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Arithmetic of generalized Dedekind sums and their modularity

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Abstract: Dedekind sums were introduced by Dedekind to study the transformation properties of Dedekind η function under the action of $\mathrm{SL}_2(\mathbb{Z})$. In this paper, we study properties of generalized Dedekind sums $s_{i,j}(p,q)$. We prove an asymptotic expansion of a function on $\mathbb{Q}$ defined in terms of generalized Dedekind sums by using its modular property. We also prove an equidistribution property of generalized Dedekind sums.

Keywords: Dedekind sum, Quantum modular form

MSC: 11F20

1 Introduction

Dedekind sums are defined by

$$s(b,c) := \sum_{h \pmod c} \overline{B}_1\left(\frac{h}{c}\right)\overline{B}_1\left(\frac{bh}{c}\right) \tag{1}$$

for coprime integers b and c, where

$$\overline{B}_1(x) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

These sums were introduced by Dedekind [1], and have been studied with applications in diverse areas of mathematics (for example, see [2–4]).

The Dedekind sum has been generalized and studied by many other authors from diverse standing (for example, see [5–7]). In this paper, we consider generalized Dedekind sums defined as follows. The Bernoulli polynomial $B_i(x)$ is defined by the exponential generating function

$$\sum_{i=0}^{\infty} B_i(x)\frac{t^i}{i!} = \frac{te^{tx}}{e^t - 1} \tag{2}$$

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and $\overline{B}_i(x)$ denotes the periodic Bernoulli polynomial

$$\overline{B}_i(x) := \begin{cases} B_i(\langle x\rangle) & \text{for } i \neq 1 \text{ or } x \notin \mathbb{Z}, \\ 0 & \text{for } i = 1 \text{ and } x \in \mathbb{Z}, \end{cases} \tag{3}$$

where $\langle x\rangle := x - [x]$ and $[x]$ denotes the greatest integer not exceeding x. Let i and j be nonnegative integers. Suppose that p is an integer and q is a positive integer with $\gcd(p, q) = 1$. Then, generalized Dedekind sums are defined as

$$s_{i,j}(p,q) := \sum_{k=1}^{q} \overline{B}_i\left(\frac{k}{q}\right)\overline{B}_j\left(\frac{pk}{q}\right) \tag{4}$$

and the number $i + j$ is called the weight of $s_{i,j}(p, q)$. For convenience, we let

$$h_{i,j}(p,q) := (-1)^{i+j}\frac{1}{i!j!}(s_{i,j}(p,q) - d_{i,j}B_iB_j), \tag{5}$$

where B_n is the nth Bernoulli number and $d_{i,j}$ is given by

$$d_{i,j} := \begin{cases} 1 & \text{if } i = 1 \text{ or } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ be the standard basis of $\mathbb{Q}^m$. We define a vector $F_N(p, q) \in \mathbb{Q}^N$ for an even positive integer N by

$$F_N(p,q) := \sum_{i=1}^{N} f_i(p,q)\mathbf{e}_i, \tag{6}$$

where $f_i(p, q) = q^{\frac{N}{2}-1}p^{-\frac{N}{2}+i-1}h_{i-1,N+1-i}(p, q)$ for $i = 1, \ldots, N$. Then we define a function $G_N : \mathbb{Q} \to \mathbb{Q}^N$ as

$$G_N(x) = G_N\left(\frac{p}{q}\right) := F_N(p,q). \tag{7}$$

We remark that any rational number x can be uniquely written as $\frac{p}{q}$ for a positive integer q and an integer p which are relatively prime. Thus, the above mapping G_N is well-defined, and has the following asymptotic expansion expressed in terms of Bernoulli numbers.

Theorem 1.1. *We have an asymptotic expansion of the form*

$$G_N\left(\frac{1}{n}\right) \sim ((-1)^{\frac{N}{2}+1}C_N)G_N(-n) + A_N\left(\frac{1}{n}\right)^{-1} E_N\left(\frac{1}{n}\right) + 2B_1^2\delta_{N,2}\mathbf{e}_2$$

as $n \to \infty$, where

$$\delta_{i,j} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j, \end{cases}$$

$$A_N(x) = A_N\left(\frac{p}{q}\right) := \left(q^{\frac{N}{2}-i+1}p^{i-\frac{N}{2}}(-1)^{i+j}\binom{j-1}{N-i}\right)_{1\le i,j\le N}, \tag{8}$$

$$C_N := \left((-1)^{j+1}\binom{N-j}{N-i}\right)_{1\le i,j\le N},$$

and $E_N(x) = \sum_{i=1}^{N} e_i(x)\mathbf{e}_i$ is a vector-valued function defined by

$$\begin{aligned} e_i(x) = e_i\left(\frac{p}{q}\right) :=& q^{1-i}p^i(-1)^i\binom{N}{N-i}\frac{B_N}{N!} + p^{i-N}q^{N-i+1}(-1)^{N-i}\binom{N}{i-1}\frac{B_N}{N!} \\ &+ \frac{B_{N-i}B_i}{(N-i)!i!}q(-1)^{i-1} + \frac{B_{N-i+1}B_{i-1}}{(N-i+1)!(i-1)!}p(-1)^{i-1}. \end{aligned} \tag{9}$$

Here, $\binom{a}{b}$ is defined by

$$\binom{a}{b} = \begin{cases} 1 & \text{if } b = 0, \\ \frac{a(a-1)\cdots(a-b+1)}{b!} & \text{if } b > 0, \end{cases} \tag{10}$$

for nonnegative integers a and b. Theorem 1.1 is proved by using the fact that G_N can be understood as a modular object, which was recently introduced by Zagier [8].

Example 1.2. *For a given N, we can compute the asymptotic expansion of G_N explicitly and it is written in terms of Bernoulli numbers B_n. For example, if $N = 2$, then*

$$G_N\left(\frac{1}{n}\right) \sim \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} G_N(-n) + \begin{pmatrix} 0 \\ \frac{1}{2}nB_2 + B_1^2 + \frac{1}{2n}B_2 \end{pmatrix}$$

as $n \to \infty$. If $N = 4$, then

$$G_N\left(\frac{1}{n}\right) \sim \begin{pmatrix} -1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -3 & 2 & -1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix} G_N(-n) + \begin{pmatrix} 0 \\ \frac{1}{24}n^2B_4 - \frac{1}{6}nB_3B_1 + \frac{1}{4}B_2^2 - \frac{1}{6n}B_3B_1 + \frac{1}{24n^2}B_4 \\ -\frac{1}{24}n^2B_4 + \frac{1}{4}B_2^2 - \frac{1}{3n}B_3B_1 + \frac{1}{8n^2}B_4 \\ \frac{1}{24}n^2B_4 - \frac{1}{6n}B_3B_1 + \frac{1}{8n^2}B_4 \end{pmatrix}$$

as $n \to \infty$.

Dedekind sums also have a special property of distribution. For an even positive integer N and positive integers p, q which are coprime, we consider another vector-valued function

$$H_N(p,q) := q^{N-2}\left(h_{1,N-1}(p,q), \dots, h_{N-1,1}(p,q)\right) \in \mathbb{Q}^{N-1}. \tag{11}$$

Let

$$\langle X \rangle := (\langle x_1 \rangle, \langle x_2 \rangle, \dots, \langle x_{N-1} \rangle) \in [0,1)^{N-1}$$

be the fractional part of the vector $X = (x_1, x_2, \dots x_{N-1})$. The following theorem states that $H_N(p,q)$ satisfies the property of equidistribution.

Theorem 1.3. *For an even positive integer N, there exists an integer R_N such that the set of rational numbers*

$$\{\langle R_N H_N(p,q) \rangle \mid (p,q) = 1,\ 0 < p < q\} \tag{12}$$

is equidistributed in $[0,1)^{N-1}$.

Remark 1.4. *Once for an integer R_N the set (12) is equidistributed in $[0,1)^{N-1}$, the same holds for any integer multiple of R_N. An example of R_N is given by*

$$R_N = N!\beta_N r_N,$$

where β_k is a positive integer such that $\frac{\alpha_k}{\beta_k}$ is the reduced fraction of $B_k \neq 0$ and

$$r_N := \operatorname{lcm}\left\{ \text{Denominator of } \beta_N \binom{N}{i+1} B_{i+1}B_{N-i-1} \,\middle|\, 0 \le i \le N-2 \right\}. \tag{13}$$

The following is the table for values of R_N for $N = 2, 4, 6, 8, 10$.

N	**2**	**4**	**6**	**8**	**10**
R_N	**12**	**720**	**60480**	**3628800**	**479001600.**

The rest of the paper is organized as follows. Section 2 summarizes the properties of generalized Dedekind sums. In Section 3, we show that generalized Dedekind sums satisfy a modular property introduced by Zagier [8] and prove Theorem 1.1. In Section 4, we describe the distribution of generalized Dedekind sums and prove Theorem 1.3.

2 Reciprocity formulas

In this section, we prove the reciprocity formulas of generalized Dedekind sums, which can be induced from those of Dedekind-Rademacher sums.

2.1 Dedekind-Rademacher sums

In this subsection, we briefly review the definition and the reciprocity formulas of Dedekind-Rademacher sums based on the paper [6]. The following is the definition of Dedekind-Rademacher sums.

Definition 2.1. *For $a, b, c \in \mathbb{N}$ and $x, y, z \in \mathbb{R}/\mathbb{Z}$, the Dedekind-Rademacher sum is defined by*

$$S_{m,n}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix} := \sum_{h \pmod{c}} \bar{B}_m\left(\frac{a(h+z)}{c} - x\right)\bar{B}_n\left(\frac{b(h+z)}{c} - y\right).$$

Since reciprocity relations mix various pairs of indices (m, n), Hall, Wilson and Zagier [6] stated them in terms of the generating function

$$\delta\begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} := \sum_{m,n\geq 0} \frac{1}{m!n!} S_{m,n}\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}\left(\frac{X}{a}\right)^{m-1}\left(\frac{Y}{b}\right)^{n-1},$$

where X and Y are nonzero variables and the variable Z is defined as $-X - Y$. The following is the precise statement of the reciprocity formula of Dedekind-Rademacher sums.

Theorem 2.2 ([6, Section 4])**.** *Let a, b, c be three positive integers with no common factor, x, y, z three real numbers, and X, Y, Z three variables with sum zero. Then*

$$\delta\begin{pmatrix} a & b & c \\ x & y & z \\ X & Y & Z \end{pmatrix} + \delta\begin{pmatrix} b & c & a \\ y & z & x \\ Y & Z & X \end{pmatrix} + \delta\begin{pmatrix} c & a & b \\ z & x & y \\ Z & X & Y \end{pmatrix} = \begin{cases} \frac{1}{4} & \text{if } (x, y, z) \in (a, b, c)\mathbb{R} + \mathbb{Z}^3, \\ 0 & \text{otherwise}. \end{cases}$$

2.2 Reciprocity formulas of generalized Dedekind sums

The following are the reciprocity formulas of generalized Dedekind sums, which are induced from Theorem 2.2.

Theorem 2.3. *Let p and q be positive integers with $\gcd(p, q) = 1$. The generalized Dedekind sums satisfy the following reciprocity formulas.*

1. *For $a, b \geq 1$ with odd $a + b$,*

$$\begin{aligned} &-q^{a-1}\sum_{i=0}^{b} h_{a-1+i,b-i}(p, q)(-p)^i\binom{a-1+i}{a-1} + p^{b-1}\sum_{j=0}^{a} h_{b-1+j,a-j}(q, p)(-q)^j\binom{b-1+j}{b-1} \\ &= \frac{B_{a-1}B_b}{(a-1)!b!}q(-1)^{b-1} + \frac{B_aB_{b-1}}{a!(b-1)!}p(-1)^{b-1}. \end{aligned} \tag{14}$$

2. *For $a = 0$ and odd $b \geq 1$,*

$$h_{b-1,0}(q, p) = h_{0,b-1}(q, p) = p^{2-b}\frac{B_{b-1}}{(b-1)!}.$$

Proof. (1) By the definition of Dedekind-Rademacher sums, we have the following equations

$$S_{m,n}\begin{pmatrix} 1 & p & q \\ 0 & 0 & 0 \end{pmatrix} = \sum_{h \pmod{q}} \bar{B}_m\left(\frac{h}{q}\right)\bar{B}_n\left(\frac{ph}{q}\right) = s_{m,n}(p, q),$$

$$S_{m,n}\begin{pmatrix} q & 1 & p \\ 0 & 0 & 0 \end{pmatrix} = \sum_{h \pmod q} \bar{B}_n\left(\frac{h}{p}\right)\bar{B}_m\left(\frac{qh}{p}\right) = s_{n,m}(q,p),$$

and

$$S_{m,n}\begin{pmatrix} p & q & 1 \\ 0 & 0 & 0 \end{pmatrix} = \bar{B}_m(0)\bar{B}_n(0).$$

Then Theorem 2.2 implies that the sum

$$\sum_{m,n\geq 0} \frac{1}{m!n!}\left(s_{m,n}(p,q)X^{m-1}\left(\frac{Y}{p}\right)^{n-1} + \bar{B}_m(0)\bar{B}_n(0)\left(\frac{Y}{p}\right)^{m-1}\left(\frac{Z}{q}\right)^{n-1} + s_{n,m}(q,p)\left(\frac{Z}{q}\right)^{m-1}X^{n-1}\right) \tag{15}$$

is equal to $\frac{1}{4}$.

Suppose that $m+n$ is even. Then, we have

$$h_{m,n}(p,q) = \frac{1}{m!n!}(s_{m,n}(p,q) + d_{m,n}B_mB_n).$$

Since $B_k = 0$ for odd integer $k > 1$ and $B_1 = -\frac{1}{2}$, we see that

$$h_{m,n}(p,q) = \frac{1}{m!n!}\left(s_{m,n}(p,q) - \frac{1}{4}e_{m,n}\right),$$

where

$$e_{m,n} := \begin{cases} 1 & \text{if } n = m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, we have

$$\bar{B}_m(0)\bar{B}_n(0) = B_mB_n - \frac{1}{4}e_{m,n}.$$

Therefore, the following can be induced from the sum (15)

$$\sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}} \left(h_{m,n}(p,q)X^{m-1}\left(\frac{Y}{p}\right)^{n-1} + \frac{1}{m!n!}B_mB_n\left(\frac{Y}{p}\right)^{m-1}\left(\frac{Z}{q}\right)^{n-1} + h_{n,m}(q,p)\left(\frac{Z}{q}\right)^{m-1}X^{n-1}\right) = 0. \tag{16}$$

If we multiply $\frac{1}{pq}XYZ$ on both sides of equation (16), then we obtain

$$\sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}} \left(h_{m,n}(p,q)X^m\left(\frac{Y}{p}\right)^n\frac{Z}{q} + \frac{1}{m!n!}B_mB_nX\left(\frac{Y}{p}\right)^m\left(\frac{Z}{q}\right)^n + h_{n,m}(q,p)X^n\left(\frac{Z}{q}\right)^m\frac{Y}{p}\right) = 0. \tag{17}$$

From the relation $X + Y + Z = 0$, the equation (17) is rewritten as

$$\begin{aligned}\sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}} &\left(h_{m,n}(p,q)(-1)^m(Y+Z)^m\left(\frac{Y}{p}\right)^n\frac{Z}{q} + h_{n,m}(q,p)(-1)^n\left(\frac{Z}{q}\right)^m(Y+Z)^n\frac{Y}{p}\right) \\ &= \sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}} \frac{1}{m!n!}B_mB_n\left(\frac{Y}{p}\right)^m\left(\frac{Z}{q}\right)^n(Y+Z).\end{aligned} \tag{18}$$

By the binomial expansion of $(Y+Z)^n$ one can see that

$$\begin{aligned}&\sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}}\sum_{k_1=0}^{m}\left(h_{m,n}(p,q)(-1)^mq^{-1}p^{-n}\binom{m}{k_1}Y^{k_1+n}Z^{m-k_1+1}\right) \\ &+ \sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}}\sum_{k_2=0}^{n}\left(h_{n,m}(q,p)(-1)^nq^{-m}p^{-1}\binom{n}{k_2}Y^{n-k_2+1}Z^{k_2+m}\right) \\ &= \sum_{\substack{m,n\geq 0 \\ m+n \text{ even}}} \frac{1}{m!n!}B_mB_n\left(\frac{Y}{p}\right)^m\left(\frac{Z}{q}\right)^n(Y+Z).\end{aligned} \tag{19}$$

Now, we compare the coefficients of Y^bZ^a on both sides of equation (19). If $a, b \geq 1$, then we have

$$\begin{aligned}&\sum_{i=0}^{b} q^{-1}p^{-(b-i)}(-1)^{a+i-1}h_{a+i-1,b-i}(p,q)\binom{a+i-1}{i}\\&+\sum_{j=0}^{a} q^{-(a-j)}p^{-1}(-1)^{b+j-1}h_{b+j-1,a-j}(q,p)\binom{b+j-1}{j}\\&=\frac{1}{(b-1)!a!}B_{b-1}B_a p^{-(b-1)}q^{-a}+\frac{1}{b!(a-1)!}B_bB_{a-1}p^{-b}q^{-(a-1)}.\end{aligned}\tag{20}$$

This gives the first result after multiplying by $q^ap^b(-1)^{b-1}$ in both sides of equation (20).

(2) For the second result, suppose that $a = 0$ and b is an odd integer with $b \geq 1$. Note that

$$\begin{aligned}s_{b-1,0}(p,q) &= \sum_{k=1}^{q}\overline{B}_{b-1}\left(\frac{k}{q}\right)\overline{B}_0\left(\frac{pk}{q}\right) = \sum_{k=1}^{q}\overline{B}_{b-1}\left(\frac{pk}{q}\right)\overline{B}_0\left(\frac{p(pk)}{q}\right) = \sum_{k=1}^{q}\overline{B}_{b-1}\left(\frac{pk}{q}\right)\overline{B}_0\left(\frac{k}{q}\right)\\&= s_{0,b-1}(p,q).\end{aligned}$$

Here, we used the fact that $B_0(x) = 1$ and $\gcd(p, q) = 1$. If we compare the coefficients of Y^b in both sides of equation (19), then we have

$$h_{b-1,0}(q,p)(-1)^{b-1}p^{-1} = \frac{1}{(b-1)!}B_{b-1}p^{-(b-1)},$$

which gives the second result. □

The reciprocity formulas of generalized Dedekind sums can be expressed in terms of the vectors $F_N(p, q)$.

Corollary 2.4. *Let p and q be positive integers with* $\gcd(p, q) = 1$, *and N an even positive integer. Then*

$$A_N\left(\frac{p}{q}\right)F_N(p,q) + B_N(p,q)F_N(p,q) = E_N\left(\frac{p}{q}\right),$$

where $A_N\left(\frac{p}{q}\right)$, $F_N(p, q)$ and $E_N\left(\frac{p}{q}\right)$ are given as in (8), (6), and (9), respectively. Here, we define $B_N(p, q)$ by $B_N(p, q) = (\beta_{i,j}(p, q))_{1\leq i,j\leq N}$ and

$$\beta_{i,j}(p,q) = q^{\frac{N}{2}-i+1}p^{i-\frac{N}{2}}(-1)^{i+j}\binom{j-1}{i-1}.\tag{21}$$

Proof. Let $N = a + b - 1$ for $a, b \geq 1$. By Theorem 2.3 (2), we have

$$\begin{aligned}&-q^{a-1}\sum_{i=0}^{b}h_{a-1+i,b-i}(p,q)(-p)^i\binom{a-1+i}{a-1}\\&=\sum_{j=0}^{b-1}q^{-\frac{N}{2}+a}p^{\frac{N}{2}-a+1}f_{a+j}(p,q)(-1)^{j+1}\binom{a-1+j}{a-1}-q^{a-1}p^b(-1)^b\binom{N}{a-1}h_{N,0}(p,q)\\&=\sum_{j=0}^{b-1}q^{-\frac{N}{2}+a}p^{\frac{N}{2}-a+1}f_{a+j}(p,q)(-1)^{j+1}\binom{a-1+j}{a-1}+q^{a-N}p^b(-1)^{b+1}\binom{N}{a-1}\frac{B_N}{N!}\end{aligned}\tag{22}$$

and

$$\begin{aligned}&p^{b-1}\sum_{j=0}^{a}h_{b-1+j,a-j}(q,p)(-q)^j\binom{b-1+j}{b-1}\\&=\sum_{k=0}^{a-1}p^{-\frac{N}{2}+b}q^{\frac{N}{2}-b+1}f_{b+k}(q,p)(-1)^k\binom{b-1+k}{b-1}+p^{b-1}q^a(-1)^a\binom{N}{b-1}h_{N,0}(q,p)\\&=\sum_{k=0}^{a-1}p^{-\frac{N}{2}+b}q^{\frac{N}{2}-b+1}f_{b+k}(q,p)(-1)^k\binom{b-1+k}{b-1}+p^{b-N}q^a(-1)^a\binom{N}{b-1}\frac{B_N}{N!}.\end{aligned}\tag{23}$$

If we change variables $a \mapsto N-i+1, j \mapsto j-a, b \mapsto i$, and $k \mapsto j-b$ in (22) and (23), then (14) can be written as

$$\begin{aligned}&\sum_{j=0}^{N} q^{\frac{N}{2}-i+1} p^{i-\frac{N}{2}} (-1)^{i+j} \binom{j-1}{N-i} f_j(p,q) + \sum_{j=0}^{N} p^{i-\frac{N}{2}} q^{\frac{N}{2}-i+1} (-1)^{i+j} \binom{j-1}{i-1} f_j(q,p) \\ &= q^{1-i} p^{i} (-1)^{i} \binom{N}{N-i} \frac{B_N}{N!} + p^{i-N} q^{N-i+1} (-1)^{N-i} \binom{N}{i-1} \frac{B_N}{N!} \\ &+ \frac{B_{N-i}B_i}{(N-i)!i!} q(-1)^{i-1} + \frac{B_{N-i+1}B_{i-1}}{(N-i+1)!(i-1)!} p(-1)^{i-1}.\end{aligned}$$

This gives the desired result. □

Besides the reciprocity formulas, generalized Dedekind sums satisfy the following properties.

Theorem 2.5. *Let N be an even positive integer, and i, j be nonnegative integers with $i+j=N$. Then we have the following.*

1. $h_{i,j}(-p,q) = \begin{cases} (-1)^j h_{i,j}(p,q) - 2B_1^2, & \text{if } i=j=1; \\ (-1)^j h_{i,j}(p,q), & \text{otherwise}. \end{cases}$
2. $h_{i,j}(p+q,q) = h_{i,j}(p,q)$.

Proof. (1) By the definition of $s_{i,j}(p,q)$, we see that

$$s_{i,j}(-p,q) = \sum_{k=0}^{q-1} \overline{B}_i\left(\frac{k}{q}\right)\overline{B}_j\left(\frac{-pk}{q}\right).$$

It is known that $B_j(1-x) = (-1)^j B_j(x)$ for $j \geq 0$. Therefore, we have

$$\overline{B}_j\left(\frac{-pk}{q}\right) = \overline{B}_j\left(1-\frac{pk}{q}\right) = (-1)^j \overline{B}_j\left(\frac{pk}{q}\right)$$

for $j \geq 0$. From this, we obtain

$$s_{i,j}(-p,q) = (-1)^j s_{i,j}(p,q).$$

If $i=j=1$, then we have

$$h_{i,j}(-p,q) = s_{i,j}(-p,q) - B_1^2 = -s_{i,j}(p,q) - B_1^2 = -h_{i,j}(p,q) - 2B_1^2.$$

Otherwise, we see that

$$h_{i,j}(-p,q) = (-1)^{i+j}\frac{1}{i!j!} s_{i,j}(-p,q) = (-1)^j (-1)^{i+j} \frac{1}{i!j!} s_{i,j}(p,q) = (-1)^j h_{i,j}(p,q).$$

(2) Note that $s_{i,j}(p+q,q)$ is equal to

$$\sum_{k=0}^{q-1} \overline{B}_i\left(\frac{k}{q}\right)\overline{B}_j\left(\frac{(p+q)k}{q}\right) = \sum_{k=0}^{q-1} \overline{B}_i\left(\frac{k}{q}\right)\overline{B}_j\left(\frac{pk}{q}+k\right) = \sum_{k=0}^{q-1} \overline{B}_i\left(\frac{k}{q}\right)\overline{B}_j\left(\frac{pk}{q}\right),$$

which is $s_{i,j}(p,q)$ by its definition. From this, we obtain the desired result that $h_{i,j}(p+q,q) = h_{i,j}(p,q)$. □

3 Modular properties of generalized Dedekind sums

In this section, we show that the vector-valued function G_N defined in (7) satisfies the modular property, which was introduced by Zagier [8].

3.1 Automorphic factor

To introduce a modular property for a vector-valued function, we need an automorphic factor for a vector-valued function.

Definition 3.1. *An automorphic factor of rank m is a function* $\rho : \mathrm{SL}_2(\mathbb{Z}) \times \mathbb{H} \to \mathrm{GL}_m(\mathbb{C})$ *satisfying the following conditions.*

1. *The function* ρ *satisfies the cocycle relation*
$$\rho(\gamma_1\gamma_2, x) = \rho(\gamma_1, \gamma_2 x)\rho(\gamma_2, x) \tag{24}$$
for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$ *and* $x \in \mathbb{H}$, *where* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) x = \frac{ax+b}{cx+d}$.
2. *For a fixed* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, *entries of* $\rho(\gamma, x)$ *are rational functions of x with coefficients in* $\mathbb{Q}$.

For a fixed $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, the action of $\rho(\gamma, x)$ on $\mathbb{C}^m$ is defined by
$$\rho(\gamma, x)\mathbf{e}_j = \sum_{i=1}^{m} \rho_{ij}(\gamma, x)\mathbf{e}_i,$$
where $\rho_{i,j}(\gamma, x)$ is the (i,j)th entry of $\rho(\gamma, x)$.

Let N be an even positive integer. Now, we define an automorphic factor ρ_N of rank N as follows. Let $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. We define
$$\rho_N(S, x) := ((-1)^{\frac{N}{2}+1} C_N)^{-1} \tag{25}$$
and
$$\rho_N(T, x) := D_N(x)^{-1}, \tag{26}$$
where
$$C_N := \left((-1)^j \binom{N-j}{N-i}\right)_{1 \le i,j \le N}$$
and
$$D_N(x) := \left(\delta_{i,j}\left(\frac{x}{x+1}\right)^{-\frac{N}{2}+i-1}\right)_{1 \le i,j \le N}.$$
Here, $\delta_{i,j}$ is the Kronecker delta. Then, it induces an automorphic factor of rank N.

Note that S and T generate $\mathrm{SL}_2(\mathbb{Z})$. Therefore, from (25) and (26), we can compute $\rho_N(\gamma)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ by using the cocycle condition as in (24). To prove that ρ_N is an automorphic factor of rank N, it suffices to check that
$$\rho_N(S, Sx)\rho_N(S, x) = \rho_N(U, U^2x)\rho_N(U, Ux)\rho_N(U, x)$$
and
$$(\rho_N(S, Sx)\rho_N(S, x))^2 = I,$$
where $U := TS$, $\rho_N(U, x) = \rho_N(T, Sx)\rho_N(S, x)$ and I denotes the $N \times N$ identity matrix. To prove this, we need the following lemmas.

Lemma 3.2. *Let N be a positive integer. Let i and j be integers with* $1 \le i, j \le N$.

1. *If N is even and we let*
$$\Lambda_{i,j} := \sum_{k=1}^{N} (-1)^{k+j-1} \binom{N-k}{i-1}\binom{j-1}{N-k},$$
then $\Lambda_{i,j} = \delta_{i,j}$.
2. $\sum_{k=1}^{N} \binom{N-k}{i-1}\binom{j-1}{k-1}(-1)^{k-1} = \binom{N-j}{N-i}$.

Proof. (1) Recall that

$$\binom{a}{b} = \begin{cases} 1 & \text{if } b = 0, \\ \frac{a(a-1)\cdots(a-b+1)}{b!} & \text{if } b > 0, \end{cases} \tag{27}$$

for nonnegative integers a and b. If we consider the binomial expansion of $(1-(1-T))^{j-1}$, then we see that

$$(1-(1-T))^{j-1} = \sum_{k=0}^{N-1} \binom{j-1}{k} (-1)^k (1-T)^k.$$

In the above equality, we used the fact that $\binom{a}{b} = 0$ if $a < b$. Then we have

$$\begin{aligned}(1-(1-T))^{j-1} &= \sum_{k=1}^{N} \binom{j-1}{N-k} (-1)^{N-k} (1-T)^{N-k} = \sum_{k=1}^{N} \binom{j-1}{N-k} (-1)^{N-k} \sum_{i=0}^{N-1} \binom{N-k}{i} (-1)^i T^i \\ &= \sum_{k=1}^{N} \binom{j-1}{N-k} (-1)^{N-k} \sum_{i=1}^{N} \binom{N-k}{i-1} (-1)^{i-1} T^{i-1} \\ &= \sum_{k=1}^{N}\sum_{i=1}^{N} \binom{j-1}{N-k}\binom{N-k}{i-1} (-1)^{N-k+i-1} T^{i-1} = \sum_{i=1}^{N} \Lambda_{i,j} (-1)^{N+i+j} T^{i-1}.\end{aligned}$$

Since $(1-(1-T))^{j-1} = T^{j-1}$, we see that

$$\Lambda_{i,j}(-1)^{N+i+j} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

If $i = j$, then $(-1)^{N+i+j} = 1$ since N is even. Therefore, we obtain the desired result that $\Lambda_{i,j} = \delta_{i,j}$ for $1 \le i, j \le N$.

(2) We will use the induction on N. If $N = 1$, it is easy to see that (2) is true. Suppose that (2) is true for $N \ge 1$. Let $1 < i < N+1$ and $1 \le j < N+1$. By the recursive formula, we have

$$\binom{N+1-j}{N+1-i} = \binom{N-j}{N-(i-1)} + \binom{N-j}{N-i}.$$

Then, the induction hypothesis implies that

$$\binom{N+1-j}{N+1-i} = \sum_{k=1}^{N} \binom{N-k}{i-2}\binom{j-1}{k-1}(-1)^{k-1} + \sum_{k=1}^{N} \binom{N-k}{i-1}\binom{j-1}{k-1}(-1)^{k-1}.$$

If we use the recursive formula again, then we have

$$\binom{N+1-j}{N+1-i} = \sum_{k=1}^{N} \binom{N+1-k}{i-1}\binom{j-1}{k-1}(-1)^{k-1} = \sum_{k=1}^{N+1} \binom{N+1-k}{i-1}\binom{j-1}{k-1}(-1)^{k-1}.$$

The last equality follows from that $\binom{N+1-k}{i-1} = \binom{0}{i-1} = 0$ if $k = N+1$ since $i-1 > 0$.

Now, we will check the remaining three cases: $i = 1$, $i = N+1$, and $j = N+1$. If $i = 1$ and $1 \le j \le N+1$, then we have

$$\begin{aligned}\sum_{k=1}^{N+1} \binom{N+1-k}{0}\binom{j-1}{k-1}(-1)^{k-1} &= \sum_{k=1}^{N+1} \binom{j-1}{k-1}(-1)^{k-1} = \sum_{k=0}^{N} \binom{j-1}{k}(-1)^{k} = \delta_{j,1} \\ &= \binom{N+1-j}{N}.\end{aligned}$$

If $i = N+1$ and $1 \le j \le N+1$, then we obtain

$$\sum_{k=1}^{N+1} \binom{N+1-k}{N}\binom{j-1}{k-1}(-1)^{k-1} = \binom{N}{N} = 1 = \binom{N+1-j}{0}.$$

Suppose that $j = N+1$ and $2 \le i \le N$. Then, we have

$$\sum_{k=1}^{N+1} \binom{N+1-k}{i-1}\binom{N}{k-1}(-1)^{k-1} = \sum_{k=1}^{N+1}\binom{N}{i-1}\binom{N+1-i}{k-1}(-1)^{k-1}$$
$$= \binom{N}{i-1}\sum_{k=1}^{N+1}\binom{N+1-i}{k-1}(-1)^{k-1} = \binom{N}{i-1}\sum_{k=0}^{N}\binom{N+1-i}{k}(-1)^{k} = \binom{N}{i-1}(1-1)^{N+1-i}$$
$$= 0 = \binom{0}{N+1-i}.$$

Here, we used the identity

$$\binom{N-(k-1)}{i-1}\binom{N}{k-1} = \binom{N}{i-1}\binom{N-(i-1)}{k-1}.$$

□

Lemma 3.3. *Let i and j be integers with $N \ge j > i \ge 1$. Then*

$$\frac{x^j}{y^i}\sum_{k,l=1}^{N}\left(\frac{z}{x}\right)^k\left(\frac{y}{z}\right)^l\binom{j}{k}\binom{k}{l}\binom{l}{i} = (x+y+z)^{j-i}\frac{j!}{i!(j-i)!}.$$

Proof. If we apply the binomial expansion twice, then we obtain

$$(x+y+z)^{j-i} = \sum_{k=0}^{j-i}\sum_{l=0}^{k}\binom{j-i}{k}\binom{k}{l}x^{j-i-k}y^l z^{k-l}.$$

Then, one can see that

$$(x+y+z)^{j-i} = \sum_{k=i}^{j}\sum_{l=i}^{k}\binom{j-i}{k-i}\binom{k-i}{l-i}x^{j-k}y^{l-i}z^{k-l}$$
$$= \frac{i!(j-i)!}{j!}\frac{x^j}{y^i}\sum_{k=i}^{j}\sum_{l=i}^{k}\left(\frac{z}{x}\right)^k\left(\frac{y}{z}\right)^l\binom{j}{k}\binom{k}{l}\binom{l}{i}.$$

By the definition of $\binom{a}{b}$ as in (27), we obtain the desired result. □

Lemma 3.4. *Suppose that N is an even positive integer. Let i and j be integers with $1 \le i, j \le N$.*

1. $\sum_{k=1}^{N}\binom{N-k}{N-i}\binom{N-j}{N-k}(-1)^{j+k} = \delta_{i,j}$.
2. *Let*

$$\Delta_{i,j} := \sum_{k,l=1}^{N}\left(\frac{x-1}{x}\right)^{\frac{N}{2}+1-i}x^{\frac{N}{2}+1-l}\left(\frac{1}{1-x}\right)^{\frac{N}{2}+1-k}(-1)^{l+k+j}\binom{N-l}{N-i}\binom{N-k}{N-l}\binom{N-j}{N-k}.$$

Then

$$\Delta_{i,j} = (-1)^{\frac{N}{2}+1}\delta_{i,j}.$$

Proof. (1) One can obtain (1) from Lemma 3.2 (1) by replacing i (resp. j) with $N-i+1$ (resp. $N-j+1$).

(2) Note that by the definition of $\binom{a}{b}$ as in (27), we have

$$\binom{N-l}{N-i}\binom{N-k}{N-l}\binom{N-j}{N-k} \ne 0$$

only when $i \ge l \ge k \ge j$. Therefore, if $i < j$, then $\Delta_{i,j} = 0$. If $i = j$, then a nonzero term in the summation appears only when $k = l = i = j$. So, we have $\Delta_{i,j} = (-1)^{\frac{N}{2}+1}$. In the case of $i > j$, one can see that

$$\Delta_{i,j} = \left(\frac{x-1}{x}\right)^{\frac{N}{2}+1-i}x^{-\frac{N}{2}+1}\left(\frac{1}{1-x}\right)^{-\frac{N}{2}+1}(-1)^j$$
$$\times\sum_{k,l=1}^{N}\left(\frac{-1}{1-x}\right)^{N-k}\left(\frac{x}{-1}\right)^{N-l}\binom{N-l}{N-i}\binom{N-k}{N-l}\binom{N-j}{N-k}.$$

By Lemma 3.3, $\Delta_{i,j}$ is equal to

$$\frac{(N-j)!}{(N-i)!(i-j)!}\left(\frac{x-1}{x}\right)^{\frac{N}{2}+1-i} x^{\frac{N}{2}+1-i}\left(\frac{1}{1-x}\right)^{\frac{N}{2}+1-j}(-1)^j \times ((1-x)+x+(-1))^{i-j} = 0.$$

This completes the proof. □

Now, we prove that ρ_N induces an automorphic factor of rank N satisfying the cocycle relation as in (24).

Proposition 3.5. *Let ρ_N be defined by (25) and (26). Then*
1. $\rho_N(S,Sx)\rho_N(S,x) = I$,
2. $\rho_N(U,U^2x)\rho_N(U,Ux)\rho_N(U,x) = I$.

Proof. (1) Lemma 3.4 (1) implies that $C_N^2 = I$, and hence we have

$$\rho_N(S,Sx)\rho_N(S,x) = I.$$

(2) For $U = TS$, we see that

$$\rho_N(U,x) = (-1)^{\frac{N}{2}+1}\left(\left(\frac{1}{1-x}\right)^{\frac{N}{2}+1-i}\binom{N-j}{N-i}(-1)^j\right)_{1\le i,j\le N},$$

$$\rho_N(U,Ux) = (-1)^{\frac{N}{2}+1}\left(x^{\frac{N}{2}+1-i}\binom{N-j}{N-i}(-1)^j\right)_{1\le i,j\le N},$$

and

$$\rho_N(U,U^2x) = (-1)^{\frac{N}{2}+1}\left(\left(\frac{x-1}{x}\right)^{\frac{N}{2}+1-i}\binom{N-j}{N-i}(-1)^j\right)_{1\le i,j\le N}.$$

By Lemma 3.4 (2), we have

$$\rho_N(U,U^2x)\rho_N(U,Ux)\rho_N(U,x) = (-1)^{\frac{N}{2}+1}(\Delta_{i,j})_{1\le i,j\le N} = I.$$

This is the desired result. □

Remark 3.6. *These kinds of automorphic factors are closely related with a monomial times a modular form. For example, we consider $F(z) := z^6\Delta(z)$, where $\Delta(z)$ is defined by*

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

and $q = e^{2\pi iz}$ for $z \in \mathbb{H}$. Then, it is known that $\Delta(z)$ is a modular form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. Note that

$$F(z+1) = \frac{(z+1)^6}{z^6}F(z)$$

and

$$F\left(-\frac{1}{z}\right) = F(z).$$

Hence, we see that $F(z)$ is a modular form associated with ρ, which is defined by

$$\rho(T,z) = \left(\frac{z+1}{z}\right)^6,\ \rho(S,z) = 1.$$

3.2 Modular property of G_N

With the automorphic factor ρ_N, we can state the transformation property of the vector-valued function G_N. We define a slash operator associated with ρ_N as follows. Let f be a vector-valued function on $\mathbb{Q}$, i.e., f is a sum of functions $f = \sum_{i=1}^m f_i \mathbf{e}_i$, where f_i is a function on $\mathbb{Q}$ for $i = 1, \ldots, m$. Then we define

$$(f|_{\rho_N}\gamma)(x) := \sum_{i=1}^{m} f_i(\gamma x)\rho_N^{-1}(\gamma, x)\mathbf{e}_i$$

for $x \in \mathbb{Q}$ and $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Lemma 3.7. *The vector-valued function $G_N : \mathbb{Q} \to \mathbb{C}^N$ satisfies the functional equations*

$$\left(G_N - G_N|_{\rho_N} S\right)(x) = A_N(x)^{-1}E_N(x) + 2B_1^2\delta_{N,2}\mathbf{e}_2$$

for all positive rational numbers x and

$$\left(G_N - G_N|_{\rho_N} T\right)(x) = 0$$

for all $x \in \mathbb{Q}$.

Proof. In this proof, we will use the properties of generalized Dedekind sums. For positive integers p and q which are relatively prime, by Corollary 2.4, we have

$$A_N\left(\frac{p}{q}\right)F_N(p,q) + B_N(p,q)F_N(q,p) = E_N\left(\frac{p}{q}\right),$$

where $A_N\left(\frac{p}{q}\right)$, $B_N(p,q)$, $F_N(p,q)$, and $E_N\left(\frac{p}{q}\right)$ are defined as in (8), (21), (6), and (9), respectively.

Lemma 3.2 (1) implies that

$$A_N\left(\frac{p}{q}\right)^{-1} = (v_{i,j}(p,q))_{1\le i,j\le N}, \quad v_{i,j} = -q^{-\frac{N}{2}+j-1}p^{\frac{N}{2}-j}\binom{N-j}{i-1}$$

since $(v_{i,j}(p,q))_{1\le i,j\le N} \times A_N\left(\frac{p}{q}\right) = (\Lambda_{i,j})_{1\le i,j\le N}$, which is the identity matrix. Lemma 3.2 (2) implies that

$$A_N\left(\frac{p}{q}\right)^{-1}B_N(p,q) = C_N.$$

Therefore, we see that

$$G_N(x) - (-C_N)G_N\left(\frac{1}{x}\right) = A_N(x)^{-1}E_N(x)$$

for any positive rational number x since $G_N(x) = F_N(p,q)$, where $x = \frac{p}{q}$. By Theorem 2.5 (1), we see that

$$G_N(-x) = \begin{cases} (-1)^{\frac{N}{2}}G_N(x) - 2B_1^2\mathbf{e}_2 & \text{if } N = 2, \\ (-1)^{\frac{N}{2}}G_N(x) & \text{if } N \ge 4. \end{cases}$$

Therefore, we have the first desired result.

The second result directly comes from Theorem 2.5 (2) and the definition of $\rho_N(T)$ as in (26). □

Note that $A_N(x)^{-1}E_N(x)$ is a rational function of p and q, where $x = \frac{p}{q} \in \mathbb{Q}$. But $A_N(x)$ and $E_N(x)$ are homogeneous of degree 1, and hence $A_N(x)^{-1}E_N(x)$ is homogeneous of degree 0. Therefore, $A_N(x)^{-1}E_N(x)$ is actually a rational function of x. This implies that this vector-valued function can be extended to $\mathbb{R} \setminus \{0\}$. The vector-valued function G_N can be understood as a quantum modular form, which is a new modular object on $\mathbb{Q}$ introduced by Zagier [8]. Quantum modular forms were studied in connection with Maass forms, mock modular forms and Eichler integrals (for example, see [9, 10]).

3.3 Proof of Theorem 1.1

By Lemma 3.7, we have

$$G_N(x) - ((-1)^{\frac{N}{2}+1}C_N)G_N\left(-\frac{1}{x}\right) = A_N(x)^{-1}E_N(x) + 2B_1^2\delta_{N,2}\mathbf{e}_2$$

for positive rational numbers x. If we let $x = \frac{1}{n}$ for $n \in \mathbb{N}$, then

$$G_N\left(\frac{1}{n}\right) \sim ((-1)^{\frac{N}{2}+1}C_N)G_N(-n) + A_N\left(\frac{1}{n}\right)^{-1}E_N\left(\frac{1}{n}\right) + 2B_1^2\delta_{N,2}C_N\mathbf{e}_2$$

as $n \to \infty$.

3.4 Application of modular property to the arithmetic of Dedekind sums

With the cocycle property of ρ_2, one can obtain an explicit expression of $G_2(\frac{p}{q})$ in terms of the negative continued fraction of $\frac{p}{q}$. For this, we recall the followings:

1. $G_2\left(\frac{p}{q}\right) = \begin{pmatrix} p^{-1}h_{0,2}(p,q) \\ h_{1,1}(p,q) \end{pmatrix} = \begin{pmatrix} \frac{1}{12}p^{-1}q^{-1} \\ s_{1,1}(p,q)+\frac{1}{4} \end{pmatrix}$,
2. $\rho_2(S,x)^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$,
3. $\rho_2(T,x)^{-1} = \begin{pmatrix} \frac{x+1}{x} & 0 \\ 0 & 1 \end{pmatrix}$,
4. $G_2(x) - (G_2|_{\rho_2}S)(x) = \begin{pmatrix} 0 \\ \frac{B_2}{2}\frac{1}{x} - 3B_1^2 + \frac{B_2}{2}x \end{pmatrix}$,
5. $G_2(x) - (G_2|_{\rho_2}T)(x) = 0$.

By the cocycle condition as in (24), we obtain

$$\begin{aligned} G_2(x) - (G_2|_{\rho_2}T^aS)(x) &= G_2(x) - \rho_2^{-1}(S,x)\rho_2(T^a,Sx)^{-1}G_2(T^aSx) = G_2(x) - \rho_2^{-1}(S,x)G_2(Sx) \\ &= \begin{pmatrix} 0 \\ \frac{B_2}{2}\frac{1}{x} - 3B_1^2 + \frac{B_2}{2}x \end{pmatrix} \end{aligned} \quad (28)$$

for a positive integer a.

Now, we express $\frac{p}{q}$ by using the negative continued fraction as follows

$$\frac{p}{q} = [a_1, a_2, \ldots, a_n] := a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \frac{1}{a_n}}}$$

and define

$$\frac{p_i}{q_i} := [a_i, a_{i+1}, \ldots, a_n]$$

for $1 \le i \le n$. Then, for $1 \le i \le n-1$, we see that

$$\begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} p_i \\ q_i \end{pmatrix}. \quad (29)$$

Moreover, for $1 \le i \le n-1$, we also have

$$G_2\left(\frac{p_{i+1}}{q_{i+1}}\right) - \rho_2^{-1}\left(\begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix}, \frac{p_{i+1}}{q_{i+1}}\right)G_2\left(\frac{p_i}{q_i}\right) = E_2\left(\frac{p_{i+1}}{q_{i+1}}\right). \quad (30)$$

For simplicity, we define

$$F_i := \rho_2^{-1}\left(\begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix}, \frac{p_{i+1}}{q_{i+1}}\right) = \begin{pmatrix} \frac{p_i}{q_{i+1}} & 0 \\ \frac{p_i}{q_{i+1}} & 1 \end{pmatrix}.$$

By combining equations in (30), we have

$$G_2\left(\frac{p}{q}\right) = F_1^{-1}F_2^{-1}\cdots F_{n-1}^{-1}G_2\left(\frac{p_n}{q_n}\right) - E_2\left(\frac{p_2}{q_2}\right) - E_2\left(\frac{p_3}{q_3}\right) - \cdots - E_2\left(\frac{p_n}{q_n}\right).$$

Note that

$$F_1^{-1}F_2^{-1}\cdots F_{n-1}^{-1} = \begin{pmatrix} \frac{a_n}{pq} & 0 \\ -\frac{a_n q'}{q} & 1 \end{pmatrix},$$

where

$$\frac{q'}{p'} = [a_2, a_3, \dots, a_{n-1}].$$

Moreover, we have

$$G_2\left(\frac{p_n}{q_n}\right) = \frac{1}{12}\begin{pmatrix} a_n^{-1} \\ 3 \end{pmatrix}, \quad G_2\left(\frac{p}{q}\right) = \frac{1}{12}\begin{pmatrix} p^{-1}q^{-1} \\ 12s_{1,1}(p,q)+3 \end{pmatrix},$$

and

$$E_2\left(\frac{p_2}{q_2}\right) + E_2\left(\frac{p_3}{q_3}\right) + \cdots + E_2\left(\frac{p_n}{q_n}\right) = \begin{pmatrix} 0 \\ \frac{1}{12}\left(\frac{-p}{q} + a_1 + a_2 + \cdots + a_n\right) - \frac{n-1}{4} \end{pmatrix}.$$

Finally, we obtain

$$12s_{1,1}(p,q) = \frac{p-q'}{q} - (a_1 + a_2 + \cdots + a_n) + 3(n-1).$$

Eventually, this formula gives a simple expression in $12s_{1,1}(p,q)$ modulo 1 that plays a crucial role of the proof for the equidistribution property of $h_{1,1}(x)$.

Remark 3.8. *We note that in above examples, the generalized Dedekind sum $s_{1,1}(p,q)$ is completely determined by the initial values and transformation formulas by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Along this line, restricting ourselves to the two dimensional case, the Apostol sum $s_{1,n}(p,q)$ is completely determined by a two term relation*

$$(n+1)\left(pq^n s_{1,n}(p,q) + p^n q s_{1,n}(q,p)\right) = \sum_{i=0}^{n-1}\binom{n+1}{i}(-1)^i B_i B_{n+1-i} p^i q^{n+1-i} + nB_{n+2}$$

together with the continued fraction of $\frac{p}{q}$ (cf. [5]).

In this paper, we consider a similar property for generalized Dedekind sums $s_{i,j}(p,q)$. Usually, generalized Dedekind sums $s_{i,j}$, $(i \neq 1, j \neq 1)$ except the outermost case (i.e. Apostol sums) do not have two term reciprocity formula. Namely,

$$\frac{1}{2!2!}s_{2,2}(p,q) - \frac{3p}{4!q^3}s_{0,4}(q,p) + \frac{p}{3!q^2}s_{1,3}(q,p) + \frac{p}{2!2!q}s_{2,2}(q,p) = f(p,q)$$

for a Laurent polynomial $f(p,q)$. In this case, the reciprocity formula for $s_{2,2}(p,q)$ cannot be reduced to a Laurent polynomial, but involves a transcendental term $s_{1,3}(p,q)$.

However, if we consider a vector consisting of generalized Dedekind sums, then they have a two term reciprocity formula as follows

$$\begin{pmatrix} \frac{1}{4!}s_{0,4}(p,q) \\ \frac{1}{3!}s_{1,3}(p,q) \\ \frac{1}{2!2!}s_{2,2}(p,q) \\ \frac{1}{3!}s_{3,1}(p,q) \end{pmatrix} + \begin{pmatrix} -\frac{p^3}{q^3} & 0 & 0 & 0 \\ -\frac{3p^2}{q^3} & \frac{p^2}{q^2} & 0 & 0 \\ -\frac{3p}{q^3} & \frac{2p}{q^2} & \frac{p}{q} & 0 \\ -\frac{1}{q^3} & \frac{1}{q^2} & -\frac{1}{q} & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{4!}s_{0,4}(q,p) \\ \frac{1}{3!}s_{1,3}(q,p) \\ \frac{1}{2!2!}s_{2,2}(q,p) \\ \frac{1}{3!}s_{3,1}(q,p) \end{pmatrix} = F(p,q)$$

for a column vector $F(p,q) = (f_1(p,q), \dots, f_4(p,q))^T$ consisting of Laurent polynomials in p and q. Thus, we are able to compute algorithmically the (vector of) generalized Dedekind sums of (p,q) of fixed weight using the continued fraction of $\frac{p}{q}$, once we obtain the reciprocity formula.

4 Distribution of generalized Dedekind sums

For an even positive integer N and positive integers p and q which are relatively prime, we consider a vector-valued function $H_N(p,q)$ defined in (11). In this section, we show that there exists an integer R_N such that fractional parts of the vectors $\langle R_N H_N(p,q)\rangle$ are equidistributed. That is the image of these vectors under the projection $\mathbb{R}^N$ onto $\mathbb{R}^N/\mathbb{Z}^N$ is equidistributed on the torus. There is a necessary and sufficient condition for this due to Weyl.

4.1 Weyl's equidistribution criterion

We recall the statement of Weyl's criterion on torus. For details, we refer to [11].

Theorem 4.1 (Weyl's equidistribution criterion). *A sequence*

$$\{s_k = (s_1^{(k)}, s_2^{(k)}, \dots, s_n^{(k)}) \in [0,1)^n\}_{k\in\mathbb{N}}$$

is equidistributed in $[0,1)^n$ *if and only if for every* $m = (m_1, m_2, \dots, m_n) \in \mathbb{Z}^n - \{0\}$,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} \mathbf{e}(m \cdot s_k) = 0,$$

where $m \cdot s_k = \sum_{i=1}^n m_i s_i^{(k)}$ *and* $\mathbf{e}(x)$ *denotes* $\exp(2\pi i x)$.

For a nonzero vector $m \in \mathbb{Z}^{N-1}$ and a positive real number x, let $E(m,x)$ be the average of the exponentials of $(2\pi i)m \cdot R_N H_N(p,q)$ defined by

$$E(m,x) := \frac{1}{\#\{(p,q) |\, \gcd(p,q)=1, p<q\le x\}} \sum_{0<q<x} \sum_{\substack{0<p<q \\ (p,q)=1}} \mathbf{e}\left(m \cdot R_N H_N(p,q)\right). \tag{31}$$

To apply Theorem 4.1, one needs to show that $E(m,x)$ tends to 0 as x goes to ∞. This is done by relating an exponential sum to

$$\sum_{\substack{0<p<q \\ (p,q)=1}} \mathbf{e}\left(m \cdot R_N H_N(p,q)\right). \tag{32}$$

4.2 Exponential sums of generalized Dedekind sums

We relate (32) to an exponential sum for a Laurent polynomial. Let us first recall the exponential sum for a Laurent polynomial $F(x) \in \mathbb{Z}[x, x^{-1}]$.

Definition 4.2. *For a positive integer q and $F(x) \in \mathbb{Z}[x,x^{-1}]$, we define the exponential sum of modulus q of $F(x)$ as*

$$K(F,q) := \sum_{x\in(\mathbb{Z}/q\mathbb{Z})^*} \mathbf{e}_q\left(F(x)\right),$$

where $\mathbf{e}_q(x) := \exp\left(2\pi i \frac{x}{q}\right)$.

Let $F_N(x)$ be the rank $(N-1)$ vector of Laurent polynomials

$$F_N(x) := \left(f_1(x), f_2(x), \dots, f_{N-1}(x)\right), \tag{33}$$

where $f_i(x) = \alpha_N r_N \left(\binom{N-1}{i} x^{-i} + \binom{N-1}{N-i} x^{N-i}\right)$. Here, α_N denotes the numerator of B_N, and r_N is the integer defined as in (13). For a nonzero integer vector m of rank $(N-1)$, we have the following Laurent polynomial in x

$$m \cdot F_N(x) = \alpha_N r_N \sum_{i=1}^{N-1} \left[m_i \binom{N-1}{i} x^{-i} + m_i \binom{N-1}{N-i} x^{N-i} \right]. \tag{34}$$

To prove the relation between the two exponential sums (32) and $K(m \cdot F_N, q)$, we need the following theorem.

Theorem 4.3. *[12, Theorem 1.1] Let N be an even positive integer. For positive integers i and j with $i + j = N$,*

$$R_N q^{N-2} h_{ij}(p,q) - \alpha_N r_N \frac{(p')^i \binom{N-1}{i} + p^j \binom{N-1}{j}}{q} \in \mathbb{Z},$$

where p' is an integer such that $p'p \equiv 1 \pmod q$, and $R_N = N!\beta_N r_N$ with β_N being the denominator of B_N.

By Theorem 4.3, one can see that

$$\sum_{\substack{0<p<q \\ (p,q)=1}} \mathbf{e}\left(m \cdot R_N H_N(p,q)\right) = K(m \cdot F_N, q).$$

Therefore, we come to the estimation of the exponential sum of $m \cdot F_N(x)$.

4.3 Bounds for exponential sums

Let q be a prime. Then, the estimation of $K(m \cdot F_N, q)$, accompanied with some reductions, will be sufficient in showing Weyl's criterion for

$$\{\langle R_N H_N(p,q)\rangle \,|\, (p,q) = 1, 0 < p < q\}$$

to be equidistributed. This will complete the proof of Theorem 1.3. To achieve the full estimation, we will follow the steps taken in [12].

The following lemma is necessary to prove that $m \cdot F_N(x)$ has a Weil type bound for all but finitely many primes p.

Lemma 4.4. *For positive integers i and j, let $F(x) = \sum_{k=-j}^{i} a_k x^k$ be a Laurent polynomial with integer coefficients such that $a_i \neq 0$ and $a_{-j} \neq 0$. Let p be any prime with $p \nmid a_k$ for some $k \neq 0$. Then*

$$K(F,p) \leq (i+j)\sqrt{p}.$$

Proof. See Theorem 1.3 in [13]. □

Since $m \cdot F_N$ is written as Laurent polynomial type in Lemma 4.4 (see equation (34)), we have the following.

Proposition 4.5. *Let m be a nonzero integer vector and $m \cdot F_N(x)$ be the Laurent polynomial as give in (33). Put $d = \gcd(m_1, m_2, \ldots, m_{N-1})$. Then for any positive integer $p \nmid \alpha_N r_N d$, we have*

$$|K(m \cdot F_N, p)| \leq 2(N-1)\sqrt{p}. \tag{35}$$

For a general modulus q, we have the following bound.

Proposition 4.6. *Suppose that q is a positive integer, and that N is an even positive integer. Let D_{F_N} be a positive integer such that $D_{F_N} || F_N$. Let $\omega(q)$ be the number of prime factors of q. Then*

$$|K(m \cdot F_N, q)| \leq D_{F_N}(12N-12)^{\omega(q)} q^{(1-\frac{1}{3N-2})}.$$

From now on, we justify Proposition 4.6.

4.4 Reduction to prime modulus

If q has many prime factors, the bound is obtained by composing the bound previously obtained for primes dividing q. This is done by the next two reduction steps.

First, we consider the case q being a power of a prime p.

Lemma 4.7. *Let $F(x)$ be a Laurent polynomial with integer coefficients. Let p be a fixed prime and $p^\beta \| F(x)$. Then for $\alpha > \beta$, we have*

$$K(F, p^\alpha) = p^\beta K(\tilde{F}, p^{\alpha-\beta}),$$

where $\tilde{F}(x) = \frac{1}{p^\beta} F(x)$.

Proof. We note that an element $z \in (\mathbb{Z}/p^\alpha\mathbb{Z})^*$ is written uniquely as $z = p^{\alpha-\beta}x + y$ for $x \in \mathbb{Z}/p^\beta\mathbb{Z}$ and $y \in (\mathbb{Z}/p^{\alpha-\beta}\mathbb{Z})^*$. Thus, we obtain

$$K(F, p^\alpha) = \sum_{z\in(\mathbb{Z}/p^\alpha\mathbb{Z})^*} \mathbf{e}\left(\frac{F}{p^\alpha}\right) = \sum_{x\in\mathbb{Z}/p^\beta\mathbb{Z}} \sum_{y\in(\mathbb{Z}/p^{\alpha-\beta}\mathbb{Z})^*} \mathbf{e}\left(\frac{\tilde{F}}{p^{\alpha-\beta}}\right) = p^\beta K(\tilde{F}, p^{\alpha-\beta}). \qquad \square$$

After the previous lemma, we can pull out p-factors out of the coefficients of $m \cdot F_N(x)$. For positive integers i and j, let $F(x) = \sum_{k=-j}^{i} a_k x^k$ be a Laurent polynomial with integer coefficients such that $a_i \neq 0$ and $a_{-j} \neq 0$. Let p be a prime such that $F(x) \not\equiv 0 \pmod p$ and α be a positive integer.

Now, the following lemma is implied by Corollary 4.1 in [14] with the trivial character.

Lemma 4.8. *With the above notation, suppose that $\alpha \geq 2$ is an integer. Then*

$$|K(F, p^\alpha)| \leq 4(i + 2j)p^{\alpha(1-\frac{1}{i+2j+1})}.$$

The previous two lemmas imply the following bound for prime powers.

Proposition 4.9. *Let p be a prime. Suppose that m is any fixed nonzero vector in $\mathbb{Z}^{N-1}$, and that $p^\beta \| m \cdot F_N(x)$ for some integer β. Suppose that $\alpha \geq 2$ is an integer such that $\alpha > \beta$. Then for any p, we have*

$$|K(m \cdot F_N, p^\alpha)| \leq p^\beta 12(N-1)p^{\alpha(1-\frac{1}{3N-2})} \leq D_{F_N} 12(N-1)p^{\alpha(1-\frac{1}{3N-2})},$$

where D_{F_N} is the largest positive integer such that $D_{F_N} | F_N$.

Let us consider the case when q has several prime factors. We have the following effect of the Chinese remainder theorem for exponential sums.

Lemma 4.10. *Let $F(x)$ be a Laurent polynomial with integer coefficients, and let $q_1 > 1$ and $q_2 > 1$ be relatively prime integers. Let $F_i(x)$ be the mod q_i Chinese remainder of $F(x)$ for $i = 1, 2$ (i.e. $F \mapsto (F_1, F_2)$ under the isomorphism $(\mathbb{Z}/q_1q_2\mathbb{Z})[x, x^{-1}] \to (\mathbb{Z}/q_1\mathbb{Z})[x, x^{-1}] \times (\mathbb{Z}/q_2\mathbb{Z})[x, x^{-1}]$). Then*

$$K(F, q_1q_2) = K(F_1, q_1)K(F_2, q_2).$$

Proof. This is a consequence of Fubini theorem. □

4.5 Proof of Theorem 1.3

For $x > 1$, let $\phi(x) := |(\mathbb{Z}/[x]\mathbb{Z})^*|$ be Euler's phi function. By using $\sum_{q<x} \phi(q) \sim x^2$ as $x \to \infty$, we obtain the proof of the main theorem from Weyl's criterion for equidistribution and Proposition 4.6.

Now, we apply Proposition 4.6 to deducing Weyl's criterion from the bound of exponential sums. Note that $\omega(q)$ in Proposition 4.6 has a well-known estimation

$$\omega(q) \leq \frac{c \log q}{\log\log q} \tag{36}$$

for some constant c. For sufficiently large q,

$$(12N-12)^{\omega(q)} \le (12N-12)^{\frac{c\log q}{\log\log q}} \le (q)^{\frac{c\log(12N-12)}{\log\log q}}.$$

Thus, we obtain that for any $\epsilon > 0$,

$$(12N-12)^{\omega(q)} \ll q^{\epsilon}.$$

Therefore, by Proposition 4.6, we have the following bound.

Proposition 4.11. *Let N be an even positive integer, and let m be a nonzero integer vector of rank $N-1$. Then*

$$|K(m\cdot F_N, q)| \ll q^{(1+\epsilon-\frac{1}{3N-2})}$$

for all $\epsilon > 0$.

Weyl's criterion for $H_N(p,q)$ comes from the following estimation

$$\sum_{0<q<x}\sum_{\substack{0<p<q\\(p,q)=1}} \mathbf{e}\left(m\cdot R_N H_N(p,q)\right) = \sum_{0<q<x} K(m\cdot F_N, q) \le x^{(2+\epsilon-\frac{1}{3N-2})}. \tag{37}$$

Consequently, Weyl's criterion is fulfilled for the fractional part of the vector $H_N(p,q)$:

$$E(m,x) = \frac{1}{\#\left\{(p,q)|\gcd(p,q)=1, p<q\le x\right\}}\sum_{0<q<x}\sum_{\substack{0<p<q\\(p,q)=1}} \mathbf{e}(m\cdot R_N H_N(p,q)) \to 0 \tag{38}$$

as $x\to\infty$. This completes the proof.

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