin enlarged the language of residuated lattice by introducing a new operator, an internal state on residuated lattice in [18].

Definition 4.8 ([18]). *A state residuated lattice is a pair (A, σ) where A is a residuated lattice and $\sigma : A \to A$ is a mapping, called state operator, such that for any $x, y \in A$ the following conditions are satisfied:*
(1) $\sigma(0) = 0$;
(2) $x \to y = 1$ *implies* $\sigma(x) \to \sigma(y) = 1$;
(3) $\sigma(x \to y) = \sigma(x) \to \sigma(x \wedge y)$;
(4) $\sigma(x \odot y) = \sigma(x) \odot \sigma(x \to x \odot y)$;
(5) $\sigma(\sigma(x) \odot \sigma(y)) = \sigma(x) \odot \sigma(y)$;
(6) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$;
(7) $\sigma(\sigma(x) \vee \sigma(y)) = \sigma(x) \vee \sigma(y)$;
(8) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

Let $(H; \odot, \to, 0, 1)$ be a bounded $\sqcup$-semihoop. For any $x, y \in H$, we set $x \sqcup y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. Then $(H, \wedge, \sqcup, \to, \odot, 0, 1)$ is a residuated lattice. (see [2, 3])

Theorem 4.9. *Let H be a bounded $\sqcup$-semihoop and $\sigma : H \to H$ be a mapping on H preserving $\to$. Then the following conditions are equivalent:*
(1) σ is an IS-map on H;
(2) (H, σ) is a state residuated lattice.

Proof. $(1) \Rightarrow (2)$ If H is a bounded $\sqcup$-semihoop and $\sigma : H \to H$ is a mapping on H preserving $\to$. Then $\sigma(x \to y) = \sigma(x \to x \wedge y) = \sigma(x) \to \sigma(x \wedge y)$. Moreover, by Proposition 3.9(5),(6), we have $\sigma(\sigma(x) \sqcup \sigma(y)) = \sigma(((\sigma(x) \to \sigma(y)) \to \sigma(y)) \wedge ((\sigma(y) \to \sigma(x)) \to \sigma(x))) = \sigma(x) \sqcup \sigma(y)$. Therefore, (H, σ) is a state residuated lattice.

$(2) \Rightarrow (1)$ Let (H, σ) be a state residuated lattice and σ preserving $\to$. We only need to prove that (SM2) holds. Since $((x \to y) \to y) \to y = x \to y$, so we have $\sigma((x \to y) \to y) \to \sigma(y) = \sigma(((x \to y) \to y) \to y) = \sigma(x \to y)$. Thus σ is an IS-map on H. □

A state operator σ on a BL-algebra L was introduced in Ciungu et al. (2011) as a mapping $\sigma : L \to L$ satisfying conditions (1) and (3)–(6) in Definition 4.8. We know that BL-algebras are special cases of residuated lattices satisfying the conditions of divisibility and prelinearity. Consequently, a BL-algebra satisfies the property: $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ for any $x, y \in L$. Therefore, in the case of BL-algebras, condition (4) implies the validity of (2) and conditions (5) and (6) imply the validity of (7) and (8). Hence the notion of a state residuated lattice essentially generalizes that of a state BL-algebra. Moreover, it has been proved (Ciungu et al. 2011) that a mapping $\sigma : L \to L$ is a state operator on an MV-algebra L (Flaminio and Montagna 2007, 2009) if and only if it is a state operator on L taken as a BL-algebra. From this point of view, the notion of a state residuated lattice also generalizes that of a state MV-algebra. Based on this, we have the following results [18].

Corollary 4.10. *Let H be a bounded basic hoop and $\sigma : H \to H$ be a mapping on H preserving $\to$. Then the following conditions are equivalent:*
(1) σ is an IS-map on H;
(2) (H, σ) is a state BL-algebra.

Proof. It follows from Proposition 2.8(1) and Theorem 4.9. □

Corollary 4.11. *Let H be a bounded Wajsberg hoop and $\sigma : H \to H$ be a mapping on H. Then the following conditions are equivalent:*
(1) σ is an IS-map on H preserving $\to$;
(2) (H, σ) is a state MV-algebra.

Proof. It follows from Proposition 2.8(2) and Theorem 4.9. □

As we know, every hoop H is a BCK-meet semilattice in which a partial order over H can be defined as usual.

Definition 4.12 ([5]). *A state BCK-meet semilattice is a pair (A, σ) where A is a BCK-meet semilattices and $\sigma : A \to A$ is a mapping, called state operator, such that for any $x, y \in A$ the following conditions are satisfied:*
(1) $x \to y = 1$ implies $\sigma(x) \to \sigma(y) = 1$;
(2) $\sigma(x \to y) = \sigma(x \to y) \to y) \to \sigma(y)$;
(3) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$;
(4) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

Proposition 4.13. *Let H be a hoop and $\sigma : H \to H$ be an IS-map on H. Then the $\{\to, \wedge\}$ subreduct of (H, σ) is a state BCK-meet semilattice.*

Proof. It follows from Definition 3.1 and Definition 4.12. □

Since the class of equality algebra and the class of BCK-$\wedge$-semilattice with meet are categorically equivalent, then we have the following result.

Definition 4.14 ([27]). *A state equality algebra is a pair (A, σ) where A is an equality algebra and $\sigma : A \to A$ is a mapping, called state operator, such that for any $x, y \in A$ the following conditions are satisfied:*
(1) $x \leq y$ implies $\sigma(x) \leq \sigma(y)$;
(2) $\sigma(x \sim x \wedge y) = \sigma(x \sim x \wedge y) \sim y) \sim \sigma(y)$;
(3) $\sigma(\sigma(x) \to \sigma(y)) = \sigma(x) \to \sigma(y)$;
(4) $\sigma(\sigma(x) \wedge \sigma(y)) = \sigma(x) \wedge \sigma(y)$.

Proposition 4.15. *Let H be a hoop and $\sigma : H \to H$ be an IS-map on H. Then the $\{\sim, \wedge\}$ subreduction of (H, σ) is a state equality algebra, where $x \sim y = x \to (x \wedge y)$.*

Proof. It follows from Definition 4.14. □

5 State map filters in semihoops

In this section, we introduce state map filters of semihoops.

Definition 5.1. *Let H_1 and H_2 be semihoops, $\sigma : H_1 \to H_2$ be a S-map from H_1 to H_2, F be a filter of H_1. If $\sigma^{-1}(\sigma(F)) \subseteq F$, we call F to be a SM-filter of (H_1, H_2, σ).*

Example 5.2. *Consider the Example 3.4, one can easily check that the SM-filter of (H_1, H_2, σ) are $\{a_1, b_1, c_1, 1\}$,$\{1_1\}$ and H_1.*

Example 5.3. *Let H_1 and H_2 be semihoops and σ be a S-map from H_1 to H_2. Then $Ker(\sigma) = \{x \in H_1 | \sigma(x) = 1_2\}$ is a SM-filter of (H_1, H_2, σ).*

Proof. Let $K = Ker(\sigma)$ and $x, y \in K$. Then $\sigma(x) = 1_2$ and $\sigma(y) = 1_2$. By Proposition 3.6(2) we have $\sigma(x \odot_1 y) \geq_2 \sigma(x) \odot_2 \sigma(y) = 1_2 \odot_2 1_2 = 1_2$. This means $x \odot_1 y \in K$. Let $x \in K$ and $x \leq y$. Then $1_2 = \sigma(x) \leq \sigma(y)$ and hence $\sigma(y) = 1_2$. This shows that $y \in K$. It follows that K is a filter of H_1. Moreover let $x \in \sigma^{-1}\sigma(K)$. Then $\sigma(x) \in \sigma(K) = \{1_2\}$ and hence $\sigma(x) = 1_2$. Therefore $x \in K$. This shows that $\sigma^{-1}\sigma(K) \subseteq K$, or K is a SM-filter of (H_1, σ). □

Definition 5.4. *Let H be a semihoop and σ be an IS-map on H.*
(1) A filter F of H is called state filter of (H, σ) if $x \in F$ implies $\sigma(x) \in F$ for all $x \in H$ [31],
(2) A filter F of H is called dual state filter of (H, σ) if $\sigma(x) \in F$ implies $x \in F$ for all $x \in H$,
(3) A filter F of H is called strong state filter of (H, σ) if it is both a state filter and a dual state filter of (H, σ).

Proposition 5.5. *Let H be a semihoop and σ be an IS-map on H. Then each SM-filter of H is a state filter on H.*

Proof. Let $x \in F$. Then $\sigma(x) \in \sigma(F)$. Therefore, $\sigma(\sigma(x)) \in \sigma(F)$, that is $\sigma(x) \in \sigma^{-1}(\sigma(F)) \subseteq F$. So $\sigma(x) \in F$. □

However, the converse of Proposition 5.5 is not true in general.

Example 5.6. *Let $H = \{0, a, b, 1\}$ with $0 \leq a, b \leq 1$. Consider the operation $\to$ and $\odot$ as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then H is semihoop. Now, we define σ follows: $\sigma 0 = a$, $\sigma a = a$, $\sigma b = 1$, $\sigma 1 = 1$. One can easily check that σ is an IS-map on H. It is clear that $\{a, 1\}$ is a state filter of (H, σ), but it is not a SM-filter of (H, σ).

Proposition 5.7. *Let H be a semihoop, σ be an IS-map on H and $F \subseteq H$. Then the following are equivalent:*
(1) F is a SM-filter of H,
(2) F is a strong state filter on H.

Proof. (1)⇒(2) Let F be a SM-filter of (H, σ). By Proposition 5.5 we only need to prove that $\sigma(x) \in F$ implies $x \in F$. Let $\sigma(x) \in F$. Then $\sigma(x) = \sigma(\sigma(x)) \in \sigma(F)$. Hence there is $t \in F$ such that $\sigma(x) = \sigma(t)$. It follows from (1) that $x \in \sigma^{-1}(\sigma(t)) \subseteq \sigma^{-1}(\sigma(F)) \subseteq F$. That is $x \in F$.
(2)⇒(1) Assume that F is a strong state filter on H. For $x \in \sigma^{-1}(\sigma(F))$, we have $\sigma(x) \in \sigma(F)$. Since F is strong filter of H, we get $x \in F$ and hence $\sigma^{-1}(\sigma(F)) \subseteq F$. □

Let H_1 and H_2 be two semihoops and σ be a S-map from H_1 to H_2. For any nonempty set X of H_1, we denote by $\langle X \rangle_\sigma$ the SM-filter of (H_1, σ) generated by X, that is, $\langle X \rangle_\sigma$ is the smallest SM-filter of (H_1, σ) containing X.

Let H be be a semihoop and σ be an IS-map on H. For any nonempty set X of H, we denote by $\langle X \rangle_S$ ($\langle X \rangle_{DS}$) the state filter (the dual state filter) of (H, σ) generated by X, that is, $\langle X \rangle_S$ ($\langle X \rangle_{DS}$) is the smallest state filter (the dual state filter) of (H, σ) containing X.

Denote $(X)_{DS} = \{x \in H \mid \sigma(x) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_m \odot \sigma(x_m), x_i \in X\}$. In the following we discuss the structures of $\langle X \rangle_S$, $\langle X \rangle_{DS}$ and $\langle X \rangle_\sigma$.

Theorem 5.8. *Let H be a semihoop, σ be an IS-map on H and $X \subseteq H$. Then*
(1) $\langle X \rangle_S = \{x \in H \mid x \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n), x_i \in X, m \in N\}$,
(2) $(X)_{DS}$ is a dual state filter of (H, σ) containing X, and hence $\langle X \rangle_{DS} \subseteq (X)_{DS}$,
(3) $\langle X \rangle_\sigma = \langle X \rangle_S \cup (X)_{DS}$.

Proof. (1) The proof is similar to that of He et al [30].(Theorem 4.13).

(2) Let $x, y \in (X)_{DS}$. Then $\sigma(x) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for some $x_i \in X, n \in N$ and $\sigma(y) \geq y_1 \odot \sigma(y_1) \odot \cdots \odot y_m \odot \sigma(y_m)$ for some $y_j \in X, m \in N$. Hence $\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n) \odot y_1 \odot \sigma(y_1) \odot \cdots \odot y_m \odot \sigma(y_m)$. So $x \odot y \in (X)_{DS}$. Assume $x \leq y$ and $x \in (X)_{DS}$. Then $\sigma(y) \geq \sigma(x) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for some $x_i \in X$. It follows that $y \in (X)_{DS}$. This shows that $(X)_{DS}$ is a filter of H. Moreover, let $\sigma(x) \in (X)_{DS}$. Then $\sigma(\sigma(x)) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for some $x_i \in X$ and hence $\sigma(x) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for some $x_i \in X$. This shows that $x \in (X)_{DS}$ and hence $(X)_{DS}$ is a dual state filter of (H, σ). Clearly $X \subseteq (X)_{DS}$.

(3) Denote $B = \langle X \rangle_S \cup (X)_{DS}$. Let $x, y \in B$. If $x, y \in \langle X \rangle_S$, then $x \odot y \in \langle X \rangle_S \subseteq B$ by (1). If $x, y \in (X)_{DS}$, then $x \odot y \in (X)_{DS} \subseteq B$ by (2). Let $x \in \langle X \rangle_S$ and $y \in (X)_{DS}$. Then $\sigma(x) \in \langle X \rangle_S$ since $\langle X \rangle_S$ is a state filter of (H, σ) by (1). Hence $\sigma(x) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for some $x_i \in X$ and $\sigma(y) \geq y_1 \odot \sigma(y_1) \odot \cdots \odot y_m \odot \sigma(y_m)$ for some $y_j \in X$ and hence $\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n) \odot y_1 \odot \sigma(y_1) \odot \cdots \odot y_m \odot \sigma(y_m)$ for some $x_i, y_j \in X$. It follows that $x \odot y \in (X)_{DS} \subseteq B$. Combining the above arguments we get that B is closed on $\odot$. It is easy to

check that if $x \in B$ and $x \leq y$ then $y \in B$. Clearly $X \subseteq B$. Now we prove that B is a state filter. Let $x \in B$. If $x \in \langle X\rangle_S$, then $\sigma(x) \in \langle X\rangle_S$ since $\langle X\rangle_S$ is a state filter. If $x \in (X)_{DS}$, then $\sigma(x) \in \langle X\rangle_S \subseteq B$. So B is a state filter. Moreover we prove that B is a dual state filter. Let $\sigma(x) \in B$. If $\sigma(x) \in \langle X\rangle_S$, then $x \in (X)_{DS} \subseteq B$. Let $\sigma(x) \in (X)_{DS}$. Then $x \in (X)_{DS}$ since $(X)_{DS}$ is a dual state filter by (2). This shows that B a dual state filter. By Proposition 5.7, B a SM-filter. Let F be a SM-filter of (H, σ) containing X and $x \in B$. If $x \in \langle X\rangle_S$, then $x \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for $x_i \in X$. Since $X \subseteq F$ and F is a SM-filter of (H, σ), we have $x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n) \in F$. So $x \in F$. If $x \in (X)_{DS}$, then $\sigma(x) \in \langle X\rangle_S$ by (1). If $\sigma(x) \geq x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n)$ for $x_i \in X$. Since F is a SM-filter of (H, σ) containing X, then $x_1 \odot \sigma(x_1) \odot \cdots \odot x_n \odot \sigma(x_n) \in F$ and hence $\sigma(x) \in F$. Note that F is also a dual state filter, we have $x \in F$. Combining the above arguments we get $B \subseteq F$. It follows that $B = \langle X\rangle_\sigma$. □

Proposition 5.9. *Let H be a semihoop, σ be an IS-map and F be state filters of (H, σ) and $a \notin F$. Then*
(1) $\langle a\rangle_\sigma = \{x \in H \mid x \geq (a \odot \sigma(a))^n, n \geq 1\} \cup \{x \in H \mid \sigma(x) \geq (a \odot \sigma(a))^n, n \geq 1\}$,
(2) $\langle F, \{a\}\rangle_\sigma = \{x \in H \mid x \geq f \odot (a \odot \sigma(a))^n, f \in F, n \geq 1\} \cup \{x \in H \mid \sigma(x) \geq f \odot (a \odot \sigma(a))^n, f \in F, n \geq 1\}$,
(3) if $a \leq b$, then $\langle b\rangle_\sigma \subseteq \langle a\rangle_\sigma$,
(4) $\langle a \odot a\rangle_\sigma = \langle a\rangle_\sigma$,
(5) $\langle \sigma(a)\rangle_\sigma = \langle a\rangle_\sigma$,
(6) $\langle a \odot \sigma(a)\rangle_\sigma = \langle a\rangle_\sigma$,
(7) if H is a $\sqcup$-semihoop, then $\langle a\rangle_\sigma \cap \langle b\rangle_\sigma = \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$.

Proof. The proofs of (1)–(4) are obvious.

(5) Let $x \in \langle \sigma(a)\rangle_\sigma$. Then $x \geq (\sigma(a) \odot \sigma^2(a))^n = (\sigma(a))^{2n} \geq (a \odot \sigma(a))^{2n}$ or $\sigma(x) \geq (a \odot \sigma(a))^{2n}$ and hence $x \in \langle a\rangle_\sigma$. Conversely, let $x \in \langle a\rangle_\sigma$. Then $x \geq (a \odot \sigma(a))^n$ or $\sigma(x) \geq (a \odot \sigma(a))^n$. Hence $\sigma(x) \geq (\sigma(a) \odot \sigma^2(a))^n$. It follows that $\sigma(x) \in \langle \sigma(a)\rangle_\sigma$. Since $\langle \sigma(a)\rangle_\sigma$ is a dual state filter we have $x \in \langle \sigma(a)\rangle_\sigma$.

(6) Since $a \odot \sigma(a) \leq a$ we have $\langle a\rangle_\sigma \subseteq \langle a \odot \sigma(a)\rangle_\sigma$ by (3). Conversely, by use of (3), (4) and (5) we have $\langle a \odot \sigma(a)\rangle_\sigma = \langle \sigma(a \odot \sigma(a))\rangle_\sigma \subseteq \langle \sigma(a) \odot \sigma^2(a)\rangle_\sigma = \langle \sigma(a) \odot \sigma(a)\rangle_\sigma = \langle \sigma(a)\rangle_\sigma = \langle a\rangle_\sigma$.

(7) Suppose that H is a $\sqcup$-semihoop. From $a \odot \sigma(a) \leq (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))$, we have that $\langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma \subseteq \langle a \odot \sigma(a)\rangle_\sigma = \langle a\rangle_\sigma$. Similarly, we can prove $\langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma \subseteq \langle b\rangle_\sigma$. Thus, $\langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma \subseteq \langle a\rangle_\sigma \cap \langle b\rangle_\sigma$. Conversely, let $x \in \langle a\rangle_\sigma \cap \langle b\rangle_\sigma$. Then there exist $n, m, s, t \geq 1$, such that $x \geq (a \odot \sigma(a))^n$ or $\sigma(x) \geq (a \odot \sigma(a))^m$, and $x \geq (b \odot \sigma(b))^s$ or $\sigma(x) \geq (b \odot \sigma(b))^t$. To complete the proof, we divide four cases as following:

(a) Let $x \geq (a \odot \sigma(a))^n$ and $x \geq (b \odot \sigma(b))^s$. Then $x \geq (a \odot \sigma(a))^n \sqcup (b \odot \sigma(b))^s \geq ((a \odot \sigma(a)) \sqcup (b \odot \sigma(b)))^{ns} \geq (((a \odot \sigma(a)) \sqcup (b \odot \sigma(b))) \odot \sigma((a \odot \sigma(a)) \sqcup (b \odot \sigma(b))))^{ns}$ by Proposition 2.6(3). We deduce that $x \in \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$.

(b) Let $\sigma(x) \geq (a \odot \sigma(a))^m$ and $\sigma(x) \geq (b \odot \sigma(b))^t$. Similarly to (a) we can get $\sigma(x) \in \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$. Since $\langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$ is a dual state filter we have $x \in \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$.

(c) Let $x \geq (a \odot \sigma(a))^n$ and $\sigma(x) \geq (b \odot \sigma(b))^t$. Then $\sigma(x) \geq (\sigma(a) \odot \sigma(a))^n \geq (a \odot \sigma(a))^{2n}$ and $\sigma(x) \geq (b \odot \sigma(b))^t$. Thus $\sigma(x) \geq (a \odot \sigma(a))^{2n} \sqcup (b \odot \sigma(b))^t \geq ((a \odot \sigma(a)) \sqcup (b \odot \sigma(b)))^{2nt} \geq (((a \odot \sigma(a)) \sqcup (b \odot \sigma(b))) \odot \sigma((a \odot \sigma(a)) \sqcup (b \odot \sigma(b))))^{2nt}$. It follows that $\sigma(x) \in \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$. Since $\langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$ is a dual state filter we have $x \in \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$.

(d) Let $\sigma(x) \geq (a \odot \sigma(a))^m$ and $x \geq (b \odot \sigma(b))^s$. Similarly to the case (c) we can get $x \in \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$.

Combining the above arguments we can prove $\langle a\rangle_\sigma \cap \langle b\rangle_\sigma \subseteq \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$. Therefore $\langle a\rangle_\sigma \cap \langle b\rangle_\sigma = \langle (a \odot \sigma(a)) \sqcup (b \odot \sigma(b))\rangle_\sigma$. □

Definition 5.10. *Let H_1 and H_2 be two semihoops and σ be a S-map from H_1 to H_2. A proper SM-filter F of (H_1, H_2, σ) is called a prime SM-filter of (H_1, H_2, σ), if for all SM-filters F_1, F_2 of (H_1, σ) such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$.*

Let H_1 and H_2 be two semihoops and σ be a S-map from H_1 to H_2. We denote by PSMF[H] the set of all prime SM-filters of (H_1, σ).

Example 5.11. *Consider the Example 3.4, one can check that $F = \{a_1, b_1, c_1, 1_1\}$ is a prime SM-filter of (H_1, σ).*

Theorem 5.12. *Let H be a $\sqcup$-semihoop, σ be an IS-map and F be a proper SM-filter of (H, σ). Then the following are equivalent:*
(1) F is a prime SM-filter of (H, σ),
(2) if $((x \odot \sigma(x)) \sqcup (y \odot \sigma(y))) \odot \sigma((x \odot \sigma(x)) \sqcup (y \odot \sigma(y))) \in F$ for some $x, y \in H$, then $x \in F$ or $y \in F$.

Proof. (1) $\Rightarrow$ (2) Let $((x \odot \sigma(x)) \sqcup (y \odot \sigma(y))) \odot \sigma((x \odot \sigma(x)) \sqcup (y \odot \sigma(y))) \in F$ for some $x, y \in H$. Then $\langle x\rangle_\sigma \cap \langle y\rangle_\sigma = \langle (x \odot \sigma(x)) \sqcup (y \odot \sigma(y))\rangle_\sigma \subseteq F$. Since F is a prime SM-filter of (H, σ), then $\langle x\rangle_\sigma \subseteq F$ or $\langle y\rangle_\sigma \subseteq F$. Therefore, $x \in F$ or $y \in F$.

(2) $\Rightarrow$ (1) Suppose that $F_1, F_2 \in SMF[L]$ such that $F_1 \cap F_2 \subseteq F$ and $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$. Then there exist $x \in F_1$ and $y \in F_2$ such that $x, y \notin F$. Since F_1, F_2 are SM-filter of (H, σ), then $x \odot \sigma(x) \in F_1$ and $y \odot \sigma(y) \in F_2$. From $x \odot \sigma(x), y \odot \sigma(y) \leq (x \odot \sigma(x)) \sqcup (y \odot \sigma(y))$, we obtain $(x \odot \sigma(x)) \sqcup (y \odot \sigma(y)) \in F_1 \cap F_2 \subseteq F$ and hence $((x \odot \sigma(x)) \sqcup (y \odot \sigma(y))) \odot \sigma((x \odot \sigma(x)) \sqcup (y \odot \sigma(y))) \in F_1 \cap F_2 \subseteq F$. By (2), we get that $x \in F$ or $y \in F$, which is a contradiction. Therefore, F is a prime SM-filter of (H, σ). □

Definition 5.13. *Let H_1 and H_2 be two semihoops and σ be a S-map from H_1 to H_2. A proper SM-filter of (H_1, H_2, σ) is called a maximal SM-filter if it not strictly contained in any proper SM-filter of (H_1, H_2, σ).*

Example 5.14. *Let H_1 and H_2 be two semihoops and σ be a S-map from H_1 to H_2 in Example 3.4. One can easily check that $F = \{a_1, b_1, c_1, 1_1\}$ is a maximal SM-filter of (H_1, H_2, σ).*

Proposition 5.15. *Let H be a bounded $\sqcup$-semihoop, σ be an IS-map and F be a proper SM-filter of (H, σ). Then the following are equivalent:*
(1) F is a maximal SM-filter of (H, σ),
(2) for any $a \notin F$, there is an integer $n \geq 1$ such that $(\sigma(a)^n)^* \in F$.

Proof. (1) $\Rightarrow$ (2) Suppose that F is a maximal SM-filter of (H, σ), and let $a \notin F$. Then $\langle F, a\rangle_\sigma = H$, which implies $0 \in \langle F, a\rangle_\sigma$. Then there is $f \in F$ and an integer $n \geq 1$ such that $0 = f \odot (a \odot \sigma(a))^n$. So we have $0 = \sigma(0) \geq \sigma(f) \odot \sigma(a)^{2n}$. Therefore, $\sigma(f) \geq (\sigma(a)^{2n})^*$. Thus, $(\sigma(a)^{2n})^* \in F$.

(2) $\Rightarrow$ (1) Let a satisfy the condition. Since $(\sigma(a)^n)^* \odot (a \odot \sigma(a))^n \leq (\sigma(a)^n)^* \odot (\sigma(a))^n = 0$ and $(\sigma(a)^n)^* \in F$, we obtain $0 \in \langle F, a\rangle_\sigma$, that is, $\langle F, a\rangle_\sigma = H$. Therefore, F is a maximal SM-filter of (H, σ). □

Proposition 5.16. *Let H_1 and H_2 be two bounded semihoops and σ be a S-map from H_1 to H_2.*
(1) If F_2 is filter of $\sigma(H_1)$, then $\sigma^{-1}(F_2)$ is a SM-filter of (H_1, H_2, σ).
(2) If σ is an IS-map on H and F is a maximal filter of σH, then $\sigma^{-1}(F)$ is a maximal SM-filter of H.

Proof. (1) Suppose that F_2 is a filter of $\sigma(H_1)$. If $x, y \in \sigma^{-1}(F_2)$, then $\sigma(x), \sigma(y) \in F_2$. It follows that $\sigma(x) \odot \sigma(y) \in F_2$. Since $\sigma(x \odot y) \geq \sigma(x) \odot \sigma(y)$ and $\sigma(x \odot y) \in \sigma(H_1)$, we have $\sigma(x \odot y) \in F_2$, that is, $x \odot y \in \sigma^{-1}(F_2)$. Let $x, y \in H_1$ such that $x \in \sigma^{-1}(F_2)$ and $x \leq y$. Then $\sigma(x) \leq \sigma(y)$. Since $\sigma(x) \in F$ and $\sigma(y) \in \sigma(H_1)$, we can obtain that $\sigma(y) \in F_2$, that is, $y \in \sigma^{-1}(F_2)$. Thus, $\sigma^{-1}(F_2)$ is a filter of H_1. Note that $\sigma(\sigma^{-1}(x) = x$ for any $x \in H_1$. Hence $\sigma^{-1}(\sigma(\sigma^{-1}(F)) = \sigma^{-1}(F)$. Thus $\sigma^{-1}(F_2)$ is a SM-filter of H_1.
(2) Now, suppose that F is a maximal filter of $\sigma(H)$. Let $a \notin \sigma^{-1}(F)$, thus $\sigma(a) \notin F$. By the maximality of F, there is an integer $n \geq 1$ such that $(\sigma(a)^n)^* \in F \subseteq \sigma(H)$. Since $\sigma((\sigma(a)^n)^*) = (\sigma(a)^n)^* \in F$, we have $(\sigma(a)^n)^* \in \sigma^{-1}(F)$. Therefore, $\sigma^{-1}(F)$ is a maximal SM-filter of H. □

Proposition 5.17. *Let H be a bounded semihoop and σ be an IS-map on H preserving $\odot$.*
(1) If F is a SM-filter of (H, σ), then $\sigma(F)$ is a SM-filter of $(\sigma(H), \sigma)$.
(2) If F is a maximal SM-filter of (H, σ), then $\sigma(F)$ is a maximal SM-filter of $(\sigma(H), \sigma)$.

Proof. (1) Let $\sigma(x), \sigma(y) \in \sigma(F)$, then $x, y \in \sigma^{-1}\sigma(F) \subseteq F$. Since F is a filter, thus $x \odot y \in F$ and hence $\sigma(x) \odot \sigma(y) = \sigma(x \odot y) \in \sigma(F)$. Let $\sigma(x), \sigma(y) \in \sigma(H)$ such that $\sigma(x) \in \sigma(F)$ and $\sigma(x) \leq \sigma(y)$. Since

$\sigma(x) \in \sigma(F)$ we have $x \in \sigma^{-1}\sigma(F) \subseteq F$. So $x \in F$. By Proposition 5.7 we have $\sigma(x) \in F$. Since $\sigma(x) \leq \sigma(y)$ we get $\sigma(y) \in F$. Using Proposition 5.7 again we obtain $y \in F$, and so $\sigma(y) \in \sigma(F)$. Thus, $\sigma(F)$ is a filter of $\sigma(H)$. Now let $x \in \sigma(F)$. Then $x = \sigma(t)$ for some $t \in F$ and hence $\sigma(x) = \sigma^2(t) = \sigma(t) = x \in \sigma(F)$. It follows that $\sigma(F)$ is a state filter of (H, σ). Let $x \in \sigma(H)$ and $\sigma(x) \in \sigma(F)$. Then $x = \sigma(t)$ for some $t \in H$. Hence $x = \sigma(t) = \sigma^2(t) = \sigma(\sigma(t)) = \sigma(x) \in \sigma(F)$. This means that $\sigma(F)$ is a dual state filter of $(\sigma(H), \sigma)$. Therefore $\sigma(H)$ is a strong state filter of $(\sigma(H), \sigma)$. By Proposition 5.7 we have that $\sigma(F)$ is a SM-filter of $(\sigma(H), \sigma)$.
(2) Now, let F be maximal and $\sigma(a) \notin \sigma(F)$. Then $a \notin F$, and there is an integer $n \geq 1$ such that $(\sigma(a)^n)^* \in F$ and hence $\sigma((\sigma(a)^n)^*) = (\sigma(a)^n)^* \in \sigma(F)$. Since $\sigma(\sigma(a)^n) \geq (\sigma\sigma(a))^n = (\sigma(a))^n$, we have $(\sigma(a)^n)^* \geq \sigma((\sigma(a)^n)^*)$. Hence $(\sigma(a)^n)^* \in \sigma(F)$. Therefore, $\sigma(F)$ is a maximal SM-filter of $(\sigma(H), \sigma)$. □

Corollary 5.18. *Let H be a bounded semihoop and σ be an IS-map on H.*
(1) If F is a (maximal) filter of $\sigma(H)$, then $\sigma^{-1}(F)$ is a strong state (maximal)filter of (H, σ).
(2) If σ is preserving $\odot$ and F is a strong state (maximal) filter of (H, σ), then $\sigma(F)$ is a strong state (maximal) filter of $(\sigma(H), \sigma)$.

Proof. (1) It follows from Proposition 5.7 and 5.16.
(2) It follows from Proposition 5.7 and 5.17. □

Now, we introduce two kinds of semihoops and give some characterizations of them.

Definition 5.19. *Let H be a semihoop and $\sigma : H \to H$ be an IS-map on H. If (H, σ) has exactly one maximal SM-filter , we call (H, σ) to be state local.*

Theorem 5.20. *Let H be a semihoop and σ be an IS-map on H. Then the following are equivalent:*
(1) (H, σ) is state local;
(2) $\sigma(H)$ is local.

Proof. (1) ⇒ (2) Let F be the only maximal SM-filter of (H, σ). We prove that $\sigma(F)$ is the only maximal filter of $\sigma(H)$. First, $\sigma(F)$ is a proper filter of $\sigma(H)$. In fact, if $\sigma(F) = \sigma(H)$, then $0 \in \sigma(F)$, which implies $0 \in F$, a contradiction. Now, let G be a filter of $\sigma(H)$, $G \neq \sigma(H)$ and let $x \in G$. It follows from Corollary 5.18(1) that $\sigma^{-1}(G)$ is a SM-filter of (H, σ). Thus $\sigma^{-1}(G)$ is a proper SM-filter of (H, σ). Moreover, if $\sigma^{-1}(G) = H$, then $0 \in \sigma^{-1}(G)$, so $0 \in G$, a contradiction. It follows that $\sigma^{-1}(G) \subseteq F$. if $x = \sigma(x) \in G$, then $x \in \sigma^{-1}(G)$, it follows that $x \in F$. But $x = \sigma(x)$, so $x \in \sigma(H)$. Thus $G \subseteq \sigma(F)$. Hence $\sigma(G)$ is the only maximal filter of $\sigma(H)$. Therefore, $\sigma(H)$ is local.
(2) ⇒ (1) Suppose that G is the only maximal filter of $\sigma(H)$. By Corollary 5.18(1), we have that $\sigma^{-1}(G)$ is a maximal SM-filter of (H, σ). We will prove that $\sigma^{-1}(G)$ is the only maximal SM-filter of (H, σ). Let G be a SM-filter of (H, σ), $F \neq L$. Then $\sigma(F)$ is a proper filter of $\sigma(H)$, so $\sigma(F) \subseteq G$. Let $x \in F$ then $\sigma(x) \in \sigma(F) \subseteq G$. Thus, $x \in \sigma^{-1}(G)$. It follows that $F \subseteq \sigma^{-1}(G)$. Therefore, (H, σ) is state local. □

Definition 5.21. *Let H a be semihoop and $\sigma : H \to H$ be an IS-map on H. If (H, σ) has two SM-filters $\{1\}$ and H, we call (H, σ) to be simple.*

Theorem 5.22. *Let H a be semihoop and $\sigma : H \to H$ be an IS-map on H such that σ preserving $\odot$. Then the following are equivalent:*
(1) (H, σ) is simple;
(2) $\sigma(H)$ is simple and $Ker(\sigma) = \{1\}$.

Proof. (1) ⇒ (2) Let F be a filter of $\sigma(H)$ and $F \neq \{1\}$. It follows from Corollary 5.18(1) that $\sigma^{-1}F$ is a SM-fiter of (H, σ). Since (H, σ) is state simple, we have that $\sigma^{-1}(F) = \{1\}$ or $\sigma^{-1}(F) = H$. Notice that $F \subseteq \sigma^{-1}F$ (if $x \in F$, then $\sigma x = x$, that is, $x \in \sigma^{-1}F$, we obtain that $\sigma^{-1}F \neq \{1\}$. Thus, $\sigma^{-1}F = H$. Then $0 \in \sigma^{-1}F$, that is, $0 = \sigma 0 \in F$. So we obtain that $F = \sigma H$. Therefore, σH is simple.

By Example 5.3 we have $\mathrm{Ker}(\sigma)$ is a SM-filter of (H, σ) and $\mathrm{Ker}(\sigma) \neq H$. It follows that $\mathrm{Ker}(\sigma) = \{1\}$.

(2) ⇒ (1) Let F be a SM-filter of (H, σ) and $F \neq \{1\}$. By Corollary 5.18(2), we obtain that σF is a filter of σH. Since σH is simple, we obtain that $\sigma F = \{1\}$ or $\sigma F = \sigma x$. Since $\mathrm{Ker}(\sigma) = \{1\}$, we have $F \neq \{1\}$. Thus, $\sigma F = \sigma x$. Then $0 \in \sigma F$, that is, $0 \in F$. It follows that $F = H$. Therefore (H, σ) is state simple. □

6 Conclusion

We observed that the states and state operators on MV-algebras, BL-algebras and BCK-algebras, are maps from an algebra X to $[0, 1]$ and X to X, respectively. From the viewpoint of universal algebras, it is meaningful to study a state map from an algebra X to anther algebra Y. Indeed, if $Y = [0, 1]$, a state can be seen as a state map from X to $[0, 1]$, and if $X = Y$, a state operator can also be seen as a state map from $X \to X$. Based on this idea, we introduce a notion of state maps on semihoops by extending the codomain of a state (or internal state) to a more general algebraic structure, that is, from a semihoop H_1 to an arbitrary semihoop H_2. We give a type of special state map from a semihoop H to H, called internal state map (or IS-map), which is a generalization of internal states (or state operators) on some types of semihoops. We try to give a unified model of states and internal states on some important logic algebras. By the arguments in the paper we can see that state maps on an semihoops are generalization of internal states on BL-algebras, MV-algebras, equality algebras and BCK-algebras. In the next work, it is worthy to portray some types of logic algebras and corresponding logics by use of state maps.

Acknowledgement: This research is partially supported by a grant of National Natural Science Foundation of China (11571281,61602359), China Postdoctoral Science Foundation (2015M582618), China 111 Project (B16037).

References

[1] Aglianò P., Ferririm I.M.A., Montagna F., Basic hoops: an algebraic study of continuous t-norms, Studia Logica, 2007, 87, 73-98.
[2] Blok W.J., Ferreirim I.M.A., Hoops and their implicational reducts, Algebraic Methods in Logic and Computer Science, Banach Center Publications, 1993, 28, 219-230.
[3] Blok W.J., Ferreirim I.M.A., On the structure of hoops, Algebra Universalis, 2000, 43, 233-257.
[4] Borzooei R.A., Aaly K.M., Local and perfect semihoops, Journal of Intelligent and Fuzzy Systems, 2015, 29(1), 223-234.
[5] Borzooei R.A., Dvurečenskij A., Zahiri O., State BCK-algebras and state-morphism BCK-algebras, Fuzzy Sets and Systems, 2014, 244, 86-105.
[6] Bosbach B., Halbgruppen K., Axiomatik und Aritmetik, Fundamenta Mathematicae, 1969, 64, 257-287.
[7] Bosbach B., Halbgruppen K., Kongruenzen and Quotienten, Fundamenta Mathematicae, 1970, 69, 1-14.
[8] Botur M., Dvurečenskij A., State-morhism algebras - General approach, Fuzzy Sets and Systems, 2013, 218, 90-102.
[9] Ciungu L.C., Bounded pseudo-hoops with internal states, Mathematica Slovaca, 2013, 63(5), 903-934.
[10] Ciungu L.C., Algebras on subintervals of pseudo-hoops, Fuzzy Sets and Systems, 2009, 160, 1099-1113.
[11] Ciungu L.C., Dvurečenskij A., Hyčko M., State BL-algebras, Soft Computing, 2011, 15, 619-634.
[12] Ciungu L.C., Bosbach and Riečan states on residuated lattices, Journal of Applied Functional Analysis, 2008, 2, 175-188.
[13] Dvurečenskij A., Rachunek J., Šalounova D., State operators on generalizations of fuzzy structures, Fuzzy Sets and Systems, 2012, 187, 58-76.
[14] Esteva F., Godo L., Hájek P., Montagna F., Hoops and fuzzy logic, Journal of Logic and Computation, 2003, 13, 532-555.
[15] Flaminio T., Montagna F., An algebraic approach to states on MV-algebras, In: Novák V. (eds) Fuzzy Logic 2. Proc of the 5th EUSFLAT Conf, Sept 11-14, Ostrava, vol. II, 2007, 2, 201-206.
[16] Flaminio T., Montagna F., MV-algebras with internal states and probabilistic fuzzy logic, International Journal of Approximate Reasoning, 2009, 50, 138-152.
[17] Hájek P., Metamathematics of fuzzy logic, Kluwer Academic Publishers, Dordrecht, 1998.
[18] He P.F., Xin X.L., Yang Y.W., On state residuated lattices, Soft Computing, 2015, 8, 2083-2094.
[19] Kondo M., States on bounded commutative residuated lattices, Mathematica Slovaca, 2014, 64, 1093-1104.
[20] Liu L.Z., On the existence of states on MTL-algebras, Information Sciences, 2013, 220, 559-567.
[21] Liu L.Z., Zhang X.Y., States on finite linearly ordered IMTL-algebras, Soft Computing, 2011, 15, 2021-2028.

[22] Liu L.Z.„ Zhang X.Y., States on R_0-algebras, Soft Computing, 2008, 12, 1099-1104.
[23] Mertanen J., Turunen E., States on semi-divisible generalized residuated lattices reduce to states on MV-algebras, Fuzzy Sets and Systems, 2008, 159, 3051-3064.
[24] Mundici D., Averaging the truth-value in Lukasiewicz sentential logic, Studia Logica, 1995, 55, 113-127.
[25] Riečan B., On the probability on BL-algebras, Acta Math Nitra, 2000, 4, 3-13.
[26] Turunen E., Mertanen J., States on semi-divisible residuated lattices, Soft Computing, 2008, 12, 353-357.
[27] Ciungu L.C., Internal states on equality algebras, Soft Computing, 2015, 19, 939-953.
[28] Burris S., Sankappanavar M.P., A course in universal Algebra, Springer-Verlag, New York, 1981.
[29] Leuştean L., Representations of many valued algebras, Ph.D,Thesis, University of Bucharest, 2003.
[30] He P.F., Zhao B., Xin X.L., States and internal states on semihoops, Soft Computing, 2016, 21, 2941-2957.
[31] Constantinescu N.M., State filters on fuzzy structures with internal states, Soft Computing, 2014, 18, 1841-1852.