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Regularization and error estimates for an inverse heat problem under the conformable derivative

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Abstract: In this paper we study an inverse time problem for the nonhomogeneous heat equation under the conformable derivative which is a severely ill-posed problem. Using the quasi-boundary value method with two regularization parameters (one related to the error in a measurement process and the other is related to the regularity of the solution) we regularize this problem and obtain a Hölder-type estimation error for the whole time interval. Numerical results are presented to illustrate the accuracy and efficiency of the method.

Keywords: Conformable derivative, Inverse heat problem, Ill-posed problem, Nonhomogeneous heat, Two regularization parameters

MSC: 35R30, 35R35, 58J35

1 Introduction

Partial differential equations (PDEs) arise in the natural sciences, and various boundary value problems for these were widely studied including inverse and ill-posed problems (see, e.g., Tikhonov and Arsenin [1] and Glasko [2]). An example is the backward heat conduction problem (BHCP) and the aim is to detect the previous status of a physical area from present information. The BHCP is a classical ill-posed problem that is difficult to solve since, in general, the solution does not always exist. Furthermore, even if the solution does exist, the continuous dependence of the solution on the data is not guaranteed and numerical calculations are difficult. The BHCP has been considered by many authors using different methods [3]-[10]. In [5], Hao, Duc and Lesnic gave an approximation for this problem using a non-local boundary value problem method, Hao and Duc in [6] used the Tikhonov regularization method to give an approximation for this problem in a Banach space, and Trong and Tuan in [11] used the method of integral equations to regularize the BHCP with a nonlinear right hand side.

Fractional calculus arises in many areas in science and engineering such as aerodynamics and control systems, signal processing, bioengineering and biomedical, viscoelasticity, finance and plasma physics, etc. (see [12]-[14]). For basic information and results we refer the reader to the monographs of Samko et al. [15], Podlubny [16] and Kilbas et al. [17]. Mathematical modeling of many real world phenomena based on definitions of fractional order integrals and derivatives is regarded as more appropriate than ones depending on integer order operators, so as a result fractional differential equations and fractional partial differential

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equations are important fields of research [18]-[21]. In the above works the definition of the fractional used is either the Riemann-Liouville or the Caputo fractional derivative and most works use an integral form for the fractional derivative. Many researchers are interested in the time-inverse problem for the heat equation where the time-derivative is in the Caputo fractional sense. In particular, they consider the problem

$$\begin{cases} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} - u_{xx} = f(x,t), & (x,t) \in (0,\pi) \times (0,T] \\ u(0,t) = u(\pi,t) = 0, & t \in [0,T] \\ u(x,T) = g(x), \end{cases} \tag{1}$$

where $\alpha \in (0,1)$ is the fractional order of derivative and

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u_s(x,s) ds.$$

By a time-inverse problem, we mean that, given information at a specific point of time, say $t = T$, the goal is to recover the corresponding structure at an earlier time $t < T$. When $\alpha = 1$, the problem (1) turns back to the classical ill-posed problem for the well-known heat equation (BCHP). Many researchers have applied different methods to regularize this problem. For example, in [9]- [10] the authors successfully applied various methods to stabilize BCHP and obtained many results on the convergent of the regularized solution to the exact one. In [7]-[8], the authors consider BCHP where the frequency domain is $\mathbb{R}$. Problem (1) with $0 < \alpha < 1$ was studied in [22]-[24] where fundamental contributions were made for problem (1) on existence and uniqueness of solution for this problem. In [25], the authors simplified the Tikhonov regularization method to stabilize problem (1). In [26] the authors consider problem (1) where the data is discrete.

However, there are some setbacks in the approaches of the Riemann-Liouville fractional and the Caputo fractional derivative when modeling real world phenomena (see [27] for a discussion). In [27] the authors gave a new well-behaved simple fractional derivative called "the conformable derivative" depending just on the basic limit definition of the derivative and this concept seems to satisfy all the requirements of the standard derivative. For a function $u : (0,\infty) \to \mathbb{R}$ the conformable derivative of order $\alpha \in (0,1)$ of u at $t > 0$ is defined by

$$D_t^{\alpha} u(t) = \lim_{h \to 0} \frac{u(t + ht^{1-\alpha}) - u(t)}{h}, \quad D_t^{\alpha} u(0) = \lim_{t \to 0^+} D_t^{\alpha} u(t). \tag{2}$$

Note that if u is differentiable, then $D_t^{\alpha} u(t) = t^{1-\alpha} u'(t)$, where $u'(t) = \lim_{h \to 0} [u(t+h) - u(t)]/h$. This concept overcomes the setbacks of the previous concept and this new theory is discussed by Atangana [28] and Abdeljawad [29]. In addition, Anderson and Ulness in [30] provide a potential application of the conformable derivative in quantum mechanics.

For PDEs concerning the conformable derivative there are several studies. In [31], Hammad and Khalil used conformable fourier series to interpret the solution for the conformable heat equation, which is a fundamental equation in mathematical physics. In [32], Chung used the conformable fractional derivative and integral to study fractional Newtonian mechanics, and in addition, the fractional Eule-Lagrange equation was constructed. In [33], Eslami applied the Kudryashov method to obtain the traveling wave solutions to the conformable fractional coupled nonlinear Schrodinger equation. In [34, 35], Çenesiz et al. studied the conformable version of the time-fractional Burgers' equation, the modified Burgers's equation, the Burgers-Korteweg-de Vries equation and the solutions of conformable derivative heat equation. Çenesiz, Kurt and Nane in [36] studied stochastic solutions of conformable fractional Cauchy problems where the space operators may correspond to fractional Brownian motion, or a Levy process. Motivated by the above studies, it is natural to consider the time-inverse problem for the heat equation under the conformable derivative. Throughout this paper, we let $\Omega = [0,a]$, T is a positive number and D_t^{α} is the conformable derivative of order α with respect to t. We begin with the inverse problem in the conformable heat equation.

1.1 The direct problem

Consider the following conformable heat equation

$$D_t^\alpha u(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{3}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{4}$$

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{5}$$

Solving this equation with the given information $f(x,t)$ and $u_0(x)$ is called the direct problem.

1.2 The inverse problem

Consider the following conformable heat equation

$$D_t^\alpha u(x,t) - u_{xx}(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{6}$$

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{7}$$

$$u(x,T) = g(x), \quad x \in \Omega, \tag{8}$$

where $f(x,t) \in C(0,T;L^2(\Omega))$ and $g(x) \in L^2(\Omega)$. From the information given at final time $t = T$, the goal of the inverse problem is to recover the information $u(x,t)$ for $0 \le t < T$. Unfortunately, the inverse problem is usually an ill-posed problem in the sense of Hadamard. An ill-posed problem in the sense of Hadamard is the one which violates at least one of the following conditions:

- *Existence:* There exists a solution of the problem.
- *Uniqueness:* The solution must be unique.
- *Stability:* The solution must depend continuously on the data, i.e., any small error in given data must lead to a corresponding small error in the solution.

Problems which satisfy these conditions are called well-posed problems. We will show that the conformable backward heat problem is an ill-posed problem.

First, let us make clear what a solution of the Problem (6) - (8) is. We call a function $u \in C^{2,1}\left((0,a)\times(0,T);L^2(\Omega)\right)$ a solution for Problem (6) - (8) if

$$D_t^\alpha \langle u(\cdot,t), w\rangle - \langle u_{xx}(\cdot,t), w\rangle = \langle f(\cdot,t), w\rangle \tag{9}$$

for all functions $w \in L^2(\Omega)$. In fact, it is enough to choose w in the orthogonal basis $\left\{\sin\left(\frac{n\pi}{a}x\right)\right\}_{n=1}^{\infty}$ and then (9) reduces to

$$u_n(t) = e^{k_n\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)} g_n - \int_t^T s^{\alpha-1} e^{k_n\left(\frac{s^\alpha - t^\alpha}{\alpha}\right)} f_n(s)\,ds,$$

and as a result, the solution of (6) - (8) can be represented by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin\left(\frac{n\pi}{a}x\right) = \sum_{n=1}^{\infty}\left(e^{k_n\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)} g_n - \int_t^T s^{\alpha-1} e^{k_n\left(\frac{s^\alpha - t^\alpha}{\alpha}\right)} f_n(s)\,ds\right)\sin\left(\frac{n\pi}{a}x\right), \tag{10}$$

where $k_n = (n\pi/a)^2$ and

$$\begin{aligned} g_n &= \frac{2}{a}\int_0^a g(x)\sin\left(\frac{n\pi}{a}x\right)dx, \\ f_n(t) &= \frac{2}{a}\int_0^a f(x,t)\sin\left(\frac{n\pi}{a}x\right)dx, \\ u_n(t) &= \frac{2}{a}\int_0^a u(x,t)\sin\left(\frac{n\pi}{a}x\right)dx. \end{aligned} \tag{11}$$

It is noted that the term $e^{k_n\left(\frac{T^\alpha - t^\alpha}{\alpha}\right)}$ tends to infinity as n tends to infinity. Hence, it causes instability in the solution.

In this paper, we will apply the quasi-boundary value method with a small modification to regularize (6) - (8). In fact, rather than using the original information, we will consider problem (6) - (8) with adjusted information so that the adjusted problem is well-posed and approximates the original one. Consider the following problem

$$D_t^\alpha u^{\varepsilon,\tau}(x,t) - u_{xx}^{\varepsilon,\tau}(x,t) = f^{\varepsilon,\tau}(x,t), \quad (x,t) \in \Omega \times (0,T], \tag{12}$$

$$u^{\varepsilon,\tau}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T], \tag{13}$$

$$u^{\varepsilon,\tau}(x,T) = g^{\varepsilon,\tau}(x), \quad x \in \Omega, \tag{14}$$

where

$$\begin{aligned} f^{\varepsilon,\tau}(x,t) &= \sum_{n=1}^{\infty} \frac{e^{-k_n\frac{(T^\alpha+\tau)}{\alpha}}}{\varepsilon k_n + e^{-k_n\frac{(T^\alpha+\tau)}{\alpha}}} f_n(t) \sin\left(\frac{n\pi}{a}x\right), \\ g^{\varepsilon,\tau}(x) &= \sum_{n=1}^{\infty} \frac{e^{-k_n\frac{(T^\alpha+\tau)}{\alpha}}}{\varepsilon k_n + e^{-k_n\frac{(T^\alpha+\tau)}{\alpha}}} g_n \sin\left(\frac{n\pi}{a}x\right). \end{aligned} \tag{15}$$

Lemma 1.1. *Let* $0 \le t \le T$, $\tau > 0$, $\varepsilon \in \mathbb{D} := \left(0, \frac{(T^\alpha+\tau)}{\alpha}\right)$ *and* $x > 0$. *Then, for* $\alpha \in (0,1)$ *the following inequality holds*

$$\frac{e^{-\frac{(t^\alpha+\tau)}{\alpha}x}}{\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}} \le (\alpha\varepsilon)^{\frac{t^\alpha - T^\alpha}{T^\alpha+\tau}} \left(\frac{T^\alpha+\tau}{\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha - t^\alpha}{T^\alpha+\tau}}. \tag{16}$$

Proof. For any $\varepsilon \in \mathbb{D}$, $x > 0$, $\alpha \in (0,1]$ and $T > 0$, the function

$$w(x) = \frac{1}{\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}}$$

maximizes at $x = \ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})/(\frac{T^\alpha+\tau}{\alpha})$. Therefore,

$$w(x) = \frac{1}{\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}} \le w\left(\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})/(\frac{T^\alpha+\tau}{\alpha})\right) = \frac{T^\alpha+\tau}{\alpha\varepsilon\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}. \tag{17}$$

Then, we obtain the following estimation

$$\begin{aligned} \frac{e^{-\frac{(t^\alpha+\tau)}{\alpha}x}}{\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}} &= \frac{e^{-\frac{(t^\alpha+\tau)}{\alpha}x}}{\left(\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}\right)^{\frac{T^\alpha - t^\alpha}{T^\alpha+\tau}} \left(\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}\right)^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}} \\ &\leqslant \frac{e^{-\frac{(t^\alpha+\tau)}{\alpha}x}}{e^{-\frac{(t^\alpha+\tau)}{\alpha}x}} \frac{1}{\left(\varepsilon x + e^{-\frac{(T^\alpha+\tau)}{\alpha}x}\right)^{\frac{T^\alpha - t^\alpha}{T^\alpha+\tau}}} \\ &\leqslant (\alpha\varepsilon)^{\frac{t^\alpha - T^\alpha}{T^\alpha+\tau}} \left(\frac{T^\alpha+\tau}{\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha - t^\alpha}{T^\alpha+\tau}}. \end{aligned}$$

The proof is complete. □

The rest of the paper is organized as follows. In Section 2, we study the well-posedness of problem (12) - (14) and provide an error estimation between solutions of these two problems. Section 3 provides a numerical example to illustrate the efficiency of our method.

2 Well-posedness of the regularized problem (12)-(14)

Theorem 2.1. *Let $f(x,t) \in C(0,T;L^2(\Omega))$ and $g(x) \in L^2(\Omega)$. Let $\tau > 0$ and $\varepsilon \in \mathbb{D}$ be given. Then,* (12)–(14) *has a unique solution $u^{\varepsilon,\tau}$ satisfying*

$$u^{\varepsilon,\tau}(x,t) = \sum_{n=1}^{\infty} \left(\frac{e^{-k_n \frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} g_n - \frac{e^{-k_n \frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} \int_t^T s^{\alpha-1} e^{k_n \left(\frac{s^\alpha - T^\alpha}{\alpha}\right)} f_n(s) ds \right) \sin\left(\frac{n\pi}{a}x\right). \tag{18}$$

The solution depends continuously on g in $L^2(\Omega)$.

Proof. First we prove the existence and uniqueness of a solution of the regularized problem (12) - (14).

Existence of solution. For all $0 \le t \le T$, we have

$$u^{\varepsilon,\tau}(x,t) = \sum_{n=1}^{\infty} u_n^{\varepsilon,\tau}(t) \sin\left(\frac{n\pi}{a}x\right),$$

where

$$u_n^{\varepsilon,\tau}(t) = \frac{e^{-k_n \frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} g_n - \frac{e^{-k_n \frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} \int_t^T s^{\alpha-1} e^{k_n \left(\frac{s^\alpha - T^\alpha}{\alpha}\right)} f_n(s) ds.$$

It follows that

$$\begin{aligned} D_t^\alpha u^{\varepsilon,\tau}(x,t) &= -\sum_{n=1}^{\infty} k_n \left(\frac{e^{-k_n \frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} g_n - \frac{e^{-k_n \frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} \int_t^T s^{\alpha-1} e^{k_n \left(\frac{s^\alpha - T^\alpha}{\alpha}\right)} f_n(s) ds \right) \sin\left(\frac{n\pi}{a}x\right) \\ &\quad + \sum_{n=1}^{\infty} \frac{e^{-k_n \frac{T^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} f_n(t) \sin\left(\frac{n\pi}{a}x\right) \\ &= u_{xx}^{\varepsilon,\tau}(x,t) + \sum_{n=1}^{\infty} \frac{e^{-k_n \frac{T^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} f_n(t) \sin\left(\frac{n\pi}{a}x\right). \end{aligned} \tag{19}$$

On the other hand, we have

$$u^{\varepsilon,\tau}(x,T) = \sum_{n=1}^{\infty} \frac{e^{-k_n \frac{T^\alpha+\tau}{\alpha}}}{\varepsilon k_n + e^{-k_n \frac{T^\alpha+\tau}{\alpha}}} g_n \sin\left(\frac{n\pi}{a}x\right). \tag{20}$$

Hence, $u^{\varepsilon,\tau}$ is the solution of the regularized problem (12)–(14), so the existence of a solution of the regularized problem (12)–(14) is proved.

Uniqueness of solution. Let $u^{\varepsilon,\tau}(x,t)$ and $v^{\varepsilon,\tau}(x,t)$ be two solutions of (12) - (14). We denote $w(x,t) = u^{\varepsilon,\tau}(x,t) - v^{\varepsilon,\tau}(x,t)$. It is clear that $w(x,T) = 0$. We expand $w_n(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin\left(\frac{n\pi}{a}x\right)$ with the coefficient

$$w_n(t) = \frac{2}{a} \int_0^a w(x,t) \sin\left(\frac{n\pi}{a}x\right) dx, \quad n = 1, 2, 3, \ldots.$$

Multiply both sides of equation (12) by $\sin\left(\frac{n\pi}{a}x\right)$ and integrate by parts with respect to x, and use the boundary condition to obtain

$$D_t^\alpha w_n(t) + k_n w_n(t) = f_n^{\varepsilon,\tau}(t), \ t \in (0,T), \tag{21}$$

where $k_n = (n\pi/a)^2$ and

$$f_n^{\varepsilon,\tau}(t) = \frac{2}{a} \int_0^a f^{\varepsilon,\tau}(x,t) \sin\left(\frac{n\pi}{a}x\right) dx.$$

The condition $w(x,T)=0$ yields $w_n(T)=0$. Then, the well-posedness for the fractional differential equation (21) with the boundary condition $w_n(T)=0$ yields $w_n(t)\equiv 0$ in $t\in[0,T]$. This infers that $w(x,t)\equiv 0$.

Stability of solution. The solution of the problem (12)–(14) depends continuously on g. In fact, let $u^{\varepsilon,\tau}$ and $v^{\varepsilon,\tau}$ be two solutions of (12)–(14) corresponding to the final data $g^{\varepsilon,\tau}$ and $h^{\varepsilon,\tau}$, and $u^{\varepsilon,\tau}$ and $v^{\varepsilon,\tau}$ are represented by

$$u^{\varepsilon,\tau}(x,t)=\sum_{n=1}^{\infty}\left(\frac{e^{-k_n\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}g_n-\frac{e^{-k_n\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}\int_t^T s^{\alpha-1}e^{k_n\left(\frac{s^\alpha-T^\alpha}{\alpha}\right)}f_n(s)ds\right)\sin\left(\frac{n\pi}{a}x\right),\tag{22}$$

$$v^{\varepsilon,\tau}(x,t)=\sum_{n=1}^{\infty}\left(\frac{e^{-k_n\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}h_n-\frac{e^{-k_n\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}\int_t^T s^{\alpha-1}e^{k_n\left(\frac{s^\alpha-T^\alpha}{\alpha}\right)}f_n(s)ds\right)\sin\left(\frac{n\pi}{a}x\right),\tag{23}$$

where

$$g_n=\frac{2}{a}\int_0^a g^{\varepsilon,\tau}(x)\sin\left(\frac{n\pi}{a}x\right)dx,\quad h_n=\frac{2}{a}\int_0^a h^{\varepsilon,\tau}(x)\sin\left(\frac{n\pi}{a}x\right)dx.$$

Direct computation leads us to

$$\left|u^{\varepsilon,\tau}(x,t)-v^{\varepsilon,\tau}(x,t)\right|=\left|\sum_{n=1}^{\infty}\frac{e^{-k_n\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}(g_n-h_n)\sin\left(\frac{n\pi}{a}x\right)\right|.$$

Applying Lemma 1.1 directly, we get

$$\begin{aligned}\left\|u^{\varepsilon,\tau}(.,t)-v^{\varepsilon,\tau}(.,t)\right\|^2&=\frac{a}{2}\sum_{n=1}^{\infty}\left|\frac{e^{-k_n\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}(g_n-h_n)\right|^2\\&\leqslant\frac{a}{2}\left((\alpha\varepsilon)^{\frac{t^\alpha-T^\alpha}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}\right)^2\sum_{n=1}^{\infty}|(g_n-h_n)|^2\\&=\left((\alpha\varepsilon)^{\frac{t^\alpha-T^\alpha}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}\right)^2\left\|g^{\varepsilon,\tau}-h^{\varepsilon,\tau}\right\|^2.\end{aligned}\tag{24}$$

Therefore,

$$\left\|u^{\varepsilon,\tau}(\cdot,t)-v^{\varepsilon,\tau}(.,t)\right\|\leqslant(\alpha\varepsilon)^{\frac{t^\alpha-T^\alpha}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}\left\|g^{\varepsilon,\tau}-h^{\varepsilon,\tau}\right\|.\tag{25}$$

The proof is complete. □

We have shown that the regularization problem (12)–(14) is a well-posed problem in the sense of Hadamard. Now, the main goal of the coming theorem is to provide an error estimation between the regularization solution and the exact solution.

Theorem 2.2. *Let $g,f,u^{\varepsilon,\tau}$ as in Theorem 2.1 and assume that problem* (6)–(8) *has a solution $u\in C^{2,1}((0,a)\times(0,T);L^2(\Omega))$ and $\sqrt{\sum_{n=1}^{\infty}\left(\int_0^T k_n s^{\alpha-1}\frac{f_n(s)}{e^{-k_n\frac{s^\alpha}{\alpha}}}ds\right)^2}<\infty$, where $k_n=(n\pi/a)^2$. Suppose that the problem* (6)–(8) *has uniquely a solution u such that $\|u(\cdot,0)\|<\infty$. Then the following estimate holds for all $0<t\leq T$,*

$$\left\|u(\cdot,t)-u^{\varepsilon,\tau}(\cdot,t)\right\|\leqslant\mathcal{C}_1\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}},\tag{26}$$

where

$$\mathcal{C}_1=\sqrt{a\sum_{n=1}^{\infty}k_n^2\left(|u_n(0)|^2+\left(\int_0^T\frac{s^{\alpha-1}f_n(s)}{e^{-k_n\frac{s^\alpha}{\alpha}}}ds\right)^2\right)},$$

and $u^{\varepsilon,\tau}$ is the unique solution of problem (12)–(14).

Proof. The exact solution satisfies

$$u(x,t)=\sum_{n=1}^{\infty}\left(e^{k_n\frac{T^\alpha-t^\alpha}{\alpha}}g_n-e^{k_n\frac{T^\alpha-t^\alpha}{\alpha}}\int_t^T\left(s^{\alpha-1}e^{k_n\frac{s^\alpha-T^\alpha}{\alpha}}f_n(s)\right)ds\right)\sin\left(\frac{n\pi}{a}x\right). \tag{27}$$

On the other hand, in terms of $u_n(0)$, we have

$$u(x,T)=\sum_{n=1}^{\infty}\left(u_n(0)e^{-k_n\frac{T^\alpha}{\alpha}}+\int_0^T\left(s^{\alpha-1}e^{-k_n\frac{T^\alpha-s^\alpha}{\alpha}}f_n(s)\right)ds\right)\sin\left(\frac{n\pi}{a}x\right), \tag{28}$$

where

$$u_n(0)=\frac{2}{a}\left\langle u(x,0),\sin\frac{n\pi}{a}x\right\rangle=\frac{2}{a}\int_0^a u(x,0)\sin\left(\frac{n\pi}{a}x\right)dx.$$

It follows that

$$g_n=u_n(0)e^{-k_n\frac{T^\alpha}{\alpha}}+\int_0^T\left(s^{\alpha-1}e^{-k_n\frac{T^\alpha-s^\alpha}{\alpha}}f_n(s)\right)ds. \tag{29}$$

Combining (18) and (27), we get

$$\begin{aligned}\left|u_n(t)-u_n^{\varepsilon,\tau}(t)\right|&=\left|\frac{\varepsilon k_n e^{-k_n\frac{t^\alpha}{\alpha}}}{e^{-k_n\frac{T^\alpha}{\alpha}}\left(\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}\right)}g_n-\frac{\varepsilon k_n e^{-k_n\frac{t^\alpha}{\alpha}}}{e^{-k_n\frac{T^\alpha}{\alpha}}\left(\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}\right)}\int_t^T s^{\alpha-1}e^{k_n\left(\frac{s^\alpha-T^\alpha}{\alpha}\right)}f_n(s)ds\right|\\&=\left|\frac{\varepsilon k_n e^{-k_n\frac{t^\alpha}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}u_n(0)+\frac{\varepsilon k_n e^{-k_n\frac{t^\alpha}{\alpha}}}{\varepsilon k_n+e^{-k_n\frac{T^\alpha+\tau}{\alpha}}}\int_0^t s^{\alpha-1}e^{k_n\frac{s^\alpha}{\alpha}}f_n(s)ds\right|\\&\leqslant\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}k_n\left|u_n(0)\right|\\&\quad+\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}k_n\int_0^T\left(\frac{s^{\alpha-1}f_n(s)}{e^{-k_n\frac{s^\alpha}{\alpha}}}\right)ds.\end{aligned} \tag{30}$$

From the inequality $(a+b)^2\leq 2(a^2+b^2)$, we have

$$\left|u_n(t)-u_n^{\varepsilon,\tau}(t)\right|^2\leq 2\left(\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}k_n\right)^2\left(\left|u_n(0)\right|^2+\left(\int_0^T\left(\frac{s^{\alpha-1}f_n(s)}{e^{-k_n\frac{s^\alpha}{\alpha}}}\right)ds\right)^2\right). \tag{31}$$

From (31) and Parseval's identity $\left\|u(\cdot,t)-u^{\varepsilon,\tau}(\cdot,t)\right\|^2=\frac{a}{2}\sum_{n=1}^{\infty}\left|u_n(t)-u_n^{\varepsilon,\tau}(t)\right|^2$, we get

$$\left\|u_n(\cdot,t)-u^{\varepsilon,\tau}(\cdot,t)\right\|^2\leqslant\mathcal{C}_1^2\left(\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}\right)^2, \tag{32}$$

where

$$\mathcal{C}_1=\sqrt{a\sum_{n=1}^{\infty}k_n^2\left(\left|u_n(0)\right|^2+\left(\int_0^T\frac{s^{\alpha-1}f_n(s)}{e^{-k_n\frac{s^\alpha}{\alpha}}}ds\right)^2\right)}. \tag{33}$$

This completes the proof of Theorem 2.2. □

Theorem 2.3. *(Error estimates in case of non-exact data) Let f, g as in Theorem 2.1. Let $\tau\geq 0$ and $\varepsilon\in\mathbb{D}$ be given. Assume u is the unique solution of problem* (6)–(8) *corresponding to the exact data g. Suppose that $g^{\varepsilon,\tau}$ is measured data such that*

$$\left\|g-g^{\varepsilon,\tau}\right\|\leq\varepsilon.$$

Then there exists an approximate solution $U^{\varepsilon,\tau}$, which links to the noisy data $g^{\varepsilon,\tau}$, satisfying

$$\left\|U^{\varepsilon,\tau}(\cdot,t)-u(\cdot,t)\right\| \leqslant \left(\alpha^{\frac{t^\alpha-T^\alpha}{T^\alpha+\tau}}+\mathcal{C}_1\right)\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}, \quad \forall t\in(0,T). \tag{34}$$

Proof. Let $U^{\varepsilon,\tau}$ be the solution of the regularized problem (12)–(14) corresponding to data $g^{\varepsilon,\tau}$ and let $u^{\varepsilon,\tau}$ be the solution of the problem (12)–(14) corresponding to the data g. Let $u(x,t)$ be the exact solution, and in view of the triangle inequality, one has

$$\left\|U^{\varepsilon,\tau}(x,t)-u(x,t)\right\| \leq \left\|U^{\varepsilon,\tau}(x,t)-u^{\varepsilon,\tau}(x,t)\right\| + \left\|u(x,t)-u^{\varepsilon,\tau}(x,t)\right\|.$$

Combining the results from Theorem 2.1 (see the proof) and Theorem 2.2, for every $t\in[0,T]$, we get

$$\begin{aligned}\left\|U^{\varepsilon,\tau}(x,t)-u(x,t)\right\| &\leq (\alpha\varepsilon)^{\frac{t^\alpha-T^\alpha}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}\left\|g^{\varepsilon,\tau}-g\right\| + \mathcal{C}_1\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}\\ &\leq \left(\alpha^{\frac{t^\alpha-T^\alpha}{T^\alpha+\tau}}+\mathcal{C}_1\right)\varepsilon^{\frac{t^\alpha+\tau}{T^\alpha+\tau}}\left(\frac{T^\alpha+\tau}{\alpha\left(1+\ln(\frac{T^\alpha+\tau}{\alpha\varepsilon})\right)}\right)^{\frac{T^\alpha-t^\alpha}{T^\alpha+\tau}}.\end{aligned}$$

The proof is complete. □

3 Numerical illustration

In this section, we illustrate the theoretical results in Section 2 through an example. Consider the space domain $\Omega=[0,a]$ in association with the final time T, and our problem is

$$D_t^\alpha u(x,t)-u_{xx}(x,t)=\left(1+\left(\frac{\pi}{a}\right)^2\right)e^{\frac{t^\alpha}{\alpha}}\sin\left(\frac{\pi}{a}x\right), \quad (x,t)\in[0,a]\times(0,T], \tag{35}$$

$$u(0,t)=u(a,t)=0, \quad t\in(0,T], \tag{36}$$

$$u(x,T)=g(x), \quad x\in[0,a], \tag{37}$$

where $g(x)=e^{\frac{T^\alpha}{\alpha}}\sin\left(\frac{\pi}{a}x\right)$. Under the above assumptions, the exact solution of the problem is

$$u^{ex}(x,t)=e^{\frac{t^\alpha}{\alpha}}\sin\left(\frac{\pi}{a}x\right). \tag{38}$$

Now, due to the error in the measuring process, the measured data is perturbed by a "noise" with level ε, i.e

$$g^{noise}(x)=e^{T^\alpha/\alpha}\sin\left(\frac{\pi}{a}x\right)+\sum_{p=1}^{P_0}\varepsilon c_p\sin\left(\frac{p\pi}{a}x\right),$$

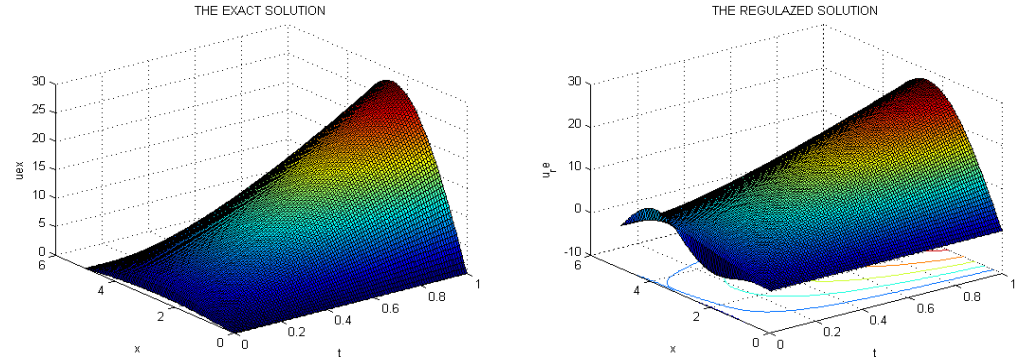
where P_0 is a natural number and c_p is a finite sequence of random normal numbers with mean 0 and variance A^2. It follows that the error in the measurement process is bounded by ε, $\left\|g^{noise}-g\right\|\leq R\varepsilon$ where R is some positive number. The error between the measured data and the exact data will tend to 0 as ε tends to 0. Regarding (18), the regularized solution corresponding to the measured data takes the following form

$$\begin{aligned}u^{\varepsilon,\tau}(x,t)=&\frac{e^{-k_1\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_1+e^{-k_1\frac{T^\alpha+\tau}{\alpha}}}\left[e^{T^\alpha/\alpha}-\left(1+\left(\frac{\pi}{a}\right)^2\right)\int_t^T s^{\alpha-1}e^{k_1\left(\frac{s^\alpha-T^\alpha}{\alpha}\right)}e^{\frac{s^\alpha}{\alpha}}ds\right]\sin\left(\frac{\pi}{a}x\right)\\ &+\sum_{p=1}^{P_0}c_p\frac{\varepsilon e^{-k_p\frac{t^\alpha+\tau}{\alpha}}}{\varepsilon k_p+e^{-k_p\frac{T^\alpha+\tau}{\alpha}}}\sin\left(\frac{p\pi}{a}x\right).\end{aligned} \tag{39}$$

Let $a=5$, $P_0=1000$, $A^2=100$. Consider the following situation:

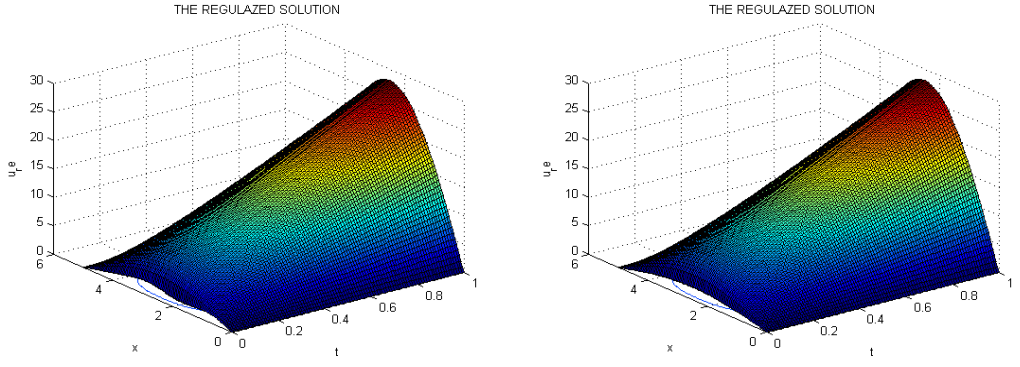
Situation 1. In this situation, the regularization parameter ε will be discussed. Fix $\alpha = 0.3$, $\tau = 0.5$. Consider $\varepsilon_1 = 10^{-1}$, $\varepsilon_2 = 10^{-3}$, $\varepsilon_3 = 10^{-5}$. We have the following figures:

Fig. 1. The exact solution with $\alpha = 0.3$ (a) and regularized solution with $\varepsilon_1 = 10^{-1}$ (b)



(a) u_{ex} (b) u^{ε_1}

Fig. 2. The regularized solution with $\varepsilon_2 = 10^{-3}$ (a) and with $\varepsilon_3 = 10^{-5}$ (b)



(a) u^{ε_2} (b) u^{ε_3}

For each point of time we evaluate the "Relative error" between the exact solution and the regularized solution which is defined by

$$RE(\varepsilon, t) = \frac{\|u^{\varepsilon,\tau}(., t) - u^{ex}(., t)\|}{\|u^{ex}(., t)\|}. \tag{40}$$

The relative error is a better representation of the difference between the exact and the approximate solution. When the value of the exact solution is large, the difference between the exact and the approximate solution does not tell us much information about the accuracy of the approximation. In this case, the relative error is a better measurement. Figure 3 shows errors for a comparison between the exact solution and the regularized solution at the initial time $t_0 = 0$ and $\tau = 1$ with various values of ε. In Table 1, we have the error table at time time $t_* = 0.1$.

Remark 3.1. *From Figure 1, Figure 2, Figure 3, Figure 4 and Table 1, it is clear that as the mearsuring error ε gets smaller, the regularized solution gets closer and closer to the exact one. It is also noted that in this situation, the noise parameter c_p varies from -249.689 to 242.4461.*

Fig. 3. At time $t_0 = 0$ and $\tau = 1$: Exact solution with $\alpha = 0.9$ (black) and Regularized solution with $\varepsilon_1 = 10^{-1}$ (blue), $\varepsilon_3 = 10^{-3}$ (green), $\varepsilon_5 = 10^{-5}$ (red)

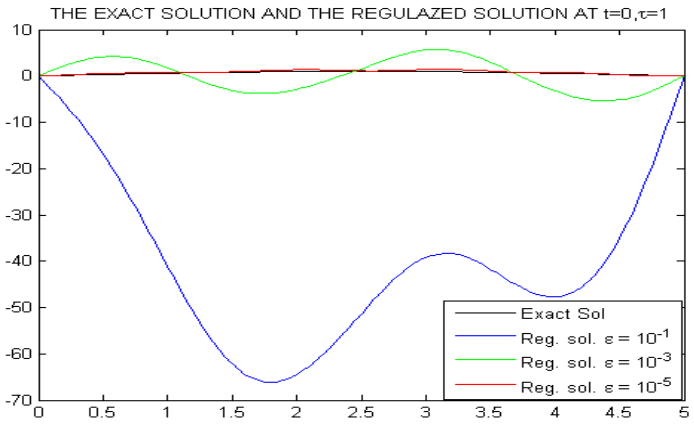
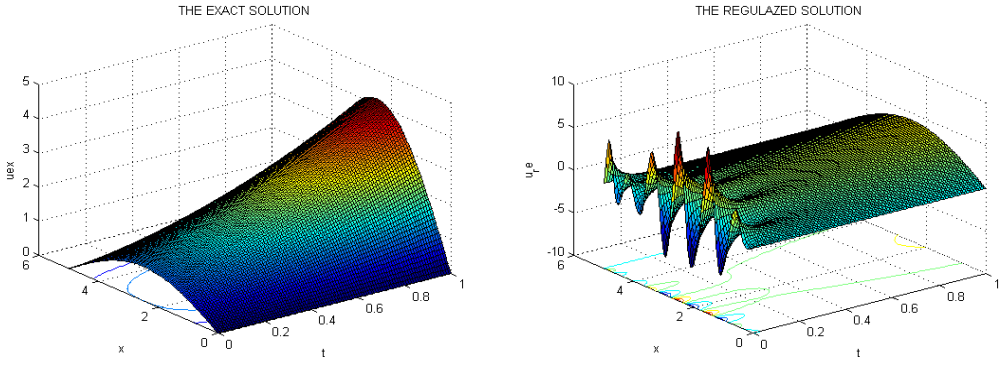


Table 1. The error and Relative error at time $t_* = 0.1$.

ϵ	$\|u^{\varepsilon}(.,0.1) - u^{ex}(.,0.1)\|$	$RE(\varepsilon, 0.1)$
$\varepsilon_1 = 10^{-1}$	2004.6002763324	40.0920055266479
$\varepsilon_2 = 10^{-2}$	93.6668715963709	1.87333743192742
$\varepsilon_3 = 10^{-3}$	11.7191083956268	0.234382167912537
$\varepsilon_4 = 10^{-4}$	0.333770043724429	0.0666754008744885
$\varepsilon_5 = 10^{-5}$	0.0558155468882481	0.00111631093776496

Situation 2. In this situation, the focusing parameter is τ. Let $\alpha = 0.1$ and fix $\varepsilon = 10^{-3}$. Consider the series of τ: $\tau_1 = 0.3, \tau_2 = 0.5, \tau_3 = 1$. We have Figure 6 to illustrate our theoretical results. It is also noted that in this situation, the noise parameter c_p varies from -220.5152 to 352.6678.

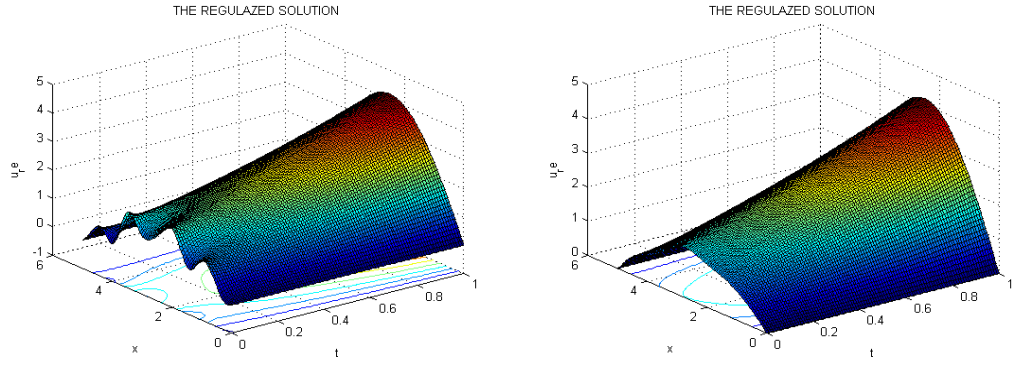
Fig. 4. The exact solution with $\alpha = 0.1$ (a) and regularized solution with $\varepsilon_1 = 10^{-1}$ (b)



(a) u_{ex} (b) u^{ε_1}

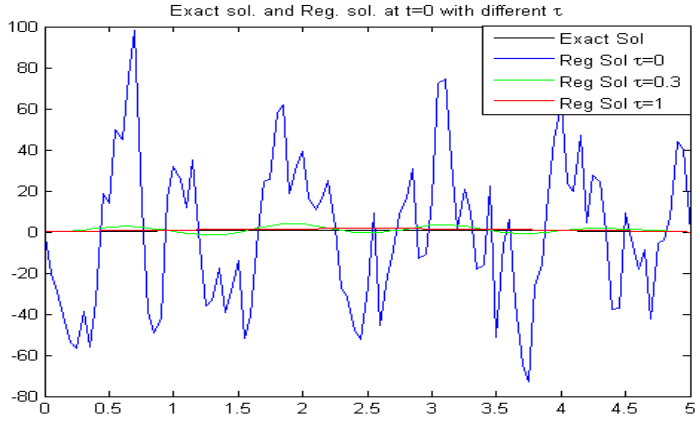
Remark 3.2. *Figure 6 agrees with the theoretical result: the regularized solution with a higher value of τ is closer to the exact one. The parameter τ is very useful if we want to get a more accurate approximation if the measuring process cannot be improved or if the cost of measuring better is very expensive. In this case, with the appearance of τ, the error can be improved without any extra cost on measuring (as we can see in Figure 6).*

Fig. 5. The regularized solution with $\varepsilon_2 = 10^{-3}$ (a) and with $\varepsilon_3 = 10^{-5}$ (b)



(a) u^{ε_2} (b) u^{ε_3}

Fig. 6. Case $\alpha = 0.5$, At time $t = 0$ and $\varepsilon_1 = 10^{-3}$: Exact solution (black) and Regularized solution with $\tau_1 = 0$ (blue), $\tau_2 = 0.3$ (green), $\tau_3 = 1$ (red)



4 Conclusion

In this paper, we have stated and discussed the quasi-boundary value regularization method for the inverse problem in the heat equation under the conformable derivative. In addition, we have also established an error estimate between exact and regularized solutions. These estimates are supported by several numerical examples. The estimate is a Hölder-type estimate ($\varepsilon^{t/T}$) for all values of t in the interval (0, T]. However, at the initial time $t = 0$, the error estimate is of logarithm type only. In the future, we hope to improve the error estimate as well as to consider the nonlinear case of f.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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