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Transitivity of the ε_m -relation on (m -idempotent) hyperrings

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Abstract: On a general hyperring, there is a fundamental relation, denoted γ^* , such that the quotient set is a classical ring. In a previous paper, the authors defined the relation ε_m on general hyperrings, proving that its transitive closure ε_m^* is a strongly regular equivalence relation smaller than the γ^* -relation on some classes of hyperrings, such that the associated quotient structure modulo ε_m^* is an ordinary ring. Thus, on such hyperrings, ε_m^* is a fundamental relation. In this paper, we discuss the transitivity conditions of the ε_m -relation on hyperrings and m -idempotent hyperrings.

Keywords: Hyperring, m -idempotent hyperring, m -complete part

MSC: 20N20, 16Y99

1 Introduction

The quotient set has played an important role in the algebraic hyperstructures theory since its beginning for at least two reasons. The first one concerns the motivation of the definition of hypergroup, very well pointed out by F. Marty in his pioneering paper on hypergroups from 1934. It is well known that the quotient of a group G by an arbitrary subgroup H of G is a group if and only if H is a normal subgroup, while Marty showed that the quotient structure G/H is always a hypergroup. More generally, as Vougiouklis proved in [1], if one factorizes the group G by any partition S of G , then the quotient G/S is an H_v -group (i.e. a reproductive hypergroupoid satisfying the weak associativity). Secondly, the quotient set represents the bridging element between the classical algebraic structures and the corresponding hyperstructures, as mentioned in [2]. The first step in this direction was made by Koskas [3], when he used the β -relation and its transitive closure β^* to obtain a group (as a quotient structure of a hypergroup modulo β^*). Later on, the study of this correspondence between classical structures and hyperstructures with similar behaviour has been extended and new equivalence relations have been defined and called *fundamental relations*. They are the smallest strongly regular relations defined on a hyperstructure such that the quotient set is a classical structure, having similar properties. If on a (semi)hypergroup one considers the β^* -relation, then the quotient set is a (semi)group. Besides, the quotient set modulo the γ^* -relation, introduced by Freni [4], is a commutative (semi)group. Similarly, other fundamental relations have been defined on hypergroups in order to obtain nilpotent groups [5], engel groups [6], or solvable groups [7]. The same approach was used also for ring-like hyperstructures. It started in 1991, when Vougiouklis [1] defined the γ -relation on a general hyperring R (addition and multiplications are both hyperoperations) such that the quotient R/γ^* is a ring. Even if they are denoted in the same way (this could

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create confusion for the new readers of the algebraic hyperstructure theory, while it is already accepted for the researchers of this field), the fundamental relation defined by Freni [4] on semihypergroups is different by the fundamental relation γ defined by Vougiouklis [1] on hyperrings. Later on, the α^* -relation [8] has been introduced to obtain a commutative ring. More recently, other fundamental relations have been defined obtaining Boolean rings [9] or commutative rings with identity [10] as associated quotient structures. We end this brief recall of the fundamental relations with those in hypermodule theory, where, for example, the θ -fundamental relation [11] leads to commutative modules by the same method of factorization.

The authors of this note proposed in [12] a new perspective of the study of fundamental relations on hyperstructures. The γ^* -relation defined on a general hyperring R is the smallest strongly regular relation such that the quotient R/γ^* is a ring. The paper [12] deals with the question: Under which conditions can a fundamental relation smaller than γ be defined on a general hyperring, such that its transitive closer behaves similar to γ^* ? To answer to this question, the ε_m -relation was defined on a special class of (semi)hyperrings, such that $\varepsilon_m \subsetneq \gamma$ and the quotient structure modulo ε_m^* is an ordinary (semi)ring. Moreover, on m -idempotent hyperrings it was proved that $\varepsilon_m^* = \gamma^*$.

In this paper, we study the transitivity property of the ε_m -relation on general hyperrings. First, we introduce the notion of m -complete parts based on the ε_m -relation and investigate their properties, which help us to show that ε_m is transitive on m -idempotent hyperfields.

2 Regular and fundamental relations on hyperstructures

In this section we review some basic definitions and properties regarding fundamental relations on general hyperrings. For further details, the readers are referred to [2], [10, 13–15, 17].

Definition 2.1 ([16]). *An algebraic system $(R, +, \cdot)$ is said to be a general hyperring (by short a hyperring), if $(R, +)$ is a hypergroup, $(R, \cdot)$ is a semihypergroup, and “ $\cdot$ ” is distributive with respect to “ $+$ ”.*

In the above definition, if $(R, +)$ is a semihypergroup, then $(R, +, \cdot)$ is called a *semihyperring*. A nonempty subset I of a hyperring $(R, +, \cdot)$ is a *hyperideal*, if $(I, +)$ is a subhypergroup of $(R, +)$ and, for all $x \in I$ and $r \in R$, we have $r \cdot x \cup x \cdot r \subseteq I$.

We recall that a subhypergroup A of $(R, \cdot)$ is said to be *invertible on the left (on the right)*, if $x \in A \cdot y$ ($x \in y \cdot A$), then $y \in A \cdot x$ ($y \in x \cdot A$), for all $x, y \in R$. A subhypergroup is *invertible*, if it is invertible on the left and on the right. Moreover, a subhypergroup B of $(R, \cdot)$ is called *closed on the left (on the right)*, if $x \in a \cdot y$ ($x \in y \cdot a$) implies that $a \in B$, for every $a \in R$ and $x, y \in B$. We say B is *closed*, if it is closed on the left and on the right. It is easy to see that every invertible subhypergroup of $(R, \cdot)$ is closed.

Let ρ be an equivalence relation on a hypergroup $(H, \circ)$. For $A, B \subseteq H$, $A\bar{\rho}B$ means that, for all $x \in A$ there exists $y \in B$ such that $x\rho y$, and for all $v \in B$ there exists $u \in A$ such that $u\rho v$. Moreover, $A\bar{\bar{\rho}}B$ means that for all $x \in A$ and for all $y \in B$, we have $x\rho y$. Accordingly, an equivalence relation ρ on a hypergroup $(H, \circ)$ is called *regular* if $a\rho b$ and $c\rho d$ imply $(a \circ c) \bar{\rho} (b \circ d)$, for $a, b, c, d \in R$. Besides, ρ is called *strongly regular* if, under the same conditions, we have $(a \circ c) \bar{\bar{\rho}} (b \circ d)$, for $a, b, c, d \in R$.

The main role of the (strongly) regular relations on hypergroups is reflected by the following result.

Theorem 2.2 ([17]). *Consider the equivalence relation ρ on the hypergroup $(H, \circ)$ and the hyperoperation $\rho(x) \otimes \rho(y) = \{\rho(z) \mid z \in \rho(x) \circ \rho(y)\}$ on the quotient $H/\rho = \{\rho(x) \mid x \in H\}$. Then ρ is regular (strongly regular) on H if and only if $(H/\rho, \otimes)$ is a hypergroup (group).*

An equivalence relation ρ is (strongly) regular on a hyperring $(R, +, \cdot)$, if it is (strongly) regular with respect to both hyperoperations “ $+$ ” and “ $\cdot$ ”. One example of strongly regular relation on (semi)hyperrings is the γ -relation defined by Vougiouklis in [1] as follows. Let $(R, +, \cdot)$ be a (semi)hyperring and $x, y \in R$. Then $x\gamma y$ if and only if $\{x, y\} \subseteq u$, where u is a finite sum of finite products of elements of R . In other words, $x\gamma y$ if

and only if $\{x, y\} \subseteq \sum_{j \in J} \left(\prod_{i \in I_j} z_i \right)$, for some finite sets of indices J and I_j and elements $z_i \in R$. Let γ^* be the transitive closure of γ , that is $x\gamma^*y$ if and only if there exist the elements $z_1, \dots, z_{n+1} \in R$, with $z_1 = x$ and $z_{n+1} = y$, such that $z_i\gamma z_{i+1}$, for $i \in \{1, \dots, n\}$. In [1] it was shown that γ^* is the smallest strongly regular relation on a hyperring R such that the quotient $(R/\gamma^*, \oplus, \odot)$ is a classical ring with the operations defined as: $\gamma^*(x) \oplus \gamma^*(y) = \gamma^*(z)$, for all $z \in \gamma^*(x) + \gamma^*(y)$ and $\gamma^*(x) \odot \gamma^*(y) = \gamma^*(t)$, for all $t \in \gamma^*(x) \cdot \gamma^*(y)$. Hence, $(R/\gamma^*, \oplus, \odot)$ is called the *fundamental ring* obtained by the factorization with the γ^* -relation.

3 The ε_m -relation on hyperrings

In [12] the authors defined on (semi)hyperrings a new relation, denoted by ε_m , smaller than the γ -relation, and which is not transitive in general. Thus they found some conditions for the transitivity of the ε_m -relation on hyperrings. In this section we recall its definition and main properties.

Let $(R, +, \cdot)$ be a semihyperring and select a constant m , such that $2 \leq m \in \mathbb{N}$. Put $\{(x, x) \mid x \in R\} \subseteq \varepsilon_m$ and for all $a, b \in R$ define

$$a\varepsilon_m b \iff \exists n \in \mathbb{N}, \exists (z_1, \dots, z_n) \in R^n : \{a, b\} \subseteq \sum_{i=1}^n z_i^m, \quad (1)$$

where $z_i^m = \underbrace{z_i \cdot z_i \cdot \dots \cdot z_i}_m$.

Now, let $(R, +, \cdot)$ be a hyperring such that $(R, \cdot)$ is commutative and the following implication holds:

$$B \subseteq \sum_{i=1}^n A_i^m \implies \exists x_i \in A_i \ (1 \leq i \leq n) : B \subseteq \sum_{i=1}^n x_i^m, \quad (2)$$

for all $B, A_1, \dots, A_n \subseteq R$. Accordingly with Theorems 3.3 and 3.4 in [12], on a hyperring R satisfying condition (2), the relation ε_m^* is the smallest strongly regular equivalence relation such that the quotient set R/ε_m^* is a ring, thus it is a fundamental relation on R . Besides, we note that relation (2) is valid if and only if, for all $A_1, \dots, A_n \subseteq R$, there exists $x_i \in A_i$ ($1 \leq i \leq n$) such that $\sum_{i=1}^n A_i^m \subseteq \sum_{i=1}^n x_i^m$.

The next result provides sufficient conditions for the transitivity of the relation ε_m .

Theorem 3.1 ([12]). *Let $(R, +, \cdot)$ be a hyperring satisfying the relation (2) such that there exists $0 \in R$ such that $x + 0 = \{x\}$ and $x \cdot 0 = \{0\}$ for all $x \in R$. If $A_1, \dots, A_n$ are hyperideals of R , then $X = \bigcup_{i=1}^n A_i^m \mid A_1, \dots, A_n \subseteq R$ is an equivalence class of ε_m^* and ε_m is transitive.*

Example 3.2. Define on $R = \{0, a, b\}$ two hyperoperations as follows:

$+$	0	a	b	$\cdot$	0	a	b
0	{0}	{a}	{b}	0	{0}	{0}	{0}
a	{a}	{a, b}	R	a	{0}	R	R
b	{b}	R	{a, b}	b	{0}	R	R

Then, $(R, +, \cdot)$ is a hyperring [18].

It is easy to check that, for all $A_1, \dots, A_n \subseteq R$, there exist $x_i \in A_i$ ($1 \leq i \leq n$), such that $\sum_{i=1}^n A_i^m \subseteq \sum_{i=1}^n x_i^m$, relation equivalently with (2).

We end this section emphasizing the fact that if the hyperring $(R, +, \cdot)$ does not satisfy condition (2), then the relation ε_m is not transitive, while its transitive closure ε_m^* is not strongly regular on $(R, \cdot)$ [12].

4 Transitivity of the relation ε_m on m -idempotent hyperfields

Since the conditions in Theorem 3.1 are not immediate, we aim to find some particular hyperrings, where the relation ε_m is transitive. For doing this, we will first define the concept of m -complete part and then we will prove that ε_m is transitive on m -idempotent hyperfields.

The main role of the complete parts of a semihypergroup, introduced by Koskas [3] and very well recalled by Antampoufis et al. in the survey [2], is played in finding the β^* class of each element. In particular, a nonempty subset A of a semihypergroup $(H, \cdot)$ is called a *complete part* of H if, for any nonzero natural number n and any elements $a_1, \dots, a_n$ of H , the following implication holds:

$$A \cap \prod_{i=1}^n a_i \neq \emptyset \Rightarrow \prod_{i=1}^n a_i \subseteq A.$$

In other words, the complete part A absorbs every hyperproduct containing at least one element of A . In particular, for any element $x \in A$, the class $\beta^*(x)$ is a complete part of H . Moreover, the intersection of all complete parts of H containing A is called the *complete closure* of A in H , denoted by $C(A)$. Besides, $\beta^*(x) = C(x)$, for any $x \in H$.

As already mentioned before, Vougiouklis [16] defined the relation γ on a hyperring R , proving that its transitive closure γ^* is the smallest strongly regular relation defined on R such that the quotient R/γ^* is a ring. Later on Mirvakili et al. [19] studied the transitivity property of this relation, introducing the notion of *complete part* on hyperrings as follows: a nonempty subset M of a hyperring R is a complete part if, for any natural number n , any $i = 1, 2, \dots, n$, any natural number k_i and arbitrary elements $z_{i1}, \dots, z_{ik_i} \in R$, we have

$$M \cap \sum_{i=1}^n \left(\prod_{j=1}^{k_i} z_{ij} \right) \neq \emptyset \Rightarrow \sum_{i=1}^n \left(\prod_{j=1}^{k_i} z_{ij} \right) \subseteq M.$$

Now we will extend these definitions to the case of hyperrings, aiming to prove that the class $\varepsilon^*(x)$ of an element x in the hyperring R is an m -complete part of R .

Definition 4.1. We say that a nonempty subset A of a (semi)hyperring $(R, +, \cdot)$ is an m -complete part of R if $A \cap \sum_{i=1}^n z_i^m \neq \emptyset$ implies that $\sum_{i=1}^n z_i^m \subseteq A$, for all $n \in \mathbb{N}$ and $z_1, \dots, z_n \in R$. The intersection of all m -complete parts of R containing a nonempty subset A of R is called the m -complete closure of A and it is denoted by $C_m(A)$.

Example 4.2. Consider the following hyperoperations on the set $R = \{a, b, c, d\}$:

$+$	a	b	c	d	$\cdot$	a	b	c	d
a	$\{b, c\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	a	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	b	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	c	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$
d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	d	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$	$\{b, d\}$

Then $(R, +, \cdot)$ is a semihyperring. For every $m \geq 2$ and for all $z_1, \dots, z_n \in R$, we have $\sum_{i=1}^n z_i^m = \{b, d\}$. It follows that the subsets $A_1 = \{b, d\}$, $A_2 = \{a, b, d\}$ and $A_3 = \{c, b, d\}$ are all proper m -complete parts of R , i.e. m -complete parts of R , different by R .

Theorem 4.3. Let ρ be a strongly regular equivalence relation on R . Then $\rho(a)$ is an m -complete part of R , for all $a \in R$.

Proof. Since ρ is a strongly regular relation on R , it follows that the quotient R/ρ is a ring (with the addition " $\oplus$ " and the multiplication " $\odot$ "). Let $a \in R$ and $\rho(a) \cap \sum_{i=1}^n z_i^m \neq \emptyset$, for arbitrary elements $z_1, \dots, z_n \in R$. Hence,

there exists $y \in \sum_{i=1}^n z_i^m$ such that $\rho(y) = \rho(a)$. Consider the strong homomorphism $\pi : R \rightarrow R/\rho$ defined by $\pi(x) = \rho(x)$, for all $x \in R$, where R/ρ is a ring (a trivial hyperring). Thus,

$$\pi\left(\sum_{i=1}^n z_i^m\right) = \bigoplus_{i=1}^n \underbrace{(\rho(z_i) \odot \dots \odot \rho(z_i))}_{m \text{ times}} = \rho(y) = \rho(a),$$

which implies that $\sum_{i=1}^n z_i^m \subseteq \rho(a)$. This completes the proof. $\square$

For a nonempty subset A of a (semi)hyperring R , denote

$$\begin{aligned} K_1^m(A) &= A \\ K_{n+1}^m(A) &= \left\{ x \in R \mid \exists (z_1, \dots, z_n) \in R^n ; x \in \sum_{i=1}^n z_i^m \text{ and } K_n^m(A) \cap \sum_{i=1}^n z_i^m \neq \emptyset \right\} \\ K^m(A) &= \bigcup_{n \geq 1} K_n^m(A). \end{aligned}$$

Moreover, for any $x \in R$ and any natural number n , for simplicity we denote $K_n^m(\{x\}) = K_n^m(x)$.

Lemma 4.4. *For any nonempty subset A of a hyperring R , the set $K^m(A)$ is an m -complete part of R .*

Proof. Let $K^m(A) \cap \sum_{i=1}^n z_i^m \neq \emptyset$, for arbitrary elements $z_1, \dots, z_n \in R$. Then, there exists $t \in \mathbb{N}$ such that $K_t^m(A) \cap \sum_{i=1}^n z_i^m \neq \emptyset$, which implies that $\sum_{i=1}^n z_i^m \subseteq K_{t+1}^m(A) \subseteq K^m(A)$. Thus, $K^m(A)$ is an m -complete part of R , containing A . $\square$

Theorem 4.5. $K^m(A) = C_m(A)$, for any nonempty subset A of R .

Proof. By Lemma 4.4, we have $C_m(A) \subseteq K^m(A)$. Now, let M be an m -complete part of R containing A . Clearly, $K_1^m(A) = A \subseteq M$. Suppose that $K_n^m(A) \subseteq M$. Let $x \in K_{n+1}^m(A)$. Then there exist the elements $z_1, \dots, z_n \in R$ such that $x \in \sum_{i=1}^n z_i^m$ and $\emptyset \neq K_n^m(A) \cap \sum_{i=1}^n z_i^m \subseteq M \cap \sum_{i=1}^n z_i^m$. Since M is an m -complete part, it follows that $\sum_{i=1}^n z_i^m \subseteq M$ and so $K_{n+1}^m(A) \subseteq M$. Hence, $K^m(A) \subseteq M$, and thus $K^m(A) \subseteq C_m(A)$. $\square$

Example 4.6. *Consider the semihyperring R in Example 4.2. One obtains that the m -complete closure of A_2 is A_2 itself, for any natural number $m \geq 2$. Moreover, if we consider the γ -relation on R , then we have $\sum \prod z_i = \{b, c\} := P$ or $\sum \prod z_i = \{b, d\} := Q$, for any finite hypersums of finite hyperproducts of elements $z_i \in R$. Since $P \cap Q \neq \emptyset$ and $P \not\subseteq Q$ and $Q \not\subseteq P$, then P and Q are not complete parts of R . Besides, $A_2 \cap P \neq \emptyset$, but $P \not\subseteq A_2$. This means that A_2 is not a complete part, but only an m -complete part.*

Theorem 4.7. *For all nonempty subsets A of R , it holds $C_m(A) = \bigcup_{a \in A} C_m(a)$.*

Proof. Clearly we have the inclusion $C_m(a) \subseteq C_m(A)$, for all $a \in A$. Hence, $\bigcup_{a \in A} C_m(a) \subseteq C_m(A)$.

Conversely, we show that $K_n^m(A) \subseteq \bigcup_{a \in A} K_n^m(a)$, by induction on " n ". For $n = 1$, we have $K_1^m(A) = A = \bigcup_{a \in A} \{a\} = \bigcup_{a \in A} K_1^m(a)$. Now, suppose that $K_n^m(A) \subseteq \bigcup_{a \in A} K_n^m(a)$ and take an arbitrary $x \in K_{n+1}^m(A)$. Then, $x \in \sum_{i=1}^n z_i^m$ and $\emptyset \neq K_n^m(A) \cap \sum_{i=1}^n z_i^m$, for some elements $z_1, \dots, z_n \in R$. Thus, there exists $y \in R$ such that $x, y \in \sum_{i=1}^n z_i^m$ and $y \in K_n^m(A) \subseteq \bigcup_{a \in A} K_n^m(a)$. This implies that there exists $a' \in A$ such that $y \in K_n^m(a') \cap \sum_{i=1}^n z_i^m \neq \emptyset$, meaning that $x \in K_{n+1}^m(a')$, and thus $K_{n+1}^m(A) \subseteq \bigcup_{a \in A} K_{n+1}^m(a)$.

If $x \in \mathcal{C}_m(A)$, it follows that $x \in K^m(A) = \bigcup_{n \geq 1} K_n^m(A) \subseteq \bigcup_{n \geq 1} \left(\bigcup_{a \in A} K_n^m(a) \right)$. Then, for some $a' \in A$ and $n \geq 1$, we have $x \in K_n^m(a') \subseteq K^m(a') = \mathcal{C}_m(a') \subseteq \bigcup_{a \in A} \mathcal{C}_m(a)$. This completes the proof. $\square$

In the following we will give an equivalent description of the relation ε_m^* on hyperrings, using the notion of m -complete part. First we will prove some properties of the m -complete parts.

Lemma 4.8. $K_n^m(K_2^m(x)) = K_{n+1}^m(x)$, for all $x \in R$ and $n \geq 2$.

Proof. We prove it by induction on " n ". For $n = 2$, we have

$$\begin{aligned} K_2^m(K_2^m(x)) &= \left\{ x \in R \mid \exists (z_1, \dots, z_n) \in R^n; x \in \sum_{i=1}^n z_i^m \text{ and } K_1^m(K_2^m(x)) \cap \sum_{i=1}^n z_i^m \neq \emptyset \right\} \\ &= \left\{ x \in R \mid \exists (z_1, \dots, z_n) \in R^n; x \in \sum_{i=1}^n z_i^m \text{ and } K_2^m(x) \cap \sum_{i=1}^n z_i^m \neq \emptyset \right\} \\ &= K_3^m(x). \end{aligned}$$

Now, suppose that $K_{n-1}^m(K_2^m(x)) = K_n^m(x)$ and take an arbitrary element $y \in K_n^m(K_2^m(x))$. Then there exist $z_1, \dots, z_n \in R$ such that $y \in \sum_{i=1}^n z_i^m$ and $\emptyset \neq K_{n-1}^m(K_2^m(x)) \cap \sum_{i=1}^n z_i^m$. Thus, $\emptyset \neq K_n^m(x) \cap \sum_{i=1}^n z_i^m$, and so $y \in K_{n+1}^m(x)$. Hence, $K_n^m(K_2^m(x)) \subseteq K_{n+1}^m(x)$. Similarly, we have $K_{n+1}^m(x) \subseteq K_n^m(K_2^m(x))$, that completes the proof. $\square$

Lemma 4.9. For all $n \geq 2$ and $x, y \in R$, $x \in K_n^m(y)$ if and only if $y \in K_n^m(x)$.

Proof. We prove the result by induction on " n ". Let $n = 2$. Then $x \in K_2^m(y)$ if and only if, for $z_1, \dots, z_n \in R$, we have $x \in \sum_{i=1}^n z_i^m$ and $\emptyset \neq \{y\} \cap \sum_{i=1}^n z_i^m$, meaning that, for $z_1, \dots, z_n \in R$, $y \in \sum_{i=1}^n z_i^m$ and $\emptyset \neq \{x\} \cap \sum_{i=1}^n z_i^m$. This is equivalent with $y \in K_2^m(x)$. Suppose now that $x \in K_{n-1}^m(y)$ if and only if $y \in K_{n-1}^m(x)$ and take $x \in K_n^m(y)$; then there exist $z, z_1, \dots, z_n \in R$ such that $x \in \sum_{i=1}^n z_i^m$, $z \in K_{n-1}^m(y)$ and $z \in \sum_{i=1}^n z_i^m$. Hence, by induction procedure, $y \in K_{n-1}^m(z)$ and $z \in K_2^m(x)$, which implies that $y \in K_{n-1}^m(K_2^m(x)) = K_n^m(x)$ by Lemma 4.8. Similarly, $y \in K_n^m(x)$ implies that $x \in K_n^m(y)$. $\square$

Define on a hyperring R the relation θ as follows: $x\theta y$ if and only if $x \in K^m(y)$, for all $x, y \in R$.

Corollary 4.10. The relation θ is an equivalence on R .

Proof. For all $x \in R$, we have $x \in K_1^m(x) \subseteq K^m(x)$. Hence, θ is reflexive. Now, let $x \in K^m(y)$, for $x, y \in R$. By Theorem 4.5, there exists $n \geq 1$ such that $x \in K_n^m(y)$, which implies that $y \in K_n^m(x)$, by Lemma 4.9. Then, $y \in K_n^m(x) \subseteq K^m(x)$. Similarly, the converse is valid. Thus, θ is symmetric. Moreover, let $x\theta y$ and $y\theta z$ for $x, y, z \in R$. Hence, $x \in \mathcal{C}_m(y)$ and $y \in \mathcal{C}_m(z)$. Let A be an m -complete part of R containing z . Since $y \in \mathcal{C}_m(z)$ and $\mathcal{C}_m(z) \subseteq A$, it follows that $y \in A$. Hence, $\mathcal{C}_m(y) \subseteq A$ and thus $x \in A$. Therefore, $x \in \bigcap_{z \in A} A = \mathcal{C}_m(z) = K^m(z)$ and so $x\theta z$. $\square$

Theorem 4.11. For all $x \in R$, $\varepsilon_m^*(x) = \theta(x)$.

Proof. If $x \varepsilon_m y$, then $x \in \sum_{i=1}^n z_i^m$ and $\emptyset \neq \{y\} \cap \sum_{i=1}^n z_i^m$, for some elements $z_1, \dots, z_n \in R$. Hence, $x \in K_2^m(y) \subseteq K^m(y)$. Thus, $\varepsilon_m \subseteq \theta$ and $\varepsilon_m^* \subseteq \theta^* = \theta$.

Conversely, let $x\theta y$, that is, $x \in K^m(y)$, which implies that $x \in K_{n+1}^m(y)$, for $n \in \mathbb{N}$. Then, there exist $x_1, z_1, \dots, z_n \in R$ such that $x \in \sum_{i=1}^n z_i^m$, $x_1 \in K_n^m(y)$ and $x_1 \in \sum_{i=1}^n z_i^m$. Hence, $\{x, x_1\} \subseteq \sum_{i=1}^n z_i^m$ and so $x \varepsilon_m x_1$. Similarly, since $x_1 \in K_n^m(y)$, there exists $x_2 \in K_{n-1}^m(y)$ such that $x_1 \varepsilon_m x_2$. By continuing this process, we can obtain $x_3 \in K_{n-2}^m(y), \dots, x_{n-1} \in K_2^m(y)$ and $x_n \in K_1^m(y) = \{y\}$ such that $x \varepsilon_m x_2 \varepsilon_m x_3 \dots \varepsilon_m x_{n-1} \varepsilon_m x_n = y$. Hence, $x \varepsilon_m^* y$ and so $\theta \subseteq \varepsilon_m^*$. Therefore the proof is completed. $\square$

Now, we recall that a hyperring $(R, +, \cdot)$ is said to be a *hyperfield*, if $(R, \cdot)$ is a hypergroup. Moreover, a *strong homomorphism* from a hyperring $(R, +, \cdot)$ to a hyperring $(S, \oplus, \odot)$ is a map $f : R \rightarrow S$ such that $f(x + y) = f(x) \oplus f(y)$ and $f(x \cdot y) = f(x) \odot f(y)$, for all $x, y \in R$. Considering the ε_m relation on R , it can be seen that the map $\varphi_m : R \rightarrow R/\varepsilon_m^*$ is a strong homomorphism.

In the following we will consider R a hyperfield satisfying relation (2) (this is a crucial assumption in the proofs of the next results) such that R/ε_m^* has a unit element denoted by $1_{R/\varepsilon_m^*}$. Set $\omega_R^m = \varphi_m^{-1}(1_{R/\varepsilon_m^*}) = \{x \in R \mid \varphi_m(x) = 1_{R/\varepsilon_m^*}\}$. We will state some properties of the m -complete parts of hyperfields satisfying relation (2).

Theorem 4.12. *If $(R, +, \cdot)$ is a hyperfield and $A \subseteq R$, then $\varphi_m^{-1}(\varphi_m(A)) = A \cdot \omega_R^m$.*

Proof. Let $x \in \varphi_m^{-1}(\varphi_m(A))$. Then there exists $y \in A$ such that $\varphi_m(x) = \varphi_m(y)$. Since $(R, \cdot)$ is a hypergroup, it follows that there exists $t \in R$ such that $x \in y \cdot t$, which implies that $\varphi_m(x) = \varphi_m(y) \odot \varphi_m(t)$. Since $(R/\varepsilon_m^*, \odot)$ is a group and $\varphi_m(x) = \varphi_m(y)$, it results $\varphi_m(t) = 1_{R/\varepsilon_m^*}$. Thus, $t \in \varphi_m^{-1}(1_{R/\varepsilon_m^*}) = \omega_R^m$, and so $x \in y \cdot t \subseteq A \cdot \omega_R^m$. Then $\varphi_m^{-1}(\varphi_m(A)) \subseteq A \cdot \omega_R^m$.

Conversely, let $x \in A \cdot \omega_R^m$. Then $x \in a \cdot y$, for some $a \in A$ and $y \in \omega_R^m$, that is $\varphi_m(y) = 1_{R/\varepsilon_m^*}$. Hence $\varphi_m(x) = \varphi_m(a \cdot y) = \varphi_m(a) \odot \varphi_m(y) = \varphi_m(a) \odot 1_{R/\varepsilon_m^*} = \varphi_m(a) \in \varphi_m(A)$, and so $x \in \varphi_m^{-1}(\varphi_m(A))$. This completes the proof. $\square$

Theorem 4.13. *If $(R, +, \cdot)$ is a hyperfield and $A \subseteq R$, then $C_m(A) = A \cdot \omega_R^m$.*

Proof. It is easy to see that $\varphi_m^{-1}(\varphi_m(A)) = \{x \in R \mid \exists a \in A : \varphi_m(x) = \varphi_m(a)\}$. Also, we have $\varphi_m(x) = \varphi_m(a)$ if and only if $\varepsilon_m^*(x) = \varepsilon_m^*(a)$, equivalently with $\theta(x) = \theta(a)$, meaning that $x \in K^m(a) = C_m(a)$, by Theorem 4.11. Hence, $x \in C_m(A)$ if and only if $x \in \varphi_m^{-1}(\varphi_m(A))$, thus $x \in A \cdot \omega_R^m$, by Theorem 4.7 and Theorem 4.12. Therefore, $C_m(A) = A \cdot \omega_R^m$. $\square$

Corollary 4.14. *Let R be a hyperfield. A is an m -complete part of R if and only if $A = A \cdot \omega_R^m$.*

Proof. If A is an m -complete part, then $C_m(A) = A$. Hence, $A = C_m(A) = A \cdot \omega_R^m$, by Theorem 4.13. Conversely, if $A = A \cdot \omega_R^m$, then $A = C_m(A)$ by Theorem 4.13, and so A is an m -complete part of R . $\square$

By Theorem 4.12, we have $\omega_R^m \cdot \omega_R^m = \varphi_m^{-1}(\varphi_m(\omega_R^m)) = \omega_R^m$. Hence, ω_R^m is an m -complete part of the hyperfield $(R, +, \cdot)$, by Corollary 4.14.

Moreover, notice that, for two subsets A and B of the hyperfield R such that one of them is an m -complete part of R (assume that A is so), we have $(A \cdot B) \cdot \omega_R^m = (A \cdot \omega_R^m) \cdot B = A \cdot B$, by Corollary 4.14. Hence, $A \cdot B$ is an m -complete part of R , by Corollary 4.14.

Theorem 4.15. *Let $(R, +, \cdot)$ be a hyperfield. Then every m -complete part subhypergroup of $(R, \cdot)$ is invertible. Moreover, it is closed.*

Proof. Let A be an m -complete part of R such that $(A, \cdot)$ is a subhypergroup of $(R, \cdot)$. Take $x \in A \cdot y$ for $x, y \in R$. Thus, $x \in a \cdot y$, for $a \in A$, which implies that $\varphi_m(x) = \varphi_m(a) \odot \varphi_m(y)$. Since $\varphi_m(A)$ is a subgroup of R/ε_m^* , we have $\varphi_m(y) = \varphi_m(a)^{-1} \odot \varphi_m(x) \in \varphi_m(A) \odot \varphi_m(x) = \varphi_m(A \cdot x)$. Besides, $A \cdot x$ is an m -complete part of R , hence $y \in \varphi_m^{-1}(\varphi_m(A \cdot x)) = C_m(A \cdot x) = A \cdot x$. Then, A is invertible on the left. Similarly, we can show that A is invertible on the right. Therefore, A is invertible, and by consequence it is also closed. $\square$

Theorem 4.16. *Let $(R, +, \cdot)$ be a hyperfield and $\mathcal{S}_{C_m}(R)$ be the set of all m -complete parts of R which are subhypergroups of $(R, \cdot)$. Then, $\omega_R^m = \bigcap_{A \in \mathcal{S}_{C_m}(R)} A$.*

Proof. We know that ω_R^m is an m -complete part of R . Let $x \in \omega_R^m$. For all $t, y \in \omega_R^m$, we have $\varphi_m(x) = 1_{R/\varepsilon_m^*} = \varphi_m(t) \odot \varphi_m(y) = \varphi_m(t \cdot y)$. Hence, $x \in \varphi_m^{-1}(\varphi_m(t \cdot y)) = C_m(t \cdot y) = t \cdot y$, and so $\omega_R^m \subseteq t \cdot \omega_R^m$ for all $t \in \omega_R^m$.

Clearly, $t \cdot \omega_R^m \subseteq \omega_R^m$. Then $\omega_R^m = t \cdot \omega_R^m$, for all $t \in \omega_R^m$, which implies that ω_R^m is a subhypergroup of $(R, \cdot)$. Thus, $\bigcap_{A \in \mathcal{S}_{C_m}(R)} A \subseteq \omega_R^m$. Now, let $A \in \mathcal{S}_{C_m}(R)$. By Corollary 4.14, $A = A \cdot \omega_R^m$. Hence, for every $x \in \omega_R^m$, there exist $a, b \in A$ such that $a \in b \cdot x$, and so $a \in A \cdot x$. By Theorem 4.15, A is invertible, and therefore $x \in A \cdot a \subseteq A$. Then $\omega_R^m \subseteq A$, and thus $\omega_R^m \subseteq \bigcap_{A \in \mathcal{S}_{C_m}(R)} A$. Hence, the proof is complete. $\square$

We recall that a hyperring $(R, +, \cdot)$ is said to be m -idempotent ([12]) if there exists a constant m , $2 \leq m \in \mathbb{N}$, such that $x \in x^m$, for all $x \in R$.

Example 4.17. [12] Consider the Krasner hyperring $R = \{0, a, b\}$ with the hyperaddition and the multiplication defined as follows [20]:

$+$	0	a	b	$\cdot$	0	a	b
0	$\{0\}$	$\{a\}$	$\{b\}$	0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{a, b\}$	R	a	$\{0\}$	$\{b\}$	$\{a\}$
b	$\{b\}$	R	$\{a, b\}$	b	$\{0\}$	$\{a\}$	$\{b\}$

- (1) For every odd number $m \in \mathbb{N}$, we have $0^m = 0$, $a^m = a$ and $b^m = b$. Hence, R is m -idempotent, for all odd natural numbers m .
- (2) Besides, since $a^2 = a \cdot a = b$, it follows that R is not an 2-idempotent hyperring. Similarly, one proves that, for all even numbers $m \in \mathbb{N}$, the hyperring R is not m -idempotent.

Example 4.18. The hyperring defined in Example 3.2 is an m -idempotent hyperring (satisfying relation (2)), for all m , $2 \leq m \in \mathbb{N}$ [12].

Example 4.19. Define on the set $R = \{0, 1\}$ two hyperoperations as follows:

$\boxplus$	0	1	$\boxminus$	0	1
0	$\{0\}$	R	0	$\{0\}$	$\{0\}$
1	R	$\{1\}$	1	$\{0\}$	R

Then, $(R, \boxplus, \boxminus)$ is an m -idempotent hyperring satisfying relation (2), for all m , $2 \leq m \in \mathbb{N}$.

Example 4.20. Similarly, take the same support set $R = \{0, 1\}$ and define on R the two hyperoperations as follows:

$\oplus$	0	1	$\odot$	0	1
0	$\{0\}$	R	0	$\{0\}$	R
1	R	R	1	$\{0\}$	R

The hyperring $(R, \oplus, \odot)$ is m -idempotent, for all m , $2 \leq m \in \mathbb{N}$, and satisfies relation (2).

Now we give an example of m -idempotent hyperfield.

Example 4.21. Define on $R = \{0, 1\}$ two hyperoperations as follows:

$+$	0	1	$\cdot$	0	1
0	$\{0\}$	R	0	$\{0\}$	R
1	R	$\{1\}$	1	R	$\{1\}$

Then, $(R, +, \cdot)$ is an m -idempotent hyperfield satisfying relation (2).

Now, for all $a \in R$, put $\mathcal{X}(a) = \bigcup \left\{ \sum_{i=1}^n A_i^m \mid a \in \sum_{i=1}^n A_i^m \right\}$, where $A_1, \dots, A_n \subseteq R$.

Theorem 4.22. Let R be an m -idempotent hyperfield. Then $\mathcal{X}(a)$ is an m -complete part of R , for all $a \in R$.

Proof. Suppose that $a \in R$ and $\sum_{i=1}^n z_i^m \cap \mathcal{X}(a) \neq \emptyset$, for some $z_1, \dots, z_n \in R$. Then there exists $z \in R$ such that $z \in \sum_{i=1}^n z_i^m$ and $z \in A$, where $A = \sum_{i=1}^n A_i^m$, for $A_1, \dots, A_n \subseteq R$. Since $(R, \cdot)$ is a hypergroup, there exist $w, b \in R$ such that $z_n \in w \cdot a$ and $a \in z \cdot b$. Also, for all $1 \leq i \leq n-1$ we have $z_i \in z_i \cdot R$. Hence,

$$\sum_{i=1}^n z_i^m \subseteq \sum_{i=1}^{n-1} z_i^m + (w \cdot a)^m \subseteq (z_1 \cdot R)^m + \dots + (z_{n-1} \cdot R)^m + (w \cdot A \cdot b)^m,$$

since $z \in A$. Moreover, R is m -idempotent and we have $b \in b^m$, thus

$$\begin{aligned} a \in z \cdot b &\subseteq \left(\sum_{i=1}^n z_i^m \right) \cdot b \subseteq z_1^m \cdot b^m + \dots + z_{n-1}^m \cdot b^m + (w \cdot a \cdot b)^m \\ &\subseteq (z_1 \cdot R)^m + \dots + (z_{n-1} \cdot R)^m + (w \cdot A \cdot b)^m. \end{aligned}$$

Therefore, $(z_1 \cdot R)^m + \dots + (z_{n-1} \cdot R)^m + (w \cdot A \cdot b)^m \subseteq \mathcal{X}(a)$ and so $\sum_{i=1}^n z_i^m \subseteq \mathcal{X}(a)$. Then, $\mathcal{X}(a)$ is an m -complete part of R . $\square$

Theorem 4.23. Let R be an m -idempotent hyperfield. Then $\mathcal{X}(a) = \omega_R^m$, for every $a \in \omega_R^m$.

Proof. It is not difficult to see that $a \cdot \omega_R^m = \omega_R^m$, for all $a \in \omega_R^m$. Hence, $\omega_R^m = a \cdot \omega_R^m = C_m(a) \subseteq \mathcal{X}(a)$, by Theorem 4.13 and Theorem 4.22. Now, let $a \in \omega_R^m$ and $x \in \mathcal{X}(a)$. Then there exists $A = \sum_{i=1}^n A_i^m \subseteq \mathcal{X}(a)$ such that $x \in A$. Since $a \in A$, then $\{x, a\} \subseteq A$. Since R satisfies relation (2), there exist $x_i \in A_i$, for $1 \leq i \leq n$, such that $\{x, a\} \subseteq \sum_{i=1}^n x_i^m$ and so $\varepsilon_m^*(x) = \varepsilon_m^*(a)$. Then $\varphi_m(x) = \varphi_m(a)$. Hence, $x \in \varphi_m^{-1}(\varphi_m(a)) = a \cdot \omega_R^m = \omega_R^m$. Therefore, the proof is complete. $\square$

Theorem 4.24. The relation ε_m is transitive on m -idempotent hyperfields.

Proof. Let R be an m -idempotent hyperfield and $x \varepsilon_m^* y$, for $x, y \in R$. Hence, $x \in \varphi_m^{-1}(\varphi_m(y)) = y \cdot \omega_R^m$. Similarly, we have $y \in x \cdot \omega_R^m$ which implies that $\{x, y\} \subseteq x \cdot \omega_R^m$. Then, there exist $t, z \in \omega_R^m$ such that $x \in x \cdot t$ and $y \in x \cdot z$. By Theorem 4.23, $z \in \omega_R^m = \mathcal{X}(t)$ and thus $\{t, z\} \subseteq A = \sum_{i=1}^n A_i^m$ for $A \subseteq \mathcal{X}(t)$. Since R is m -idempotent, $\{x, y\} \subseteq x \cdot A \subseteq \sum_{i=1}^n (x \cdot A_i)^m$. So, there exist $z_i \in x \cdot A_i$ for every $1 \leq i \leq n$ such that $\{x, y\} \subseteq \sum_{i=1}^n x_i^m$. Therefore, $x \varepsilon_m y$ and so $\varepsilon_m^* = \varepsilon_m$. $\square$

5 Conclusions

The fundamental relation γ^* defined by Vougiouklis [16] on a general hyperring R is the smallest equivalence relation on R such that the quotient structure R/γ^* is a ring. If we consider a special type of hyperrings, i.e. those satisfying relation (2), we can define another fundamental relation on R , ε_m^* -relation [12], smaller than γ^* , while on m -idempotent hyperrings satisfying relation (2), we have $\varepsilon_m^* = \gamma^*$ [12]. In general, ε_m is not transitive. This paper provides a detailed study on the transitivity property of ε_m . Using m -complete parts of a hyperring, we have proved that ε_m is transitive on m -idempotent hyperfields satisfying relation (2).

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