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Research Article

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Two asymptotic expansions for gamma function developed by Windschitl's formula

Abstract: In this paper we develop Windschitl's approximation formula for the gamma function by giving two asymptotic expansions using a little known power series. In particular, for $n \in \mathbb{N}$ with $n \geq 4$, we have

$$\Gamma(x+1) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\sum_{k=3}^{n-1} \frac{a_n}{x^{2n-1}} + R_n(x)\right)$$

with

$$|R_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)} \frac{1}{x^{2n-1}}$$

for all $x > 0$, where a_n has a closed-form expression, B_{2n} is the Bernoulli number. Moreover, we present some approximation formulas for the gamma function related to Windschitl's approximation, which have higher accuracy.

Keywords: Gamma function, Windschitl's formula, Asymptotic expansion

MSC: 33B15, 41A60, 41A10, 41A20

1 Introduction

It is known that the Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{1}$$

for $n \in \mathbb{N}$ has various applications in probability theory, statistical physics, number theory, combinatorics and other branches of science. As a generalization of the factorial function, the gamma function $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ for $x > 0$ is no exception. Thus, many scholars pay attention to find various better approximations for the factorial or gamma function, for example, Ramanujan [1, P. 339], Burnside [2], Gosper [3], Alzer [4, 5], Windschitl (see [6, 7]), Smith [6], Batir [8, 9], Mortici [10–15] Nemes [16, 17], Qi et al. [18, 19], Chen [20], Yang et al. [21–23], Lu et al. [24, 25].

As an asymptotic expansion of Stirling's formula (1), one has the Stirling's series for the gamma function [26, p. 257, Eq. (6.1.40)]

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{n=1}^\infty \frac{B_{2n}}{2n(2n-1)x^{2n-1}}\right) \tag{2}$$

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as $x \to \infty$, where B_{2n} for $n \in \mathbb{N} \cup \{0\}$ is the Bernoulli number. It was proved in [4, Theorem 8] by Alzer (see also [27, Theorem 2]) that for given integer $n \in \mathbb{N}$, the function

$$F_n(x) = \ln \Gamma(x+1) - \left(x + \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) - \sum_{k=1}^{n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}$$

is strictly completely monotonic on $(0, \infty)$ if n is even, and so is $-F_n(x)$ if n is odd. It follows that the double inequality

$$\exp\left(\sum_{k=1}^{2n} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}\right) < \frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x} < \exp\left(\sum_{k=1}^{2n-1} \frac{B_{2k}}{2k(2k-1)x^{2k-1}}\right) \tag{3}$$

holds for all $x > 0$.

Another asymptotic expansion is the Laplace series (see [26, p. 257, Eq. (6.1.37)])

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \cdots\right) \tag{4}$$

as $x \to \infty$. Other asymptotic expansions developed by some closed approximation formulas for the gamma function can be found in [28–39] and the references cited therein.

Now let us focus on the Windschitl's approximation formula given by

$$\Gamma(x+1) \sim W_0(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2}, \text{ as } x \to \infty. \tag{5}$$

As shown in [20, Eq. (3.8)], the rate of Windschitl's approximation $W_0(x)$ converging to $\Gamma(x+1)$ is like x^{-5} as $x \to \infty$, and is like x^{-7} if replacing $W_0(x)$ with

$$W_1(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \frac{1}{810x^6}\right)^{x/2}, \tag{6}$$

by an easy check. These show that $W_0(x)$ and $W_1(x)$ are excellent approximations for the gamma function. Recently, Lu, Song and Ma [32] extended Windschitl's formula to the following asymptotic expansion

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left[n \sinh\left(\frac{1}{n} + \frac{a_7}{n^7} + \frac{a_9}{n^9} + \frac{a_{11}}{n^{11}} + \cdots\right)\right]^{n/2} \tag{7}$$

as $n \to \infty$ with $a_7 = 1/810, a_9 = -67/42525, a_{11} = 19/8505, \ldots$. An explicit formula for determining the coefficients of n^{-k} $(n \in \mathbb{N})$ was given in [34, Theorem 1] by Chen. Other two asymptotic expansions

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2 + \sum_{j=0}^{\infty} r_j x^{-j}}, \tag{8}$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x} + \sum_{n=3}^{\infty} \frac{d_n}{x^{2n}}\right)^{x/2}, \tag{9}$$

as $x \to \infty$ were presented in the papers [34, Theorem 2], [36], respectively.

Inspired by the asymptotic expansions (7), (8), (9) and Windschitl's approximation formula (6), the first aim of this paper is to further present the following two asymptotic expansions related to Windschitl's one (5): as $x \to \infty$,

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\sum_{n=3}^{\infty} \frac{a_n}{x^{2n-1}}\right), \tag{10}$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \sum_{n=1}^{\infty} \frac{b_n}{x^n}\right). \tag{11}$$

It is worth pointing out that those coefficients in (10) have a closed-form expression, which is due to a little known power series expansion of $\ln\left(t^{-1} \sinh t\right)$ (Lemma 2.1). We also give an estimate of the remainder in the asymptotic expansion (10). Incidentally, we provide a more explicit coefficients formula in Chen's asymptotic expansion (8). These results (Theorems 1–4) are presented in Section 2.

The second aim of this paper is to give some closed approximation formulas for the gamma function generated by truncating five asymptotic series just mentioned, and compare the accuracy of them by numeric computations and some inequalities. These results (Table 1 and Theorem 5) are listed in Section 3.

2 Asymptotic expansions

To obtain the explicit coefficients formulas in the asymptotic expansions (10), (11) and (8), and to estimate the remainder in the asymptotic expansions (10), we first give a lemma.

Lemma 2.1. *For $|t| < \pi$, we have*

$$\ln \frac{\sinh t}{t} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{2n (2n)!} t^{2n}. \tag{12}$$

Moreover, for $n \in \mathbb{N}$, the double inequality

$$\sum_{k=1}^{2n} \frac{2^{2k} B_{2k}}{2k (2k)!} t^{2k} < \ln \frac{\sinh t}{t} < \sum_{k=1}^{2n-1} \frac{2^{2k} B_{2k}}{2k (2k)!} t^{2k} \tag{13}$$

holds for all $t > 0$.

Proof. It was listed in [26, p. 85, Eq. (4.5.64), (4.5.65), (4.5.67)] that

$$\coth t = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} t^{2n-1} \quad |t| < \pi.$$

Then we obtain that for $|t| < \pi$,

$$\ln \frac{\sinh t}{t} = \int_0^t \left(\coth x - \frac{1}{x} \right) dx = \int_0^t \left(\sum_{n=0}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} - \frac{1}{x} \right) dx = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{2n (2n)!} t^{2n}.$$

The double inequality (13) was proved in [23, Corollary 1], and the proof is completed. □

Theorem 2.2. *As $x \to \infty$, the asymptotic expansion*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \left(x \sinh \frac{1}{x} \right)^{x/2} \exp \left(\sum_{n=3}^{\infty} \frac{a_n}{x^{2n-1}} \right)$$

$$= \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \left(x \sinh \frac{1}{x} \right)^{x/2} \exp \left[\frac{1}{1620 x^5} - \frac{11}{18\,900 x^7} + \frac{143}{170\,100 x^9} - \frac{2\,260\,261}{1\,178\,793\,000 x^{11}} + \cdots \right]$$

holds with

$$a_n = \frac{2n (2n-2)! - 2^{2n-1}}{2n (2n)!} B_{2n}, \tag{14}$$

where B_{2n} is the Bernoulli number.

Proof. By the asymptotic expansion (2) and Lemma 2.1 we have that as $x \to \infty$,

$$\ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2} \right) \ln x + x \sim \sum_{n=1}^{\infty} \frac{a_n'}{x^{2n-1}},$$

$$\frac{x}{2} \ln \left(x \sinh \frac{1}{x} \right) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{a_n''}{x^{2n-1}},$$

where

$$a_n' = \frac{B_{2n}}{2n (2n-1)} \quad \text{and} \quad a_n'' = \frac{2^{2n} B_{2n}}{2n (2n)!}.$$

Let

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \left(x \sinh \frac{1}{x} \right)^{x/2} \exp(w_0(x)) \text{ as } x \to \infty.$$

Then we have that as $x \to \infty$,

$$w_0(x) = \left(\ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2} \right) \ln x + x \right) - \frac{x}{2} \ln \left(x \sinh \frac{1}{x} \right)$$

$$= \sum_{n=1}^{\infty} \frac{a'_n}{x^{2n-1}} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{a''_n}{x^{2n-1}} = \sum_{n=1}^{\infty} \frac{a'_n - a''_n/2}{x^{2n-1}} = \sum_{n=1}^{\infty} \frac{a_n}{x^{2n-1}}.$$

An easy computation yields $a_1 = a_2 = 0$ and

$$a_3 = \frac{1}{1620}, \quad a_4 = -\frac{11}{18\,900}, \quad a_5 = \frac{143}{170\,100}, \quad a_6 = -\frac{2260\,261}{1178\,793\,000}$$

which completes the proof. $\square$

The following theorem offers an estimate of the remainder in the asymptotic expansion (10).

Theorem 2.3. *For $n \in \mathbb{N}$ with $n \geq 4$, let*

$$\Gamma(x+1) = \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \exp\left(\sum_{k=3}^{n-1} \frac{a_k}{x^{2k-1}} + R_n(x)\right),$$

where a_k is given by (14). Then we have

$$|R_n(x)| \leq \frac{|B_{2n}|}{2n(2n-1)} \frac{1}{x^{2n-1}}$$

for all $x > 0$.

Proof. We have

$$R_n(x) = \ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x - \frac{x}{2} \ln\left(x \sinh \frac{1}{x}\right) - \sum_{k=3}^{n-1} \frac{a_k}{x^{2k-1}}.$$

If $n = 2m + 1$ for $m \geq 2$ then by inequalities (3) and (13) we have

$$R_{2m}(x) = \left(\ln \Gamma(x+1) - \ln \sqrt{2\pi} - \left(x + \frac{1}{2}\right) \ln x + x\right) - \frac{x}{2} \ln\left(x \sinh \frac{1}{x}\right) - \sum_{k=3}^{2m} \frac{a_k}{x^{2k-1}}$$

$$> \sum_{k=1}^{2m} \frac{a'_k}{x^{2k-1}} - \frac{1}{2} \sum_{k=1}^{2m+1} \frac{a''_k}{x^{2k-1}} - \sum_{k=3}^{2m} \frac{a_m}{x^{2k-1}} = -\frac{1}{2} \frac{a''_{2m+1}}{x^{4m+1}},$$

$$R_{2m}(x) < \sum_{k=1}^{2m+1} \frac{a'_k}{x^{2k-1}} - \frac{1}{2} \sum_{k=1}^{2m} \frac{\ddot{a}_k}{x^{2k-1}} - \sum_{k=3}^{2m} \frac{a_m}{x^{2k-1}} = \frac{a'_{2m+1}}{x^{4m+1}},$$

where the last equalities in the above two inequalities hold due to $a'_m - a''_m/2 = a_m$. It follows that

$$|R_n(x)| < \max\left(\left|-\frac{1}{2} \frac{a''_{2m+1}}{x^{4m+1}}\right|, \left|\frac{a'_{2m+1}}{x^{4m+1}}\right|\right) = \max\left(\frac{1}{2}|a''_n|, |a'_n|\right) \frac{1}{x^{2n-1}}.$$

The calculations also hold for the case when $n = 2m$, $m \geq 2$.

Since

$$\frac{|a'_n|}{|a''_n/2|} = \left|\frac{B_{2n}}{2n(2n-1)}\right| \Big/ \left|\frac{1}{2} \frac{2^{2n} B_{2n}}{2n(2n)!}\right| = 2 \frac{(2n)!}{(2n-1)2^{2n}} := a'''_n,$$

$$\frac{a'''_{n+1}}{a'''_n} - 1 = \frac{1}{2}(2n+3)(n-1) > 0,$$

it is derived that $a'''_n > a'''_4 = 45$ for $n \geq 4$, so we obtain

$$\max\left(\frac{1}{2}|a''_n|, |a'_n|\right) = |a'_n| = \frac{|B_{2n}|}{2n(2n-1)},$$

which completes the proof. $\square$

Remark 2.4. *Since $B_{2n+1} = 0$ for $n \in \mathbb{N}$, the asymptotic series $w_0(x)$ can also be written as*

$$w_0(x) = \sum_{n=3}^{\infty} \frac{2n(2n-2)! - 2^{2n-1}}{2n(2n)!} \frac{B_{2n}}{x^{2n-1}} = \sum_{n=1}^{\infty} \frac{(n+1)(n-1)! - 2^n}{(n+1)(n+1)!} \frac{B_{n+1}}{x^n} := \sum_{n=1}^{\infty} \frac{a_n^*}{x^n},$$

where

$$a_n^* = \frac{(n+1)(n-1)! - 2^n}{(n+1)(n+1)!} B_{n+1}. \tag{15}$$

Now we establish the second Windschitl type asymptotic series for the gamma function.

Theorem 2.5. *As $x \to \infty$, the asymptotic expansion (9)*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \sum_{n=1}^{\infty} \frac{b_n}{x^n}\right)$$

$$= \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{x/2} \left(1 + \frac{1}{1620x^5} - \frac{11}{18\,900x^7} + \frac{143}{170\,100x^9} + \frac{1}{5248\,800x^{10}} + \cdots\right)$$

holds with $b_0 = 1$, $b_1 = b_2 = b_3 = b_4 = 0$ and for $n \geq 5$,

$$b_n = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{2^k}{(k+1)^2 (k-1)!}\right) B_{k+1} b_{n-k}. \tag{16}$$

Proof. It was proved in [29, Lemma 3] that as $x \to \infty$,

$$\exp\left(\sum_{n=1}^{\infty} a_n x^{-n}\right) \sim \sum_{n=0}^{\infty} b_n x^{-n}$$

with $b_0 = 1$ and

$$b_n = \frac{1}{n} \sum_{k=1}^{n} k a_k b_{n-k} \quad \text{for } n \geq 1. \tag{17}$$

Substituting a_k^* given in (15) into (17) gives recurrence formula (16).

An easy verification shows that $b_n = 0$ for $1 \leq n \leq 4$, $b_6 = b_8 = 0$ and

$$b_5 = \frac{1}{1620}, \quad b_7 = -\frac{11}{18\,900}, \quad b_9 = \frac{143}{170\,100}, \quad b_{10} = \frac{1}{5248\,800},$$

which completes the proof. □

The following theorem improves Chen's result [34, Theorem 2].

Theorem 2.6. *As $x \to \infty$, the asymptotic expansion*

$$\Gamma(x+1) \sim \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{(x/2)\left(1 + \sum_{n=2}^{\infty} c_n x^{-2n}\right)} \tag{18}$$

$$= \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \left(x \sinh \frac{1}{x}\right)^{\frac{x}{2}\left(1 + \frac{1}{135x^4} - \frac{191}{28\,350x^6} + \frac{25\,127}{2551\,500x^8} - \frac{19\,084\,273}{841\,995\,000x^{10}} + \cdots\right)}$$

holds with $c_0 = 1$, $c_1 = 0$ and for $n \geq 2$,

$$c_n = \frac{6B_{2n+2}}{(n+1)(2n+1)} - 6 \sum_{k=1}^{n} \frac{2^{2k+2} B_{2k+2}}{2(k+1)(2k+2)!} c_{n-k}.$$

Proof. The asymptotic expansion (8) can be written as

$$\ln \Gamma(x+1) - \ln \sqrt{2\pi x} - x \ln x + x \sim \left(\frac{x}{2} + \sum_{j=0}^{\infty} \frac{r_j}{x^j}\right) \ln\left(x \sinh \frac{1}{x}\right),$$

which, by (2) and (12), is equivalent to

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}} \sim \left(\frac{x}{2} + \sum_{j=0}^{\infty} \frac{r_j}{x^j}\right)\left(\sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{2n(2n)!} \frac{1}{x^{2n}}\right).$$

Since the left hand side and the second factor of the right hand side are odd and even, respectively, the asymptotic expansion $x/2 + \sum_{j=0}^{\infty} r_j x^{-j}$ has to be odd, and so $r_{2n} = 0$ for $n \in \mathbb{N} \cup \{0\}$. Then, the asymptotic expansion (8) has the form of (18), which is equivalent to

$$\sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1}} \sim \frac{x}{2}\left(\sum_{n=0}^{\infty} \frac{c_n}{x^{2n}}\right)\left(\sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{2n(2n)!} \frac{1}{x^{2n}}\right).$$

It can be written as

$$\sum_{n=0}^{\infty} \frac{B_{2n+2}}{(n+1)(2n+1)} \frac{1}{x^{2n}} \sim \left(\sum_{n=0}^{\infty} \frac{c_n}{x^{2n}}\right)\left(\sum_{n=1}^{\infty} \frac{2^{2n+2}B_{2n+2}}{2(n+1)(2n+2)!} \frac{1}{x^{2n}}\right).$$

Comprising coefficients of x^{-2n} gives

$$\frac{B_{2n+2}}{(n+1)(2n+1)} = \sum_{k=0}^{n} \frac{2^{2k+2}B_{2k+2}}{2(k+1)(2k+2)!} c_{n-k},$$

which yields $c_0 = 1$ and for $n \geq 1$,

$$c_n = \frac{6B_{2n+2}}{(n+1)(2n+1)} - 6\sum_{k=1}^{n} \frac{2^{2k+2}B_{2k+2}}{2(k+1)(2k+2)!} c_{n-k}.$$

A straightforward computation leads to

$$c_1 = 0, \ c_2 = \frac{1}{135}, \ c_3 = -\frac{191}{28\,350}, \ c_4 = \frac{25\,127}{2\,551\,500}, \ c_5 = -\frac{19\,084\,273}{841\,995\,000},$$

which ends the proof.   □

Remark 2.7. *Chen's recurrence formula of coefficients r_j given in [34, Theorem 2] may be complicated, since he was unaware of the power series (12).*

3 Numeric comparisons and inequalities

If the series in (10), (11), (9) (18) are truncated at $n = 3, 5, 3, 2$, respectively, then we obtain four Windschitl type approximation formulas:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \exp\left(\frac{1}{1620x^5}\right) := W_{01}(x), \tag{19}$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2} \left(1 + \frac{1}{1620x^5}\right) := W_{01}^*(x), \tag{20}$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} = W_1(x),$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{\frac{x}{2}\left(1+\frac{1}{135x^4}\right)} := W_{c1}(x), \tag{21}$$

as $x \to \infty$. Also, we denote Lu et al.'s one [32, Theorem 1.8] by

$$W_{l1}(x) = \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\left(\frac{1}{x} + \frac{1}{810x^7}\right)\right)^{x/2}. \tag{22}$$

In this section, we aim to compare the five closed approximation formulas listed above.

3.1 Numeric comparisons

We easily obtain

$$\lim_{x \to \infty} \frac{\ln \Gamma (x + 1) - \ln W_1 (x)}{x^{-7}} = -\frac{163}{340\,200},$$
$$\lim_{x \to \infty} \frac{\ln \Gamma (x + 1) - \ln W_{c1} (x)}{x^{-7}} = -\frac{191}{340\,200},$$

$$\lim_{x \to \infty} \frac{\ln \Gamma (x + 1) - \ln W_{01} (x)}{x^{-7}} = \lim_{x \to \infty} \frac{\ln \Gamma (x + 1) - \ln W_{01}^* (x)}{x^{-7}} = -\frac{198}{340\,200},$$
$$\lim_{x \to \infty} \frac{\ln \Gamma (x + 1) - \ln W_{c1} (x)}{x^{-7}} = -\frac{268}{340\,200}.$$

These show that the rates of the five approximation formulas converging to $\Gamma (x + 1)$ are all like x^{-7} as $x \to \infty$, and $W_1 (x)$ are the best of all five approximations, which can also be seen from the following Table 1.

Table 1. Comparisons among $W_1 (x)$, $W_{c1} (x)$, $W_{01} (x)$, $W_{l1} (x)$

x	$\left\|\frac{W_1 (x) - \Gamma (x+1)}{\Gamma (x+1)}\right\|$	$\left\|\frac{W_{c1} (x) - \Gamma (x+1)}{\Gamma (x+1)}\right\|$	$\left\|\frac{W_{01} (x) - \Gamma (x+1)}{\Gamma (x+1)}\right\|$	$\left\|\frac{W_{l1} (x) - \Gamma (x+1)}{\Gamma (x+1)}\right\|$
1	1.832×10^{-4}	2.562×10^{-4}	2.754×10^{-4}	4.686×10^{-4}
2	2.668×10^{-6}	3.292×10^{-6}	3.449×10^{-6}	5.030×10^{-6}
5	5.743×10^{-9}	6.791×10^{-9}	7.054×10^{-9}	9.681×10^{-9}
10	4.710×10^{-11}	5.532×10^{-11}	5.738×10^{-11}	7.794×10^{-11}
20	3.727×10^{-13}	4.370×10^{-13}	4.531×10^{-13}	6.138×10^{-13}
50	6.129×10^{-16}	7.182×10^{-16}	7.446×10^{-16}	1.008×10^{-15}
100	4.791×10^{-18}	5.614×10^{-18}	5.819×10^{-18}	7.877×10^{-18}

3.2 Three lemmas

As is well known, analytic inequality [40–42] is playing a very important role in different branches of modern mathematics. To further compare $W_1 (x)$, $W_{c1} (x)$, $W_{01}^* (x)$, $W_{01} (x)$ and $W_{l1} (x)$, we first give the following inequality.

Lemma 3.1. *The inequality*

$$\psi' \left(x + \frac{1}{2} \right) < \frac{1}{x} \frac{x^4 + \frac{67}{36} x^2 + \frac{256}{945}}{x^4 + \frac{35}{18} x^2 + \frac{407}{1008}} \tag{23}$$

holds for all $x > 0$.

Proof. Let

$$g (x) = \psi \left(x + \frac{1}{2}, 1 \right) - \frac{1}{x} \frac{x^4 + \frac{67}{36} x^2 + \frac{256}{945}}{x^4 + \frac{35}{18} x^2 + \frac{407}{1008}}.$$

Then we have

$$g (x + 1) - g (x) = \psi \left(x + \frac{3}{2}, 1 \right) - \frac{1}{x + 1} \frac{(x + 1)^4 + \frac{67}{36} (x + 1)^2 + \frac{256}{945}}{(x + 1)^4 + \frac{35}{18} (x + 1)^2 + \frac{407}{1008}}$$

$$- \psi \left(x + \frac{1}{2}, 1 \right) + \frac{1}{x} \frac{x^4 + \frac{67}{36} x^2 + \frac{256}{945}}{x^4 + \frac{35}{18} x^2 + \frac{407}{1008}}$$

$$= \frac{921\,600}{x (x + 1) (2x + 1)^2 (1008x^4 + 1960x^2 + 407) (1008x^4 + 4032x^3 + 8008x^2 + 7952x + 3375)} > 0.$$

Hence, we conclude that

$$g(x) < g(x+1) < \cdots < \lim_{n\to\infty} g(x+n) = 0,$$

which proves (23), and the proof is done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

The second lemma offers a simple criterion to determine the sign of a class of special polynomial on a given interval contained in $(0, \infty)$ without using Descartes' Rule of Signs, which plays an important role in the study of certain special functions, see for example [43, 44]. A series version can be found in [45].

Lemma 3.2 ([43, Lemma 7]). *Let $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ with $n > m$ and let $P_n(t)$ be an n degrees polynomial defined by*

$$P_n(t) = \sum_{i=m+1}^{n} a_i t^i - \sum_{i=0}^{m} a_i t^i, \tag{24}$$

where $a_n, a_m > 0$, $a_i \geq 0$ for $0 \leq i \leq n - 1$ with $i \neq m$. Then, there is a unique number $t_{m+1} \in (0, \infty)$ to satisfy $P_n(t) = 0$ such that $P_n(t) < 0$ for $t \in (0, t_{m+1})$ and $P_n(t) > 0$ for $t \in (t_{m+1}, \infty)$.

Consequently, for given $t_0 > 0$, if $P_n(t_0) > 0$ then $P_n(t) > 0$ for $t \in (t_0, \infty)$ and if $P_n(t_0) < 0$ then $P_n(t) < 0$ for $t \in (0, t_0)$.

Lemma 3.3. *Let $W_{01}(x)$, $W_{01}^*(x)$, $W_1(x)$, $W_{c1}(x)$ and $W_{l1}(x)$ be defined by (19), (20), (6), (21) and (22), respectively. Then we have*

$$W_1(x) < W_{c1}(x) < W_{01}^*(x) < W_{01}(x) < W_{l1}(x)$$

for all $x \geq 1$.

Proof. (i) The first inequality $W_1(x) < W_{c1}(x)$ is equivalent to

$$h_1(t) = \ln\left(\frac{\sinh t}{t} + \frac{1}{810}t^6\right) - \left(1 + \frac{1}{135}t^4\right)\ln\left(\frac{\sinh t}{t}\right) < 0$$

for $t = 1/x \in (0, 1]$. We have

$$\frac{d}{dy}\left(\ln\left(y + \frac{1}{810}t^6\right) - \left(1 + \frac{1}{135}t^4\right)\ln y\right) = -\frac{1}{135}t^4 \frac{810y + 135t^2 + t^6}{y(810y + t^6)} < 0$$

for $y > 1$, which together with

$$y = \frac{\sinh t}{t} > 1 + \frac{1}{6}t^2$$

yields

$$h_1(t) < \ln\left(1 + \frac{1}{6}t^2 + \frac{1}{810}t^6\right) - \left(1 + \frac{1}{135}t^4\right)\ln\left(1 + \frac{1}{6}t^2\right) := h_{11}(t).$$

Differentiation leads us to

$$\frac{135}{2t^3}h_{11}'(t) = -2\ln\left(\frac{1}{6}t^2 + 1\right) - t^2 \frac{t^6 - 135t^2 - 1620}{(t^2 + 6)(t^6 + 135t^2 + 810)} := h_{12}(t),$$

$$h_{12}'(t) = -4\frac{t^3\left(t^{12} + 9t^{10} + 540t^8 + 8505t^6 + 47\,385t^4 + 328\,050t^2 + 1312\,200\right)}{(t^2 + 6)^2(t^6 + 135t^2 + 810)^2} < 0$$

for $t > 0$. Therefore, we obtain $h_{12}(t) < h_{12}(0) = 0$, and so $h_{11}(t) < h_{11}(0) = 0$, which implies $h_1(t) < 0$ for $t > 0$.

(ii) The second inequality $W_{c1}(x) < W_{01}^*(x)$ is equivalent to

$$\frac{x}{2}\left(1 + \frac{1}{135x^4}\right)\ln\left(x\sinh\frac{1}{x}\right) < \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) + \ln\left(1 + \frac{1}{1620x^5}\right),$$

or equivalently,

$$h_2(t) = \frac{1}{270}t^3\ln\left(\frac{\sinh t}{t}\right) - \ln\left(1 + \frac{1}{1620}t^5\right) < 0$$

for $t = 1/x \in (0, 1]$. Taking $n = 2$ in the inequalities (13) gives

$$\ln\left(\frac{\sinh t}{t}\right) < \frac{1}{6}t^2 - \frac{1}{180}t^4 + \frac{1}{2835}t^6,$$

which is applied to the expression of $h_2(t)$:

$$h_2(t) < \frac{1}{270}t^3\left(\frac{1}{6}t^2 - \frac{1}{180}t^4 + \frac{1}{2835}t^6\right) - \ln\left(1 + \frac{1}{1620}t^5\right) := h_{21}(t).$$

Differentiation yields

$$h'_{21}(t) = \frac{t^6}{340\,200}\,\frac{4t^7 - 49t^5 + 1050t^3 + 6480t^2 - 79\,380}{t^5 + 1620} < 0$$

for $t \in (0, 1]$, which proves $h_2(t) < 0$ for $t \in (0, 1]$.

   (iii) The third inequality $W^*_{01}(x) < W_{01}(x)$ is equivalent to

$$1 + \frac{1}{1620x^5} < \exp\left(\frac{1}{1620x^5}\right),$$

which follows by a simple inequality $1 + y < e^y$ for $y \neq 0$.

   (iv) The fourth inequality $W_{01}(x) < W_{l1}(x)$ is equivalent to

$$\frac{x}{2}\ln\left(x\sinh\left(\frac{1}{x} + \frac{1}{810x^7}\right)\right) > \frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right) + \frac{1}{1620x^5},$$

or equivalently,

$$h_3(t) = \ln\sinh\left(t + \frac{1}{810}t^7\right) - \ln\sinh t - \frac{1}{810}t^6 > 0$$

for $t = 1/x > 0$. Denote $h_{30}(t) = \ln\sinh t$. Then by Taylor formula we have

$$h_3(t) = h_{30}\left(t + \frac{1}{810}t^7\right) - h_{30}(t) - \frac{1}{810}t^6$$

$$= \frac{t^7}{810}h'_{30}(t) + \frac{1}{2!}\frac{t^{14}}{810^2}h''_{30}(t) + \frac{1}{3!}\frac{t^{21}}{810^3}h'''_{30}(\xi) - \frac{1}{810}t^6,$$

where $t < \xi < t + t^7/810$. Since $h'''_{30}(t) = 2(\cosh t)/\sinh^3 t > 0$, we get

$$h_3(t) > \frac{1}{810}t^7\frac{\cosh t}{\sinh t} - \frac{t^{14}}{2 \times 810^2}\frac{1}{\sinh^2 t} - \frac{1}{810}t^6 := \frac{t^6 \times h_{31}(t)}{2 \times 810^2 \sinh^2 t},$$

where

$$h_{31}(t) = 810t\sinh 2t - 810\cosh 2t + 810 - t^8.$$

Due to

$$h_{31}(t) = 540t^4 + 144t^6 + \frac{101}{7}t^8 + 810\sum_{n=5}^{\infty}\frac{(n-1)\,2^{2n}}{(2n)!}t^{2n} > 0,$$

we conclude that $h_3(t) > 0$ for $t > 0$, which completes the proof. $\qquad\square$

3.3 The comparison theorem for $W_1(x)$, $W_{c1}(x)$, $W^*_{01}(x)$, $W_{01}(x)$ and $W_{l1}(x)$

Theorem 3.4. *(i) The function*

$$f_1(x) = \ln\Gamma(x+1) - \ln\sqrt{2\pi} - \left(x + \frac{1}{2}\right)\ln x + x - \frac{x}{2}\ln\left(x\sinh\frac{1}{x} + \frac{1}{810x^6}\right)$$

is strictly increasing and concave on $[1, \infty)$.

(ii) For $x \geq 1$, we have

$$\beta_0 \left(x \sinh \frac{1}{x} + \frac{1}{810x^6} \right)^{x/2} < \frac{\Gamma(x+1)}{\sqrt{2\pi x} \, (x/e)^x} < \left(x \sinh \frac{1}{x} + \frac{1}{810x^6} \right)^{x/2}$$

$$< \left(x \sinh \frac{1}{x} \right)^{\frac{x}{2}\left(1 + \frac{1}{135x^4}\right)} < \left(x \sinh \frac{1}{x} \right)^{x/2} \left(1 + \frac{1}{1620x^5} \right) \tag{25}$$

$$< \left(x \sinh \frac{1}{x} \right)^{x/2} \exp\left(\frac{1}{1620x^5} \right) < \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \left(x \sinh \left(\frac{1}{x} + \frac{1}{810x^7} \right) \right)^{x/2}$$

with the best constant

$$\beta_0 = \frac{e}{\sqrt{2\pi \sinh 1 + \pi/405}} \approx 0.999\,81.$$

Proof. (i) Differentiation yields

$$f_1'(x) = \psi(x+1) - \ln x - \frac{1}{2x} - \frac{1}{2} \ln\left(x \sinh \frac{1}{x} + \frac{1}{810x^6} \right)$$
$$+ 3 \frac{135x^6 \cosh \frac{1}{x} - 135x^7 \sinh \frac{1}{x} + 1}{810x^7 \sinh \frac{1}{x} + 1},$$

$$f_1''(x) = \psi'(x+1) - \frac{1}{x} + \frac{1}{2x^2}$$

$$- \frac{3\left(109\,350x^{14} \sinh^2 \frac{1}{x} + 5940x^7 \sinh \frac{1}{x} + 135x^5 \sinh \frac{1}{x} - 1890x^6 \cosh \frac{1}{x} - 109\,350x^{12} - 1 \right)}{x\left(810x^7 \sinh \frac{1}{x} + 1 \right)^2}.$$

Replacing x by $x + 1/2$ in inequality (23) yields

$$\psi'(x+1) < \frac{1}{30} \frac{3780x^4 + 7560x^3 + 12\,705x^2 + 8925x + 3019}{(2x+1)\left(63x^4 + 126x^3 + 217x^2 + 154x + 60 \right)},$$

for $x > -1/2$, and and applying the above inequality combined with the change of variable $x = 1/t \in (0, 1]$ yield

$$f_1''(x) < \frac{1}{30} \frac{t\left(3019t^4 + 8925t^3 + 12\,705t^2 + 7560t + 3780 \right)}{(t+2)\left(60t^4 + 154t^3 + 217t^2 + 126t + 63 \right)} - t + \frac{1}{2}t^2$$

$$- 3t \frac{109\,350 \sinh^2 t - 1890t^8 \cosh t + 5940t^7 \sinh t + 135t^9 \sinh t - 109\,350t^2 - t^{14}}{\left(810 \sinh t + t^7 \right)^2}$$

$$:= \frac{810^2 t \times f_{11}(t)}{(t+2)\left(126t + 217t^2 + 154t^3 + 60t^4 + 63 \right)\left(810 \sinh t + t^7 \right)^2},$$

where

$$f_{11}(t) = p_6(t) \sinh^2 t + p_{13}(t) \cosh t - p_{14}(t) \sinh t + p_{20}(t),$$

$$p_6(t) = 30t^6 + 47t^5 - \frac{718}{15}t^4 - 210t^3 - 259t^2 - \frac{315}{2}t - 63,$$

$$p_{13}(t) = \frac{7}{810}t^8 (t+2)\left(60t^4 + 154t^3 + 217t^2 + 126t + 63 \right),$$

$$p_{14}(t) = t^7 \left(\frac{1}{27}t^7 + \frac{77}{810}t^6 + \frac{2857}{1620}t^5 + \frac{45\,973}{6075}t^4 + \frac{1547}{108}t^3 + \frac{12\,341}{810}t^2 + \frac{77}{9}t + \frac{154}{45} \right),$$

$$p_{20}(t) = \frac{1}{21\,870}t^{20} + \frac{257}{656\,100}t^{19} + \frac{13\,667}{9\,841\,500}t^{18} + \frac{217}{87\,480}t^{17} + \frac{7}{2700}t^{16}$$
$$+ \frac{7}{4860}t^{15} + \frac{7}{12\,150}t^{14} + 30t^7 + 137t^6 + \frac{525}{2}t^5 + 280t^4 + \frac{315}{2}t^3 + 63t^2.$$

To prove $f_{11}(t) < 0$ for $t \in (0, 1]$, we use formula $\sinh^2 t = \cosh^2 t - 1$ to write $f_{11}(t)$ as

$$f_{31}(t) = [p_6(t)\cosh t + p_{13}(t)]\cosh t - p_{14}(t)\sinh t + p_{20}(t) - p_6(t).$$

Since the coefficients of polynomial $-p_6(t)$ satisfy those conditions of Lemma 3.2, and $-p_6(1) = 19\,811/30 > 0$, we see that $-p_6(t) > 0$ for $t \in (0, 1]$. It then follows from that

$$p_6(t)\cosh t + p_{13}(t) < p_6(t) + p_{13}(t)$$

$$= \frac{14}{27}t^{13} + \frac{959}{405}t^{12} + \frac{245}{54}t^{11} + \frac{392}{81}t^{10} + \frac{49}{18}t^9 + \frac{49}{45}t^8 + 30t^6$$
$$+ 47t^5 - \frac{718}{15}t^4 - 210t^3 - 259t^2 - \frac{315}{2}t - 63 := p_{13}^*(t).$$

Application of Lemma 3.2 again with $-p_{13}^*(1) = 173\,959/270 > 0$ yields $-p_{13}^*(t) > 0$ for $t \in (0, 1]$, and so $p_6(t)\cosh t + p_{13}(t) < 0$ for $t \in (0, 1]$. Since $p_{14}(t) > 0$ for $t > 0$, using the inequalities

$$\cosh t > \sum_{n=0}^{4} \frac{t^{2n}}{(2n)!} = \frac{1}{40\,320}t^8 + \frac{1}{720}t^6 + \frac{1}{24}t^4 + \frac{1}{2}t^2 + 1,$$

$$\sinh t > \sum_{n=1}^{4} \frac{t^{2n-1}}{(2n-1)!} = \frac{1}{5040}t^7 + \frac{1}{120}t^5 + \frac{1}{6}t^3 + t,$$

we have

$$f_{11}(t) = [p_6(t)\cosh t + p_{13}(t)]\cosh t - p_{14}(t)\sinh t + p_{20}(t) - p_6(t)$$

$$< \left[p_6(t)\sum_{n=0}^{4}\frac{t^{2n}}{(2n)!} + p_{13}(t)\right]\sum_{n=0}^{4}\frac{t^{2n}}{(2n)!} - p_{14}(t)\sum_{n=1}^{4}\frac{t^{2n-1}}{(2n-1)!} + p_{20}(t) - p_6(t)$$

$$= \frac{1}{54\,190\,080}t^{22} + \frac{9007}{1\,625\,702\,400}t^{21} + \frac{9\,615\,889}{109\,734\,912\,000}t^{20} + \frac{5\,351\,449}{9\,405\,849\,600}t^{19}$$
$$+ \frac{739\,363\,013}{282\,175\,488\,000}t^{18} + \frac{17\,347\,597}{2\,508\,226\,560}t^{17} + \frac{62\,875\,199}{2\,090\,188\,800}t^{16} + \frac{25\,247}{3\,483\,648}t^{15}$$
$$- \frac{32\,887}{3\,732\,480}t^{14} - \frac{232\,765}{193\,536}t^{13} - \frac{3\,620\,941}{870\,912}t^{12} - \frac{292\,093}{34\,560}t^{11} - \frac{292\,093}{86\,400}t^{10} := t^{10}p_{12}(t).$$

From Lemma 3.2 and $-p_{12}(1) = 67\,766\,507\,802\,179/3\,950\,456\,832\,000 > 0$ it follows that $-p_{12}(t) > 0$ for $t \in (0, 1]$, and so $f_{11}(t) < 0$ for $t \in (0, 1]$, which implies $f_1''(x) < 0$ for $x \geq 1$.

(ii) Using the increasing property of f_1 and noting that

$$f_1(1) = \ln\frac{e}{\sqrt{2\pi\sinh 1 + \pi/405}} \quad \text{and} \quad \lim_{x\to\infty} f_1(x) = 0,$$

we have

$$\ln\frac{e}{\sqrt{2\pi\sinh 1 + \pi/405}} < \ln\frac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} - \ln\left(x\sinh\frac{1}{x} + \frac{1}{810x^6}\right)^{x/2} < 0,$$

which imply the first and second inequalities of (25).

The other ones of (25) follow from Lemma 3.3, which completes the proof. □

4 Conclusions

In this paper, by a little known power series expansion of $\ln\left(t^{-1}\sinh t\right)$, that is, (12), we establish an asymptotic expansion (10) for the gamma function related to Windschitl's formula, in which its coefficients have a closed-form expression (14). Moreover, we give an estimate of the remainder in the asymptotic expansion

(10) by means of inequalities (3) and (13). Due to (12), we also give other two asymptotic expansions (11) and (18), but their coefficients formulas are of recursive form. Despite all that, the latter improves Chen's result [34, Theorem 2]

Furthermore, we compare the accuracy of all five approximation formulas for the gamma function generated by truncating five asymptotic series (10), (11), (9), (18) and (7) by numeric computations and some inequalities. These show that the approximation formula (6) is the best. Some general properties of truncated series (truncated polynomials) can refer to [46].

References

[1]　Ramanujan S., The Lost Notebook and Other Unpublished Papers, 1988, Berlin: Springer.

[2]　Burnside W., A rapidly convergent series for log $N!$, Messenger Math., 1917, 46, 157–159.

[3]　Gosper R. W., Decision procedure for indefinite hypergeometric summation, Proc. Nat. Acad. Sci. U.S.A., 1978, 75, 40–42.

[4]　Alzer H., On some inequalities for the gamma and psi functions, Math. Comp., 1997, 66, 373–389.

[5]　Alzer H., Sharp upper and lower bounds for the gamma function, Proc. Roy. Soc. Edinburgh Sect. A, 2009, 139, 709–718.

[6]　Smith W. D., The gamma function revisited, http://schule.bayernport.com/gamma/gamma05. pdf, 2006.

[7]　http://www.rskey.org/gamma.htm.

[8]　Batir N., Sharp inequalities for factorial n, Proyecciones, 2008, 27, 97–102.

[9]　Batir N., Inequalities for the gamma function, Arch. Math., 2008, 91, 554–563.

[10]　Mortici C., An ultimate extremely accurate formula for approximation of the factorial function, Arch. Math., 2009, 93, 37–45.

[11]　Mortici C., New sharp inequalities for approximating the factorial function and the digamma functions, Miskolc Math. Notes, 2010, 11, 79–86.

[12]　Mortici C., A new Stirling series as continued fraction, Numer. Algor., 2011, 56, 17–26.

[13]　Mortici C., Improved asymptotic formulas for the gamma function, Comput. Math. Appl., 2011, 61, 3364–3369.

[14]　Mortici C., Further improvements of some double inequalities for bounding the gamma function, Math. Comput. Model., 2013, 57, 1360–1363.

[15]　Mortici C., A continued fraction approximation of the gamma function, J. Math. Anal. Appl., 2013, 402, 405–410.

[16]　Nemes G., New asymptotic expansion for the Gamma function, Arch. Math. (Basel), 2010, 95, 161–169.

[17]　Nemes G., More accurate approximations for the gamma function, Thai J. Math., 2011, 9, 21–28.

[18]　Guo B.-N., Zhang Y.-J., Qi F., Refinements and sharpenings of some double inequalities for bounding the gamma function, J. Inequal. Pure Appl. Math., 2008, 9, Article 17.

[19]　Qi F., Integral representations and complete monotonicity related to the remainder of Burnside's formula for the gamma function, J. Comput. Appl. Math., 2014, 268, 155–167.

[20]　Chen Ch.-P., A more accurate approximation for the gamma function, J. Number Theory, 2016, 164, 417–428.

[21]　Yang Zh.-H., Tian J.-F., Monotonicity and inequalities for the gamma function, J. Inequal. Appl., 2017, 2017, 317.

[22]　Yang Zh.-H., Tian J.-F., An accurate approximation formula for gamma function, J. Inequal. Appl., 2018, 2018, 56.

[23]　Yang Zh.-H., Approximations for certain hyperbolic functions by partial sums of their Taylor series and completely monotonic functions related to gamma function, J. Math. Anal. Appl., 2016, 441, 549–564.

[24]　Lu D., A new sharp approximation for the Gamma function related to Burnside's formula, Ramanujan J., 2014, 35, 121–129.

[25]　Lu D., Song L., Ma C., Some new asymptotic approximations of the gamma function based on Nemes' formula, Ramanujan's formula and Burnside's formula, Appl. Math. Comput., 2015, 253, 1–7.

[26]　Abramowttz M., Stegun I. A., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, 1972, New York: Dover.

[27]　Koumandos S., Remarks on some completely monotonic functions, J. Math. Anal. Appl., 2006, 324, 1458–1461.

[28]　Chen C.-P., Lin L., Remarks on asymptotic expansions for the gamma function, Appl. Math. Lett., 2012, 25, 2322–2326.

[29]　Chen C.-P., Elezović N., Vukšić L., Asymptotic formulae associated with the Wallis power function and digamma function, J. Classical Anal., 2013, 2, 151–166.

[30]　Chen C.-P., Unified treatment of several asymptotic formulas for the gamma function, Numer. Algor., 2013, 64, 311–319.

[31]　Lu D., A generated approximation related to Burnside's formula, J. Number Theory, 2014, 136, 414–422.

[32] Lu D., Song L., Ma C., A generated approximation of the gamma function related to Windschitl's formula, J. Number Theory, 2014, 140, 215–225.

[33] Hirschhorn M. D., Villarino M. B., A refinement of Ramanujan's factorial approximation, Ramanujan J., 2014, 34, 73–81.

[34] Chen C.-P., Asymptotic expansions of the gamma function related to Windschitl's formula, Appl. Math. Comput., 2014, 245, 174–180.

[35] Chen C.-P., Liu J.-Y., Inequalities and asymptotic expansions for the gamma function, J. Number Theory, 2015, 149, 313–326.

[36] Chen C.-P., Paris R. B., Inequalities, asymptotic expansions and completely monotonic functions related to the gamma function, Appl. Math. Comput., 2015, 250, 514–529.

[37] Lin L., Chen C.-P., Asymptotic formulas for the gamma function by Gosper, J. Math. Inequal., 2015, 9, 541–551.

[38] Mortici C., A new fast asymptotic series for the gamma function, Ramanujan J., 2015, 38, 549–559.

[39] Yang Zh., Tian J.-F., A comparison theorem for two divided differences and applications to special functions, J. Math. Anal. Appl., 2018, 464, 580–595.

[40] Tian J., Wang W., Cheung W.-S., Periodic boundary value problems for first-order impulsive difference equations with time delay, Adv. Difference Equ., 2018, 2018, 79.

[41] Tian J.-F., Triple Diamond-Alpha integral and Hölder-type inequalities, J. Inequal. Appl., 2018, 2018, 111.

[42] Tian J.-F., Ha M.-H., Wang Ch., Improvements of generalized Hölder's inequalities and their applications, J. Math. Inequal., 2018, 12, 459–471.

[43] Yang Zh.-H., Chu Y.-M., Tao X.-J, A double inequality for the trigamma function and its applications, Abstr. Appl. Anal., 2014, 2014, Art. ID 702718, 9 pages.

[44] Yang Zh.-H., Tian J., Monotonicity and sharp inequalities related to gamma function, J. Math. Inequal., 2018, 12, 1–22.

[45] Yang Zh.-H., Tian J., Convexity and monotonicity for the elliptic integrals of the first kind and applications, arXiv:1705.05703 [math.CA], https://arxiv.org/abs/1705.05703.

[46] Dattoli G., Cesarano C., Sacchetti D., A note on truncated polynomials, Appl. Math. Comput., 2003, 134, 595–605.