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On the recursive properties of one kind hybrid power mean involving two-term exponential sums and Gauss sums

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Abstract: The main purpose of this paper is to study the computational problem of one kind hybrid power mean involving two-term exponential sums and quartic Gauss sums using the analytic method and the properties of the classical Gauss sums, and to prove some interesting fourth-order linear recurrence formulae for this problem. As an application of our result, we can also obtain an exact computational formula for one kind congruence equation mod p, an odd prime.

Keywords: The quartic Gauss sums, Two-term exponential sums, Hybrid power mean, The fourth-order linear recurrence formula

MSC: 11L05, 11L07

1 Introduction

Let $p \geq 3$ be an odd prime. For any integer m with $(m,p) = 1$, the quartic Gauss sums $B(m) = B(m,p)$ is defined as

$$B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right),$$

where as usual, $e(y) = e^{2\pi iy}$.

Recently, some scholars have studied the hybrid power mean problems of various trigonometric sums, and obtained many interesting results. For example, Chen Li and Hu Jiayuan [1] studied the computational problem of the hybrid power mean

$$S_k(p) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^3}{p}\right)\right)^k \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{mc+\overline{c}}{p}\right)\right|^2,$$

where $\overline{c}$ denotes the multiplicative inverse of c mod p. That is, $c \cdot \overline{c} \equiv 1 \bmod p$.
For $p \equiv 1 \bmod 3$, they used the elementary method to obtain an interesting third-order linear recurrence formula for $S_k(p)$.

Li Xiaoxue and Hu Jiayuan [2] studied the computational problem of the hybrid power mean

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc+\overline{c}}{p}\right)\right|^2, \tag{1}$$

and proved an exact computational formula for (1).

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Zhang Han and Zhang Wenpeng [3] proved the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+na}{p}\right)\right|^4 = \begin{cases} 2p^3-p^2 & \text{if } 3 \nmid p-1, \\ 2p^3-7p^2 & \text{if } 3 \mid p-1. \end{cases}$$

Other related results can also be found in references [4-13].

In this paper, we will consider the calculating problem of the following hybrid power mean:

$$V_k(p) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^k \cdot \left(\sum_{b=0}^{p-1} e\left(\frac{mb^4+b}{p}\right)\right)^3, \tag{2}$$

where $k \geq 0$ is an integer.

If $p = 4h+3$, then from the properties of the Legendre's symbol mod p we have (see [14], formula (30) in Chapter 9)

$$\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1}(1+\chi_2(a))\, e\left(\frac{ma^2}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right) = i\chi_2(m)\sqrt{p},$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

So in this case, the problem we considered in (2) is trivial. If $p = 4h+1$, then the situation is more complicated. We will use the analytic method and the properties of classical Gauss sums to study this problem, and prove some new interesting fourth-order linear recurrence formulae for (2) with $p = 4h+1$. That is, we will give the following four results.

Theorem 1.1. *Let p be a prime with $p = 24h+1$. Then for any integer $k \geq 4$, we have the fourth-order linear recurrence formula*

$$V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p-4\alpha^2\right)V_{k-4}(p),$$

where the first four values are $V_0(p) = p^2 - 6p\alpha$, $V_1(p) = p\left(p^2-16p-4\alpha^2\right)$, $V_2(p) = p^2\left(2p\alpha+3p-58\alpha\right)$ and $V_3(p) = p^2\left(7p^2+4p\alpha-92p-72\alpha^2\right)$, $\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)$ is an integer, which satisfies the identity (see Theorem 4-11 in [15])

$$p = \alpha^2+\beta^2 \equiv \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}}\left(\frac{a+r\overline{a}}{p}\right)\right)^2,$$

which r is any quadratic non-residue mod p.

Theorem 1.2. *Let p be a prime with $p = 24h+17$. Then for any integer $k \geq 4$, we have the fourth-order linear recurrence formula*

$$V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p-4\alpha^2\right)V_{k-4}(p),$$

where the first four values are $V_0(p) = -p^2 - 6p\alpha$, $V_1(p) = p\left(p^2-18p-4\alpha^2\right)$, $V_2(p) = p^2\left(2p\alpha-3p-62\alpha\right)$ and $V_3(p) = p^2\left(7p^2-4p\alpha-106p-72\alpha^2\right)$.

Theorem 1.3. *Let p be a prime with $p = 24h+5$. Then for any integer $k \geq 4$, we have the fourth-order linear recurrence formula*

$$V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p-4\alpha^2\right)V_{k-4}(p),$$

where the first four terms are $V_0(p) = -\left(p^2+6p\alpha\right)$, $V_1(p) = -p\left(p^2-8p+4\alpha^2\right)$, $V_2(p) = -p^2\left(2p\alpha-p-22\alpha\right)$ and $V_3(p) = p^2\left(5p^2-6p\alpha-28p-36\alpha^2\right)$.

Theorem 1.4. *Let p be a prime with $p = 24h + 13$. Then for any integer $k \geq 4$, we have the fourth-order linear recurrence formula*

$$V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right)V_{k-4}(p),$$

where the first four terms are $V_0(p) = p^2 - 6p\alpha$, $V_1(p) = -p\left(p^2 - 6p + 4\alpha^2\right)$, $V_2(p) = -p^2\left(2p\alpha + p - 18\alpha\right)$ and $V_3(p) = p^2\left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right)$.

From our theorems we may immediately deduce the following:

Corollary 1.5. *Let p be a prime with $p \equiv 1 \bmod 4$, then we have the identity*

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^3 \cdot \left(\sum_{b=0}^{p-1} e\left(\frac{mb^4+b}{p}\right)\right)^3 = \begin{cases} p^2\left(7p^2 + 4p\alpha - 92p - 72\alpha^2\right) & \text{if } p = 24h+1, \\ p^2\left(7p^2 - 4p\alpha - 106p - 72\alpha^2\right) & \text{if } p = 24h+17, \\ p^2\left(5p^2 - 6p\alpha - 28p - 36\alpha^2\right) & \text{if } p = 24h+5, \\ p^2\left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right) & \text{if } p = 24h+13. \end{cases}$$

Note that the estimate $|\alpha| \leq \sqrt{p}$, from Corollary 1.5 we also have the following:

Corollary 1.6. *Let p be a prime with $p \equiv 1 \bmod 8$, then we have the asymptotic formula*

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^3 \cdot \left(\sum_{b=0}^{p-1} e\left(\frac{mb^4+b}{p}\right)\right)^3 = 7p^4 + O\left(p^{\frac{7}{2}}\right).$$

Corollary 1.7. *Let p be a prime with $p \equiv 5 \bmod 8$, then we have the asymptotic formula*

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^3 \cdot \left(\sum_{b=0}^{p-1} e\left(\frac{mb^4+b}{p}\right)\right)^3 = 5p^4 + O\left(p^{\frac{7}{2}}\right).$$

For any prime p with $p \equiv 1 \bmod 4$ and any positive integer k, let $M_k(p)$ denote the number of the solutions of the congruence equation

$$x_1^4 + x_2^4 + \cdots + x_k^4 + y_1^4 + y_2^4 + y_3^4 \equiv 0 \bmod p, \quad y_1 + y_2 + y_3 \equiv 0 \bmod p,$$

where $0 \leq x_i,\ y_j \leq p-1$, $i = 1,\ 2, \cdots, k$, $j = 1,\ 2,\ 3$.

Then from our theorems we can give an exact computational formula for $M_k(p)$. For example, let $H_s(p)$ denote the number of the congruence equation

$$x_1^4 + x_2^4 + \cdots + x_s^4 \equiv 0 \bmod p, \quad 0 \leq x_i \leq p-1, \ i = 1,\ 2, \cdots s.$$

Then we have the identity

$$V_k(p) = \frac{p^2}{p-1} \cdot M_k(p) - \frac{p}{p-1} \cdot H_k(p).$$

Since $H_k(p)$ has a fourth-order linear recurrence formula (see [8]), so from the above formula and our theorems we can deduce the exact value of $M_k(p)$.

2 Several lemmas

To complete the proofs of our theorems, we need to prove four simple lemmas. Hereafter, we will use many properties of the classical Gauss sums and the fourth-order character mod p, all of which can be found in books concerning Elementary Number Theory or Analytic Number Theory, such as references [7], [14] or [15]. Some important results related to Gauss sums can also be found in [16] and [17]. These contents will not be repeated here. First we have the following:

Lemma 2.1. *Let p be a prime with $p \equiv 1 \bmod 4$, λ be any fourth-order character $\bmod\ p$, then we have*

$$\tau^2(\lambda) + \tau^2\left(\overline{\lambda}\right) = \sqrt{p} \cdot \sum_{a=1}^{p-1} \left(\frac{a + \overline{a}}{p} \right) = 2\sqrt{p} \cdot \alpha,$$

where $\tau(\lambda) = \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{a}{p}\right)$ denotes the classical Gauss sums, and $\left(\frac{*}{p}\right)$ is the Legendre's symbol $\bmod\ p$.

Proof. In fact this is Lemma 2 of [18], so its proof is omitted. □

Lemma 2.2. *Let p be a prime with $p \equiv 1 \bmod 4$, then for any fourth-order character $\lambda \bmod p$, we have the identity*

$$\sum_{m=1}^{p-1} \lambda(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p} \right) \right)^3 = \begin{cases} -5p\tau(\lambda) - 2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right) & \text{if } p \equiv 1 \bmod 8, \\ -p\tau(\lambda) - 2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right) & \text{if } p \equiv 5 \bmod 8, \end{cases}$$

where α is the same as in Lemma 2.1.

Proof. First applying trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n \end{cases} \tag{3}$$

and note that $\lambda^4 = \chi_0$, the principal character mod p, we have

$$\begin{aligned} \sum_{m=1}^{p-1} \lambda(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p} \right) \right)^3 &= \sum_{m=1}^{p-1} \lambda(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p} \right) \right)^2 \\ &+ \sum_{m=1}^{p-1} \lambda(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p} \right) \right)^2 \left(\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p} \right) \right) \\ &= \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\lambda}\left(a^4 + b^4 + 1\right) \sum_{c=1}^{p-1} e\left(\frac{c(a+b+1)}{p} \right) \\ &+ \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\lambda}\left(a^4 + b^4\right) e\left(\frac{a+b}{p} \right) \\ &= \tau(\lambda) p \sum_{\substack{a=0 \\ a+b+1 \equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \overline{\lambda}\left(a^4 + b^4 + 1\right) - \tau(\lambda) \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \overline{\lambda}\left(a^4 + b^4 + 1\right) \\ &- \tau(\lambda) + \tau(\lambda) \sum_{a=0}^{p-1} \overline{\lambda}\left(a^4 + 1\right) \sum_{b=1}^{p-1} e\left(\frac{b(a+1)}{p} \right). \end{aligned} \tag{4}$$

From (3) we have

$$\begin{aligned} \tau(\lambda) \sum_{a=0}^{p-1} \overline{\lambda}\left(a^4 + 1\right) \sum_{b=1}^{p-1} e\left(\frac{b(a+1)}{p} \right) &= \overline{\lambda}(2)\tau(\lambda)(p-1) - \tau(\lambda) \sum_{a=0}^{p-2} \overline{\lambda}\left(a^4 + 1\right) \\ &= \overline{\lambda}(2)\tau(\lambda)p - \tau(\lambda) \sum_{a=0}^{p-1} \overline{\lambda}\left(a^4 + 1\right). \end{aligned} \tag{5}$$

Note that the identity $\lambda\chi_2 = \overline{\lambda}$ and

$$B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p} \right) = \chi_2(m)\sqrt{p} + \overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right). \tag{6}$$

From (6) we have

$$\tau(\lambda)\sum_{a=0}^{p-1}\overline{\lambda}\left(a^4+1\right)=\sum_{b=1}^{p-1}\lambda(b)\sum_{a=0}^{p-1}e\left(\frac{b\left(a^4+1\right)}{p}\right)$$
$$=\sum_{b=1}^{p-1}\lambda(b)\left(\chi_2(b)\sqrt{p}+\overline{\lambda}(b)\tau(\lambda)+\lambda(b)\tau\left(\overline{\lambda}\right)\right)e\left(\frac{b}{p}\right)$$
$$=\sqrt{p}\tau\left(\overline{\lambda}\right)-\tau(\lambda)+\sqrt{p}\tau\left(\overline{\lambda}\right)=2\sqrt{p}\tau\left(\overline{\lambda}\right)-\tau(\lambda). \tag{7}$$

If $p\equiv 5 \bmod 8$, then note that $\lambda(-1)=-1$ and $\tau(\lambda)\tau\left(\overline{\lambda}\right)=-p$, applying (6) and Lemma 2.1 we also have

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}\left(a^4+b^4+1\right)=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}e\left(\frac{ca^4+cb^4+c}{p}\right)$$
$$=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)e\left(\frac{c}{p}\right)\left(\sum_{a=0}^{p-1}e\left(\frac{ca^4}{p}\right)\right)^2$$
$$=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)e\left(\frac{c}{p}\right)\left(\chi_2(b)\sqrt{p}+\overline{\lambda}(b)\tau(\lambda)+\lambda(b)\tau\left(\overline{\lambda}\right)\right)^2$$
$$=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)\left(2\chi_2(c)\sqrt{p}\alpha-p+2\lambda(c)\sqrt{p}\tau(\lambda)+2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)e\left(\frac{c}{p}\right)$$
$$=p+\frac{2\sqrt{p}\left(\alpha-1\right)\tau\left(\overline{\lambda}\right)}{\tau(\lambda)}. \tag{8}$$

Note that $\lambda^2=\chi_2=\overline{\lambda}^2$ and the congruence $a+b+1\equiv 0 \bmod p$ implies the congruence $a^4+b^4+1\equiv 2\left(a^2+a+1\right)^2 \bmod p$. So we have

$$\sum_{\substack{a=0\\ a+b+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}\left(a^4+b^4+1\right)=\sum_{a=0}^{p-1}\overline{\lambda}\left(2\left(a^2+a+1\right)^2\right)$$
$$=\overline{\lambda}(2)\sum_{a=0}^{p-1}\chi_2\left(a^2+a+1\right)=\overline{\lambda}(2)\sum_{a=0}^{p-1}\chi_2\left(4a^2+4a+4\right)$$
$$=\overline{\lambda}(2)\sum_{a=0}^{p-1}\chi_2\left((2a+1)^2+3\right)=\overline{\lambda}(2)\sum_{a=0}^{p-1}\chi_2\left(a^2+3\right)=-\overline{\lambda}(2). \tag{9}$$

Combining (4), (5), (7), (8) and (9) we have the identity

$$\sum_{m=1}^{p-1}\lambda(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3=-p\tau(\lambda)-2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right). \tag{10}$$

If $p\equiv 1 \bmod 8$, then $\lambda(-1)=1$ and $\tau(\lambda)\tau\left(\overline{\lambda}\right)=p$, from the method of proving (8) we have

$$\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\overline{\lambda}\left(a^4+b^4+1\right)=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}e\left(\frac{ca^4+cb^4+c}{p}\right)$$
$$=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)e\left(\frac{c}{p}\right)\left(\chi_2(b)\sqrt{p}+\overline{\lambda}(b)\tau(\lambda)+\lambda(b)\tau\left(\overline{\lambda}\right)\right)^2$$
$$=\frac{1}{\tau(\lambda)}\sum_{c=1}^{p-1}\lambda(c)\left(3p+2\chi_2(c)\sqrt{p}\alpha+2\lambda(c)\sqrt{p}\tau(\lambda)+2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)e\left(\frac{c}{p}\right)$$
$$=5p+\frac{2\sqrt{p}\left(\alpha-1\right)\tau\left(\overline{\lambda}\right)}{\tau(\lambda)}. \tag{11}$$

Combining (4), (5), (7), (8) and (11) we have the identity

$$\sum_{m=1}^{p-1}\lambda(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3=-5p\tau(\lambda)-2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right). \tag{12}$$

Now Lemma 2.2 follows from (10) and (12). □

Lemma 2.3. *Let p be a prime with $p \equiv 1 \bmod 4$, then we have the identity*

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 = \begin{cases} p^2-6p\alpha & \text{if } p=24h+1 \text{ or } p=24h+13,\\ -p^2-6p\alpha & \text{if } p=24h+5 \text{ or } p=24h+17.\end{cases}$$

Proof. From (3) we have

$$\begin{aligned}
\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 &= p \sum_{\substack{a=0\\ a^4+b^4+c^4\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=0}^{p-1} e\left(\frac{a+b+c}{p}\right)\\
&= p \sum_{\substack{a=0\\ a^4+b^4\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} e\left(\frac{a+b}{p}\right) + p \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1}\sum_{c=1}^{p-1} e\left(\frac{c(a+b++1)}{p}\right)\\
&= p \sum_{\substack{a=0\\ a^4\equiv 0 \bmod p}}^{p-1} e\left(\frac{a}{p}\right) + p \sum_{\substack{a=0\\ a^4+1\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{b(a+1)}{p}\right)\\
&\quad + p^2 \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p\\ a+b+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 - p \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1\\
&= p - p \sum_{\substack{a=0\\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 + p^2 \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p\\ a+b+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 - p \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1. \qquad (13)
\end{aligned}$$

Now we calculate each term in (13). If $p \equiv 5 \bmod 8$, then note that $\lambda(-1) = -1$ we have

$$p \sum_{\substack{a=0\\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 = 0. \qquad (14)$$

Applying (6) and Lemma 2.1 we have

$$\begin{aligned}
p \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 &= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4+1\right)}{p}\right)\\
&= p^2 + \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^2 e\left(\frac{m}{p}\right)\\
&= p^2 + \sum_{m=1}^{p-1}\left(2\chi_2(c)\sqrt{p}\alpha - p + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right) e\left(\frac{m}{p}\right)\\
&= p^2 + 2p\alpha + p + 2\sqrt{p}\tau^2(\lambda) + 2\sqrt{p}\tau^2\left(\overline{\lambda}\right) = p^2 + p + 6p\alpha. \qquad (15)
\end{aligned}$$

It is clear that the congruences $a^4 + b^4 + 1 \equiv 0 \bmod p$ and $a + b + 1 \equiv 0 \bmod p$ implies that $ab \equiv 1 \bmod p$ and $a^3 \equiv b^3 \equiv 1 \bmod p$ with $a \neq b$. So we have

$$p^2 \sum_{\substack{a=0\\ a^4+b^4+1\equiv 0 \bmod p\\ a+b+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} 1 = p^2 \sum_{\substack{a=2\\ a^3\equiv b^3\equiv 1 \bmod p\\ ab\equiv 1 \bmod p}}^{p-1}\sum_{b=2}^{p-1} 1 = \begin{cases} 2p^2 & \text{if } p\equiv 1 \bmod 3,\\ 0 & \text{if } p \equiv 2 \bmod 3.\end{cases} \qquad (16)$$

Applying (13), (14), (15) and (16) we have the identity

$$\sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 = \begin{cases} p^2-6p\alpha & \text{if } p=24h+13,\\ -p^2-6p\alpha & \text{if } p=24h+5.\end{cases} \qquad (17)$$

If $p \equiv 1 \bmod 8$, then we also have

$$p \sum_{\substack{a=0 \\ a^4+1\equiv 0 \bmod p}}^{p-1} 1 = 4p. \tag{18}$$

$$\begin{aligned}
p \sum_{\substack{a=0 \\ a^4+b^4+1\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 &= \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^4+b^4+1\right)}{p}\right) \\
&= p^2 + \sum_{m=1}^{p-1} \left(3p + 2\chi_2(c)\sqrt{p}\alpha + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right) e\left(\frac{m}{p}\right) \\
&= p^2 + 2p\alpha - 3p + 2\sqrt{p}\tau^2(\lambda) + 2\sqrt{p}\tau^2\left(\overline{\lambda}\right) = p^2 - 3p + 6p\alpha.
\end{aligned} \tag{19}$$

$$p^2 \sum_{\substack{a=0 \\ a^4+b^4+1\equiv 0 \bmod p \\ a+b+1\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} 1 = p^2 \sum_{\substack{a=2 \\ a^3\equiv b^3\equiv 1 \bmod p \\ ab\equiv 1 \bmod p}}^{p-1} \sum_{b=2}^{p-1} 1 = \begin{cases} 2p^2 & \text{if } p = 24h+1, \\ 0 & \text{if } p = 24h+17. \end{cases} \tag{20}$$

Applying (13), (18), (19) and (20) we have

$$\sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 = \begin{cases} p^2 - 6p\alpha & \text{if } p = 24h+1, \\ -p^2 - 6p\alpha & \text{if } p = 24h+17. \end{cases} \tag{21}$$

It is clear that Lemma 2.3 follows from (17) and (21). □

Lemma 2.4. *Let p be a prime with $p \equiv 1 \bmod 4$, then we have the identity*

$$\sum_{m=1}^{p-1} \chi_2(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 = \begin{cases} p^{\frac{3}{2}}(p-6) & \text{if } p = 24h+1, \\ p^{\frac{3}{2}}(p-8) & \text{if } p = 24h+17, \\ -p^{\frac{3}{2}}(p-4) & \text{if } p = 24h+13, \\ -p^{\frac{3}{2}}(p-6) & \text{if } p = 24h+5. \end{cases}$$

Proof. From the properties of the Legendre's symbol mod p we have

$$\begin{aligned}
\sum_{m=1}^{p-1} \chi_2(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 &= \sum_{m=1}^{p-1} \chi_2(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2 \\
&+ \sum_{m=1}^{p-1} \chi_2(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^2 \left(\sum_{c=1}^{p-1} e\left(\frac{mc^4+c}{p}\right)\right) \\
&= \sqrt{p}\sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) + \sqrt{p}\sum_{a=0}^{p-1} \chi_2\left(a^4+1\right)\sum_{b=1}^{p-1} e\left(\frac{b(a+1)}{p}\right) \\
&+ \sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} \chi_2\left(a^4+b^4+1\right)\sum_{c=1}^{p-1} e\left(\frac{c(a+b+1)}{p}\right) \\
&= -\sqrt{p} + \chi_2(2)p^{\frac{3}{2}} - \sqrt{p}\sum_{a=0}^{p-1} \chi_2\left(a^4+1\right) - \sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1} \chi_2\left(a^4+b^4+1\right) \\
&+ p^{\frac{3}{2}} \sum_{\substack{a=0 \\ a+b+1\equiv 0 \bmod p}}^{p-1} \sum_{b=0}^{p-1} \chi_2\left(a^4+b^4+1\right).
\end{aligned} \tag{22}$$

From the properties of fourth-order mod p and Lemma 2.1 we have

$$\sum_{a=0}^{p-1} \chi_2\left(a^4+1\right) = 1 + \sum_{a=1}^{p-1} \chi_2(a+1)\left(1 + \lambda(a) + \chi_2(a) + \overline{\lambda}(a)\right)$$

$$= \frac{1}{\sqrt{p}} \cdot \left(\tau^2(\lambda) + \tau^2\left(\overline{\lambda}\right) - 1 = 2\alpha - 1. \tag{23}$$

$$\begin{aligned}
&\sum_{\substack{a=0 \\ a+b+1\equiv 0 \bmod p}}^{p-1}\sum_{b=0}^{p-1} \chi_2\left(a^4+b^4+1\right) = \sum_{a=0}^{p-1}\chi_2\left(a^4+(a+1)^4+1\right) \\
&= \chi_2(2)\sum_{a=0}^{p-1}\chi_2\left(a^4+2a^3+3a^2+2a+1\right) = \chi_2(2)\sum_{a=0}^{p-1}\chi_2\left(\left(a^2+a+1\right)^2\right) \\
&= \begin{cases} \chi_2(2)p & \text{if } p = 12h+1, \\ \chi_2(2)(p-2) & \text{if } p = 12h+5. \end{cases}
\end{aligned} \tag{24}$$

Note that $\tau(\lambda)\tau\left(\overline{\lambda}\right) = -p$, if $p = 8h+5$. $\tau(\lambda)\tau\left(\overline{\lambda}\right) = p$, if $p = 8h+1$. From the method of proving (15) and (19) we have

$$\begin{aligned}
&\sqrt{p}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\chi_2\left(a^4+b^4+1\right) = \sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{ma^4+mb^4+m}{p}\right) \\
&= \sum_{m=1}^{p-1}\chi_2(m)\left(\chi_2\sqrt{p}+\lambda(m)\tau\left(\overline{\lambda}\right)+\overline{\lambda}(m)\tau(\lambda)\right)^2 e\left(\frac{m}{p}\right) \\
&= \begin{cases} 7p^{\frac{3}{2}} - 2\sqrt{p}\alpha & \text{if } p = 8h+1, \\ -5p^{\frac{3}{2}} - 2\sqrt{p}\alpha & \text{if } p = 8h+5. \end{cases}
\end{aligned} \tag{25}$$

Combining (22), (23), (24) and (25) we have

$$\sum_{m=1}^{p-1}\chi_2(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3 = \begin{cases} p^{\frac{3}{2}}(p-6) & \text{if } p = 24h+1, \\ p^{\frac{3}{2}}(p-8) & \text{if } p = 24h+17, \\ -p^{\frac{3}{2}}(p-4) & \text{if } p = 24h+13, \\ -p^{\frac{3}{2}}(p-6) & \text{if } p = 24h+5. \end{cases}$$

This proves Lemma 2.4. □

3 Proofs of the theorems

Now we prove our main results. First we prove Theorem 1.1. If $p = 24h+1$, then from Lemmas 2.1, 2.2 and 2.4 we have

$$\begin{aligned}
V_1(p) &= \sum_{m=1}^{p-1}B(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3 \\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\lambda}(m)\tau(\lambda)+\lambda(m)\tau\left(\overline{\lambda}\right)\right)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3 \\
&= p^2(p-6) - 5p^2 - 2\sqrt{p}\alpha\tau^2(\lambda) - 5p^2 - 2\sqrt{p}\alpha\tau^2\left(\overline{\lambda}\right) \\
&= p\left(p^2 - 16p - 4\alpha^2\right).
\end{aligned} \tag{26}$$

Applying Lemmas 2.1–2.4 we also have

$$\begin{aligned}
V_2(p) &= \sum_{m=1}^{p-1}B^2(m)\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3 \\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\lambda}(m)\tau(\lambda)+\lambda(m)\tau\left(\overline{\lambda}\right)\right)^2\left(\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right)^3
\end{aligned}$$

$$= \sum_{m=1}^{p-1} \left(3p + 2\chi_2(m)\sqrt{p}\alpha + 2\lambda(m)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(m)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3$$
$$= p^2\left(2p\alpha + 3p - 58\alpha\right). \tag{27}$$

If $p = 8h + 1$, then from (6) we have

$$\begin{aligned} B^3(m) &= \left(\chi_2(m)\sqrt{p} + \overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right)\right)^3 \\ &= 7\chi_2(m)p^{\frac{3}{2}} + 4p\alpha + 5p\left(\overline{\lambda}(m)\tau(\lambda) + \lambda(m)\tau\left(\overline{\lambda}\right)\right) \\ &\quad +2\left(\overline{\lambda}(m)\tau\left(\overline{\lambda}\right) + \lambda(m)\tau(\lambda)\right)\sqrt{p}\alpha. \end{aligned} \tag{28}$$

So if $p = 24h + 1$, then from (28), Lemmas 2.1–2.4 we have

$$\begin{aligned} V_3(p) &= \sum_{m=1}^{p-1} B^3(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 \\ &= 7p^3(p-6) + 4p\alpha\left(p^2 - 6p\alpha\right) - 5p\tau\left(\overline{\lambda}\right)\left(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right) \\ &\quad -5p\tau(\lambda)\left(5p\tau\left(\overline{\lambda}\right) + 2\sqrt{p}\alpha\tau(\lambda)\right) - 2\sqrt{p}\alpha\tau(\lambda)\left(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right) \\ &\quad -2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\left(5p\tau\left(\overline{\lambda}\right) + 2\sqrt{p}\alpha\tau(\lambda)\right) \\ &= p^2\left(7p^2 + 4p\alpha - 92p - 72\alpha^2\right). \end{aligned} \tag{29}$$

If $p = 24h + 17$, then from Lemmas 2.1–2.4 we have

$$\begin{aligned} V_1(p) &= \sum_{m=1}^{p-1} B(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 \\ &= p^2(p-8) - 5p^2 - 2\sqrt{p}\alpha\tau^2(\lambda) - 5p^2 - 2\sqrt{p}\alpha\tau^2\left(\overline{\lambda}\right) \\ &= p\left(p^2 - 18p - 4\alpha^2\right). \end{aligned} \tag{30}$$

$$V_2(p) = \sum_{m=1}^{p-1} B^2(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3$$
$$= \sum_{m=1}^{p-1} \left(3p + 2\chi_2(c)\sqrt{p}\alpha + 2\lambda(c)\sqrt{p}\tau(\lambda) + 2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3$$
$$= p^2\left(2p\alpha - 3p - 62\alpha\right). \tag{31}$$

Applying (28) and the method of proving (29) we also have

$$\begin{aligned} V_3(p) &= \sum_{m=1}^{p-1} B^3(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 \\ &= 7p^3(p-8) - 4p\alpha\left(p^2 + 6p\alpha\right) - 5p\tau\left(\overline{\lambda}\right)\left(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right) \\ &\quad -5p\tau(\lambda)\left(5p\tau\left(\overline{\lambda}\right) + 2\sqrt{p}\alpha\tau(\lambda)\right) - 2\sqrt{p}\alpha\tau(\lambda)\left(5p\tau(\lambda) + 2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right) \\ &\quad -2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\left(5p\tau\left(\overline{\lambda}\right) + 2\sqrt{p}\alpha\tau(\lambda)\right) \\ &= p^2\left(7p^2 - 4p\alpha - 106p - 72\alpha^2\right). \end{aligned} \tag{32}$$

Similarly, if $p = 24h + 5$, then we have

$$\begin{aligned} V_1(p) &= \sum_{m=1}^{p-1} B(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3 \\ &= -p^2(p-6) + p^2 - 2\sqrt{p}\alpha\tau^2(\lambda) + p^2 - 2\sqrt{p}\alpha\tau^2\left(\overline{\lambda}\right) \\ &= -p\left(p^2 - 8p + 4\alpha^2\right). \end{aligned} \tag{33}$$

$$\begin{aligned}V_2(p) &= \sum_{m=1}^{p-1} B^2(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\lambda}(m)\tau(\lambda)+\lambda(m)\tau\left(\overline{\lambda}\right)\right)^2\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= \sum_{m=1}^{p-1}\left(2\chi_2(c)\sqrt{p}\alpha-p+2\lambda(c)\sqrt{p}\tau(\lambda)+2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= -p^2\left(2p\alpha-p-22\alpha\right).\end{aligned} \tag{34}$$

If $p = 24h+5$, then from (6) we have

$$\begin{aligned}B^3(m) &= \left(\chi_2(m)\sqrt{p}+\overline{\lambda}(m)\tau(\lambda)+\lambda(m)\tau\left(\overline{\lambda}\right)\right)^3\\
&= -5\chi_2(m)p^{\frac{3}{2}}+6p\alpha+p\left(\overline{\lambda}(m)\tau(\lambda)+\lambda(m)\tau\left(\overline{\lambda}\right)\right)\\
&\quad +2\left(\overline{\lambda}(m)\tau\left(\overline{\lambda}\right)+\lambda(m)\tau(\lambda)\right)\sqrt{p}\alpha.\end{aligned} \tag{35}$$

So from (35) and the method of proving (29) we have

$$\begin{aligned}V_3(p) &= \sum_{m=1}^{p-1} B^3(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= 5p^3(p-6)-6p\alpha\left(p^2+6p\alpha\right)-p\tau\left(\overline{\lambda}\right)\left(p\tau(\lambda)+2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right)\\
&\quad -p\tau(\lambda)\left(p\tau\left(\overline{\lambda}\right)+2\sqrt{p}\alpha\tau(\lambda)\right)-2\sqrt{p}\alpha\tau(\lambda)\left(p\tau(\lambda)+2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right)\\
&\quad -2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\left(p\tau\left(\overline{\lambda}\right)+2\sqrt{p}\alpha\tau(\lambda)\right)\\
&= p^2\left(5p^2-6p\alpha-28p-36\alpha^2\right).\end{aligned} \tag{36}$$

If $p = 24h+13$, then (35), Lemmas 2.1–2.4 we have

$$\begin{aligned}V_1(p) &= \sum_{m=1}^{p-1} B(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= -p^2(p-4)+p^2-2\sqrt{p}\alpha\tau^2(\lambda)+p^2-2\sqrt{p}\alpha\tau^2\left(\overline{\lambda}\right)\\
&= -p\left(p^2-6p+4\alpha^2\right).\end{aligned} \tag{37}$$

$$\begin{aligned}V_2(p) &= \sum_{m=1}^{p-1} B^2(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\lambda}(m)\tau(\lambda)+\lambda(m)\tau\left(\overline{\lambda}\right)\right)^2\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= \sum_{m=1}^{p-1}\left(2\chi_2(c)\sqrt{p}\alpha-p+2\lambda(c)\sqrt{p}\tau(\lambda)+2\overline{\lambda}(c)\sqrt{p}\tau\left(\overline{\lambda}\right)\right)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= -p^2\left(2p\alpha+p-18\alpha\right).\end{aligned} \tag{38}$$

$$\begin{aligned}V_3(p) &= \sum_{m=1}^{p-1} B^3(m)\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right)^3\\
&= 5p^3(p-4)+6p\alpha\left(p^2-6p\alpha\right)-p\tau\left(\overline{\lambda}\right)\left(p\tau(\lambda)+2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right)\\
&\quad -p\tau(\lambda)\left(p\tau\left(\overline{\lambda}\right)+2\sqrt{p}\alpha\tau(\lambda)\right)-2\sqrt{p}\alpha\tau(\lambda)\left(p\tau(\lambda)+2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\right)\\
&\quad -2\sqrt{p}\alpha\tau\left(\overline{\lambda}\right)\left(p\tau\left(\overline{\lambda}\right)+2\sqrt{p}\alpha\tau(\lambda)\right)\\
&= p^2\left(5p^2+6p\alpha-18p-36\alpha^2\right).\end{aligned} \tag{39}$$

Finally, note that if $p = 8h+1$, then from (6) and direct calculation (or see Lemma 3 in [7]) we have the identity

$$B^4(m) = 6pB^2(m) + 8p\alpha B(m) - p\left(p - 4\alpha^2\right). \tag{40}$$

For any prime $p = 24h + 1$ and integer $k \geq 4$, from (26), (27), (29) and (40) we may immediately deduce the fourth-order linear recurrence formula

$$\begin{aligned} V_k(p) &= \sum_{m=1}^{p-1} B^k(m) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3 \\ &= \sum_{m=1}^{p-1} B^{k-4}(m) \left(6pB^2(m) + 8p\alpha B(m) - p\left(p - 4\alpha^2\right)\right) \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right)^3 \\ &= 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right) V_{k-4}(p), \end{aligned}$$

where the first four values $V_0(p) = p^2 - 6p\alpha$, $V_1(p) = p\left(p^2 - 16p - 4\alpha^2\right)$, $V_2(p) = p^2\left(2p\alpha + 3p - 58\alpha\right)$ and $V_3(p) = p^2\left(7p^2 + 4p\alpha - 92p - 72\alpha^2\right)$.

This proves Theorem 1.1.

If $p = 24h + 17$, then from (30), (31), (32) and (40) we have

$$V_k(p) = 6pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(p - 4\alpha^2\right) V_{k-4}(p),$$

where the first four values $V_0(p) = -p^2 - 6p\alpha$, $V_1(p) = p\left(p^2 - 18p - 4\alpha^2\right)$, $V_2(p) = p^2\left(2p\alpha - 3p - 62\alpha\right)$ and $V_3(p) = p^2\left(7p^2 - 4p\alpha - 106p - 72\alpha^2\right)$.

This proves Theorem 1.2.

If $p = 8h + 5$, then from (6) and direct calculation (or see Lemma 3 in [7]) we also have

$$B^4(m) = -2pB^2(m) + 8p\alpha B(m) - p\left(9p - 4\alpha^2\right). \tag{41}$$

For any prime $p = 24h + 5$ and integer $k \geq 4$, from (33), (34), (35) and (41) we can deduce the fourth-order linear recurrence formula

$$V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right) V_{k-4}(p),$$

where the first four terms are $V_0(p) = -\left(p^2 + 6p\alpha\right)$, $V_1(p) = -p\left(p^2 - 8p + 4\alpha^2\right)$, $V_2(p) = -p^2\left(2p\alpha - p - 22\alpha\right)$ and $V_3(p) = p^2\left(5p^2 - 6p\alpha - 28p - 36\alpha^2\right)$.

This proves Theorem 1.3.

If $p = 24h + 13$, then from (37), (38), (39) and (41) we also have

$$V_k(p) = -2pV_{k-2}(p) + 8p\alpha V_{k-3}(p) - p\left(9p - 4\alpha^2\right) V_{k-4}(p),$$

where the first four terms are $V_0(p) = p^2 - 6p\alpha$, $V_1(p) = -p\left(p^2 - 6p + 4\alpha^2\right)$, $V_2(p) = -p^2\left(2p\alpha + p - 18\alpha\right)$ and $V_3(p) = p^2\left(5p^2 + 6p\alpha - 18p - 36\alpha^2\right)$.

This completes the proofs of our all results.

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